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Schwarz Reflexion Principle in 3-Space

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Introduction

The Schwarz reflexion principle is well-known in the theory of harmonic functions in a plane. In the three dimensional euclidean space (=3-space), however, it seems that some problems remain to be discussed.[1] In this paper, we shall show that any harmonic function h, defined in a domain D within an open ball V and having vanishing normal derivative on a part E of $\partial D \cap \partial V$, can always be continued across E but in general only *radially*.

J. W. Green [2] treated the case where D coincides with V. He showed that h is continued harmonically through E to the entire outside of V if and only if $\int_0^R h(r, \theta, \varphi)dr$ is constant as a function of (θ, φ) on the set $\{(\theta, \varphi); (R, \theta, \varphi) \in E\}$, and that there is a case where h cannot be continued harmonically to the entire outside of V.

§**1.** First we explain notation. Throughout this paper, V means the open ball with center at the origin 0 and radius R in the 3-space, $S=\partial V$ its boundary, D a subdomain of V, ∂D its boundary, E a two dimensional open set on $\partial D \cap S$ which contains no point of accumulation of $\partial D - E$, h a harmonic function in D, and, for a point $P \in D$, P' the symmetric point of P with respect to S. This point is called also the point of reflexion or the mirror image of P.

The case when h vanishes on E is known and stated as

Proposition. *If h is continuous on $D \cup E$ and vanishes on E, then h is extended through E to a harmonic function in the domain D' which is the reflexion of D with respect to S.*

Proof. Choose any $Q \in S$ and let Σ be the spherical surface with center Q and radius R_0. Invert the space with respect to Σ and denote by P^* the image of P by the inversion. The image of S is a plane, and P^* and P'^* are symmetric with respect to the plane. Define a function $h^*(P^*)$ by $\overline{OQ} \cdot h(P)/R_0$

1) O. D. Kellogg suggested to "derive results similar to (the result in the case where $h=0$ on E), where $\cdots$ it is assumed that the normal derivative of U vanishes on that portion" in Exercise 4 at p. 262 of [3]. It is stated at p. 244 in Lichtenstein [4] that "$\cdots$ (plane case) $\cdots$. Analoge Sätze gelten im Raume." However, this turns out not to be the case.

on the image of $D\cup E$ and by $-\overline{OQ}\cdot h(P)/R_0$ on the image of D'. The function is harmonic on the image of $\hat{D}=D\cup E\cup D'$. Therefore, if h is extended to $P'\in D'$ by $h(P')=(R_0/\overline{QP'})h^*(P'^*)=-(\overline{QP}/\overline{QP'})h(P)=-\overline{OP}\cdot h(P)/R$, then h is harmonic in $\hat{D}$.

§**2.** Our interest in the subject of the present paper lies in the case where the normal derivative $\partial h/\partial n$ vanishes on E. The situation is less simple in this case than in the case where h vanishes.

The case where $\partial h/\partial n=$const. c on E is reduced to the case $c=0$ if h is replaced by $h+cR^2/r$ in $D-\{0\}$. However, in case D coincides with V and $\partial h/\partial n=c\neq 0$ on E, h can never be continued through E to the entire outside of V as is shown in Theorem 3 of [2].

We begin with

LEMMA 1 ([2]). *The function $r\partial h/\partial r$ is harmonic in D.*

PROOF. If the origin is not included in D, we have, with polar coordinates,

$$r^2\Delta\left(r\frac{\partial h}{\partial r}\right)=\frac{\partial}{\partial r}\left(r^2\frac{\partial(r\partial h/\partial r)}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial(r\partial h/\partial r)}{\partial\theta}\right)$$
$$+\frac{1}{\sin^2\theta}\frac{\partial^2(r\partial h/\partial r)}{\partial\varphi^2}=r\frac{\partial}{\partial r}(r^2\Delta h)=0.$$

If the origin is included in D, it is a removable singularity for $r\partial h/\partial r$.

Hereafter we assume that h is continuous on $D\cup E$ together with its partial derivatives $\partial h/\partial x$, $\partial h/\partial y$, $\partial h/\partial z$ and that $\partial h/\partial n=0$ on E. Denote by D'_E the set of points of D' which can be connected to points of E radially by segments lying on $D'\cup E$, and by $\hat{D}_E$ the domain $D\cup E\cup D'_E$. We shall prove

THEOREM 1. *One can continue h to a harmonic function in $\hat{D}_E$.*

PROOF. By the proposition, $r\partial h/\partial r$ is extended to a harmonic function H in $\hat{D}=D\cup E\cup D'$. It is equal at $P'\in D'$ to the value of $-r^2R^{-1}\partial h/\partial r$ at P. Define $\hat{h}$ in $\hat{D}_E$ by

$$\hat{h}(r,\theta,\varphi)=h(r_0,\theta,\varphi)+\int_{r_0}^{r}\frac{H}{r}dr$$

where $(r_0,\theta,\varphi)\in D$ is chosen so that the segment between this point and (r,θ,φ) is contained in $\hat{D}_E$. The definition of $\hat{h}$ is independent of the choice of r_0 and $\hat{h}=h$ at (r_0,θ,φ). Let us show that h is harmonic in $\hat{D}_E$.

Denote by Δ_Θ the operator

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

This is not defined on the axis $\theta=0,\ \pi$ but the values $\Delta_{\circledcirc}f$ for any C^2 function f are independent of the choice of an axis, because

$$\Delta_{\circledcirc}f=r^2\Delta f-\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right).$$

We have

$$r^2\Delta\hat{h}=\Delta_{\circledcirc}h\Big|_{r=r_0}+\int_{r_0}^{r}\Delta_{\circledcirc}\left(\frac{H}{r}\right)dr+\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\int_{r_0}^{r}\frac{H}{r}dr\right)\right)$$

$$=\Delta_{\circledcirc}h\Big|_{r=r_0}+\int_{r_0}^{r}\frac{1}{r}\left(-\frac{\partial}{\partial r}\left(r^2\frac{\partial H}{\partial r}\right)dr\right)+\frac{\partial}{\partial r}(rH).$$

Since

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial H}{\partial r}\right)=r\frac{\partial^2(rH)}{\partial r^2},$$

we have

$$r^2\Delta\hat{h}=\Delta_{\circledcirc}h\Big|_{r=r_0}+\frac{\partial}{\partial r}(rH)\Big|_{r=r_0}-\frac{\partial}{\partial r}(rH)+\frac{\partial}{\partial r}(rH)$$

$$=\Delta_{\circledcirc}h\Big|_{r=r_0}+\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)\Big|_{r=r_0}=r^2\Delta h\Big|_{r=r_0}=0.$$

Thus $\hat{h}$ is harmonic in $\hat{D}_E$.

Being different from the case in plane, $\hat{h}$ is not always symmetric with respect to S. Actually, if $\hat{h}(r',\theta,\varphi)=h(r,\theta,\varphi)$ with $r'=R^2/r>R$, then

$$0=r'^2\Delta\hat{h}=\frac{\partial}{\partial r'}\left(r'^2\frac{\partial h}{\partial r'}\right)+\Delta_{\circledcirc}h=\frac{\partial}{\partial r'}\left(r'^2\frac{\partial h}{\partial r'}\right)-\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right).$$

By a simple computation we see that the right hand side is equal to $-2r\partial h/\partial r$. It follows that h is independent of r in D.

On the other hand, if the Kelvin transform $\overline{OP}\cdot h(P)/R$ is the harmonic continuation, its normal derivative must vanish on E. On E we have

$$\frac{1}{R}\frac{\partial}{\partial r'}(rh(r,\theta,\varphi))\Big|_{r=R}=-\frac{R}{r^2}\frac{\partial}{\partial r}(rh)\Big|_{r=R}$$

$$=-\frac{R}{r^2}\left(h+r\frac{\partial h}{\partial r}\right)\Big|_{r=R}=-\frac{h}{R}=0.$$

Thus h vanishes on E. Therefore $-\overline{OP}\cdot h(P)/R$ is the harmonic extension into D' as was seen in the proof of the proposition. Thus $\overline{OP}\cdot h(P)=-\overline{OP}\cdot h(P)$ for every $P\in D$ and hence $h\equiv 0$ in D.

It is not always possible to extend h harmonically to the entire symmetric domain of an arbitrary domain D as an example will show it later. However, we have

THEOREM 2. *Let H be the harmonic extension of $r\partial h/\partial r$ in $\hat{D}$. Let $P_0'=(r_0', \theta_0, \varphi_0)$ be in D', and h' be a function harmonic in a neighborhood U of P_0' such that $r'\partial h'/\partial r\Big|_{r=r'}=H(r', \theta, \varphi)$ in U. Let P_1' be a point of D' such that the segment $P_0'P_1'$ is included in D' and lies on a ray issuing from the origin. Then h' is defined harmonically in a neighborhood of $P_0'P_1'$.*

PROOF. Define h' in a neighborhood of $P_0'P_1'$ by

$$h'(r', \theta, \varphi)=h'(r_0', \theta, \varphi)+\int_{r_0'}^{r'}\frac{H}{r}dr$$

where (r_0', θ, φ) is in U. As in the proof of Theorem 1 we have $r'^2\Delta h'=0$.

COROLLARY. *If h is extended harmonically to $P'\in D'$, then it is extended harmonically to P_1' so far as $P'P_1'$ is included in D' and lies on a ray issuing from the origin.*

We give a condition for extensibility in a special case. First we give a lemma which is similar to Lemma 1 of [2].

LEMMA 2. *Let $(r_0, \theta, \varphi)\in D$ and suppose h is extended harmonically to the point $(R^2/r_0, \theta, \varphi)$. Denote the extension of h by $\hat{h}$. Then*

$$\Delta_{\circledcirc}(R\hat{h}(r_0')-r_0h(r_0))=r_0^2\frac{\partial h}{\partial r}\Big|_{r=r_0},$$

where $h(r, \theta, \varphi)$ is written simply as $h(r)$ and $\hat{h}(r, \theta, \varphi)$ as $\hat{h}(r)$.

PROOF. We have

$$\partial\hat{h}(r)/\partial r'=-r^3R^{-3}\partial h(r)/\partial r. \tag{1}$$

Hence

$$\begin{aligned}\hat{h}(r')&=\hat{h}(r_0')+\int_{r_0'}^{r'}\frac{\partial\hat{h}}{\partial r'}dr'=\hat{h}(r_0')+\frac{1}{R}\int_{r_0}^{r}r\frac{\partial h}{\partial r}dr\\&=\hat{h}(r_0')+\frac{r}{R}h(r)-\frac{r_0}{R}h(r)-\frac{1}{R}\int_{r_0}^{r}hdr.\end{aligned}\tag{2}$$

It follows that

$$\begin{aligned}\Delta_{\circledcirc}\hat{h}(r')&=\Delta_{\circledcirc}\hat{h}(r_0')+\frac{r}{R}\Delta_{\circledcirc}h(r)-\frac{r_0}{R}\Delta_{\circledcirc}h(r_0)-\frac{1}{R}\int_{r_0}^{r}\Delta_{\circledcirc}hdr\\&=\Delta_{\circledcirc}\hat{h}(r_0')-\frac{r_0}{R}\Delta_{\circledcirc}h(r_0)-\frac{r}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)+\frac{1}{R}\int_{r_0}^{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)dr.\end{aligned}$$

On the other hand, we have by (1)

$$\Delta_{\circledcirc}\hat{h}(r')=-\frac{\partial}{\partial r'}\left(r'^2\frac{\partial\hat{h}(r')}{\partial r'}\right)=-\frac{r^2}{R}\frac{\partial}{\partial r}\left(r\frac{\partial h(r)}{\partial r}\right).$$

Hence

$$\Delta_{\circledcirc}\hat{h}(r_0')-\frac{r_0}{R}\Delta_{\circledcirc}h(r_0)$$

$$=-\frac{r^2}{R}\frac{\partial h}{\partial r}-\frac{r^3}{R}\frac{\partial^2 h}{\partial r^2}+2\frac{r^2}{R}\frac{\partial h}{\partial r}+\frac{r^3}{R}\frac{\partial^2 h}{\partial r^2}-\frac{r^2}{R}\frac{\partial h}{\partial r}+\frac{r_0^2}{R}\frac{\partial h}{\partial r}\bigg|_{r_0}$$

$$=\frac{r_0^2}{R}\frac{\partial h}{\partial r}\bigg|_{r_0}.$$

§**3.** In this section we assume that $\partial D\cap S$ contains a two dimensional open set $B\supsetneqq E$ which has no point of accumulation of $\partial D-B$, that every point P of B can be connected with a point in D radially by a segment which is contained in D except for P, and that $\partial h/\partial x$, $\partial h/\partial y$, $\partial h/\partial z$ are continuously extended to $D\cup B$. Assume furthermore that $\partial^2 h/\partial\theta^2$ and $\partial^2 h/\partial\varphi^2$ can be continuously extended to $D\cup B$.

Suppose that h is extended harmonically to a function $\hat{h}$ in $D\cup E\cup D_B'$, where D_B' is defined in the same way as D_E'. Then $\partial^2\hat{h}/\partial\theta^2$ and $\partial^2\hat{h}/\partial\varphi^2$ are also continuously extended to B from D_B' by (2). By Lemma 2 we obtain immediately

Lemma 3. $$\Delta_{\circledcirc}(\check{h}-h)=R\frac{\partial h}{\partial r}\qquad \textit{on } B,$$
where $\check{h}(R,\theta,\varphi)=\lim_{r'\downarrow R}\hat{h}(r',\theta,\varphi)$.

Theorem 3. *h can be continued to a harmonic function $\hat{h}$ in $D\cup E\cup D_B'$ if and only if there is a solution g of $\Delta_{\circledcirc}g=R\partial h/\partial n$ on B such that g vanishes on E.*

Proof. Suppose such a g exists. Set $p=g+h$ on B and

$$h'(r,\theta,\varphi)=p(R,\theta,\varphi)+\int_R^{r'}\frac{H}{r'}dr'\qquad \text{in } D_B'.$$

On E, $p=h$ and hence h' is the harmonic extension of h into D_E'. Let us show that h' is harmonic in D_B'. For $(r',\theta,\varphi)\in D_B'$ we have by the same computation as in the proof of Theorem 1

$$r'^2\Delta h'=\Delta_{\circledcirc}p+\frac{\partial}{\partial r'}(r'H)\bigg|_{r'=R}=R\frac{\partial h}{\partial r}\bigg|_{r=R}+\Delta_{\circledcirc}h\bigg|_{r=R}+R\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right)\bigg|_{r=R}$$

$$=R\frac{\partial h}{\partial r}\bigg|_{r=R}-\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)\bigg|_{r=R}+R^2\frac{\partial^2 h}{\partial r^2}\bigg|_{r=R}+R\frac{\partial h}{\partial r}\bigg|_{r=R}=0.$$

Thus h' is harmonic in D'_B.

Conversely, suppose $\hat{h}$ is a harmonic extension of h in $D \cup E \cup D'_B$. Denote $\lim_{r' \downarrow R} \hat{h}(r', \theta, \varphi)$ by $\check{h}(R, \theta, \varphi)$ as before and set $g(R, \theta, \varphi) = \check{h}(R, \theta, \varphi) - h(R, \theta, \varphi)$. Then, on account of Lemma 3, g satisfies $\Delta_{\circledcirc} g = R\partial h/\partial n$ on B and vanishes on E. Our theorem is now proved.

COROLLARY. *Consider the case that D coincides with V. In order that h be extended across E to a harmonic function outside V, it is necessary and sufficient that there exists a solution g of $\Delta_{\circledcirc} g = R\partial h/\partial n$ on S such that g vanishes on E.*

This condition must be equivalent to the already quoted Green's condition in [2] that $\int_0^R h dr$ is constant on E. Actually one can show the equivalence directly as follows:[2]

If there exists g satisfying $\Delta_{\circledcirc} g = R\partial h/\partial n$ on S and $g=0$ on E, then $\int_0^R h dr =$ const. on E because

$$\Delta_{\circledcirc}\left(\int_0^R h dr + Rg\right) = \int_0^R \Delta_{\circledcirc} h dr + R^2 \frac{\partial h}{\partial n} = -\int_0^R \frac{\partial}{\partial r}\left(r^2 \frac{\partial h}{\partial r}\right) dr + R^2 \frac{\partial h}{\partial n} = 0 \text{ on } S^{3)}$$

and hence $\int_0^R h dr = -Rg +$ const. $=$ const. on E. Conversely, assume $\int_0^R h dr = c$ ($=$const.) on E. Then $g = -\dfrac{1}{R}\left(\int_0^R h dr - c\right)$ satisfies $\Delta_{\circledcirc} g = R\partial h/\partial n$ on S and $g=0$ on E.

Finally, we shall prove a theorem by means of which we can show that Theorem 1 is the best possible in case the (two dimensional) boundary of E is smooth.

THEOREM 4. *Suppose there is a C^4 function f on S with the following properties:*

(i) $\int_S \Delta_{\circledcirc} f dS = 0$,

(ii) $\Delta_{\circledcirc} f = 0$ *on E,*

(iii) *there exists no two dimensional domain $B \subset S$ which satisfies $B \not\subset E$ and $B \cap E \neq \emptyset$, and on which a function f_1 is defined so that $\Delta_{\circledcirc} f_1 = 0$ on B and $f_1 = f$ on $B \cap E$.*

Then the solution h of the Neumann problem in $D = V$ for the boundary condition $\partial h/\partial n = R^{-1}\Delta_{\circledcirc} f$ can never be continued harmonically to any point of $\hat{D} - \hat{D}_E$.[4]

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2) The author owes this remark to Professor H. Lewy.

3) Let f be a function on S which is twice continuously differentiable with respect to θ and φ and satisfies $\Delta_{\circledcirc} f = 0$. Then the extension f^* of f by $f^*(r, \theta, \varphi) = f(R, \theta, \varphi)$ to the whole space is harmonic because $r^2 \Delta f^* = \Delta_{\circledcirc} f^* + \partial(r^2 \partial f^*/\partial r)/\partial r = 0$. By the maximum principle it is concluded that f^* is constant.

4) cf. Theorem 1.

PROOF. It is known that the partial derivatives of second order of h have limits on S; see [5]. Suppose h is extended harmonically to a point $P' \in \hat{D} - \hat{D}_E$. Then, by the corollary of Theorem 2, there exists a two dimensional domain B on S such that $B \not\subset E$, $B \cap E \neq \emptyset$ and h is continued harmonically to $D \cup E \cup D'_B$. Theorem 3 implies that there exists g on B such that $\Delta_{\circledcirc} g = R\partial h/\partial n$ on B and $g=0$ on E. The function $f_1 = f - g$ satisfies $\Delta_{\circledcirc} f_1 = 0$ on B and $f_1 = f$ on E. This contradicts (iii).

Let us see that a function like f exists actually in case E is a two dimensional subdomain of S bounded by a finite number of closed analytic curves. Let ψ be a sufficiently smooth function which is defined on the boundary ∂E of E and which is nowhere analytic with respect to the defining parameter of ∂E, and f_0 be the function which satisfies $\Delta_{\circledcirc} f_0 = 0$ in E and $f_0 = \psi$ on ∂E. It follows that f_0 is of C^4 class on $E \cup \partial E$; see [5]. Extend f_0 to a function f of C^4 class on S so that condition (i) is satisfied. If there exist B and f_1 with the properties as decribed in (iii), then f_1 as a solution of $\Delta_{\circledcirc} f_1 = 0$ is analytic in B and hence on $\partial E \cap B$. This contradicts our assumption that, on $\partial E \cap B$, $f_1 = f = \psi$ is nowhere analytic with respect to the defining parameter of ∂E. Thus (iii) is satisfied too.

To the contrary, if a part F of $S - E$ is small, *e.g.*, if F is a closed set of logarithmic capacity zero such that $S - E - F$ is closed, then h can be continued harmonically to the set A consisting of points of $\hat{D} - D$ which can be connected to F radially in $\hat{D}$. This follows from the fact that A is of Newtonian capacity zero (cf. [1], p. 92) and hence removable for the extension of h in $\hat{D}_E$.

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