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On Certain Classes of Malcev Algebras

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This note is a sequel to the author's earlier paper [5]. For brevity we adopt the notations and definitions employed in [5] without explaining them here again. This note is concerned only with Malcev algebras (finite-dimensional) belonging to the classes of general algebras dealt with in [5]. As is well-known (see e.g. [6]), a *Malcev algebra* A is an anticommutative algebra satisfying the identity (x, y, z in A):

$$xy\cdot xz=(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y.$$

In [5] we proved: *A solvable ideal* B *of an* (A_3')*-algebra* A, *whose derived series consists of ideals, is contained in the annihilator ideal of* A. For Lie algebras, which are a priori Malcev algebras, the derived series consists of ideals and the above result reduces to a known result as generalized by us earlier (see [5, Corollary 2.9]): *A solvable ideal* B *of a Lie* (A_3)*-algebra* A *is contained in the center of* A. On the other hand, for a Malcev algebra, the derived series need not in general consist of ideals (see [6, Example 3.4] for an example of such a Malcev algebra). However, we show (Proposition 1) that the above result for Lie algebras can be extended to Malcev algebras over fields of characteristic $\neq 2$, by using a recent result of Kuz'min ([2, Lemma 2]). The main interest in this extension lies in its application to the proof of a characterizing result (Theorem 4) for Malcev (A_k)-algebras ($k\geqslant 3$). This result considered along with some known results leads to an interesting conclusion (Theorem 5): *A Malcev* (A_3)*-algebra over an algebraically closed field of characteristic zero is abelian.*

In what follows, unless otherwise stated, A is a Malcev algebra over a field of characteristic $\neq 2$.

1. If B is an ideal of A, we write $B^{[1]}=B$, $B^{[2]}=BB+(BB)A$, $\dots$, $B^{[n]}=B^{[n-1]}B^{[n-1]}+(B^{[n-1]}B^{[n-1]})A$, $\dots$. $B^{[k]}$ are ideals of A. B is said to be L-*solvable* (see [2]) if there exists an integer n such that $B^{[n]}=0$. For such of those Malcev algebras as we here consider, L-solvability of an ideal is equivalent to its solvability in the usual sense. It is this result of Kuz'min ([2, Lemma 2]) which we employ for proving

PROPOSITION 1. *A solvable ideal* B *of a Malcev* (A_3)*-algebra* A *is contained in the center of* A.

PROOF. B is L-solvable. Let $B^{[n]} \neq 0$, $B^{[n+1]} = 0$ or $B^{[n]}B^{[n]} + (B^{[n]}B^{[n]})A = 0$. Then $B^{[n]}B^{[n]} = 0$. Since $B^{[n]}$ is an ideal of A, this means that $B^{[n]}(B^{[n]}A) = 0$. In other words, $L_x^2 = 0$ for x in $B^{[n]}$. But the (A_3)-algebra A is also an (A_2)-algebra (see chart in Section 1 of [5]), so that $L_x = 0$, i.e., $B^{[n]}A = 0$. In particular $(B^{[n-1]}B^{[n-1]})A = 0 = (B^{[n-1]}B^{[n-1]})B^{[n-1]}$. $B^{[n-1]}$ being an ideal of A, we then have, for x in $B^{[n-1]}$, $AL_x^3 \subseteq B^{[n-1]}(B^{[n-1]}B^{[n-1]}) = 0$, and by the (A_3)-property of A, $B^{[n-1]}A = 0 = B^{[n-1]}B^{[n-1]}$. Consequently $B^{[n]} = 0$, contradicting our assumption. Thus we should have $B^{[2]} = 0 = B^2$. Then $B(BA) = 0$, and by (A_2)-property, we are led to $BA = 0$, which is the desired conclusion, viz. that the center of A contains B.

COROLLARY 2. *The radical (maximal solvable ideal) of a Malcev (A_3)-algebra is the center of A.*

Corollary 2 in effect reduces the study of the structure of Malcev (A_3)-algebras to that of Malcev algebras whose radical and center coincide. The structure of a subclass (viz., that of algebras over a field of characteristic zero) of the latter class of Malcev algebras is given by Lemma 3 below, which plays a major role in the proof of Theorem 4 following it.

LEMMA 3 (cf. [7, Definition 3]). *For a Malcev algebra A over a field of characteristic zero, the following properties are equivalent:*

(i) *The radical of A is contained in the center of A.*

(ii) *For a central ideal I, A/I is semisimple.*

(iii) *A is the direct sum of its center and the semisimple ideal A^2.*

We call a Malcev algebra which satisfies any one of the above three properties, a *reductive* algebra.

The analogue of Lemma 3 for Lie algebras referred to against it needs for its proof the Levi factor theorem. In the absence of a theorem of the same type for Malcev algebras in general, we require alternative tools, in the form of the following results, for the proof of the lemma. Let T_A denote the Lie triple system associated to a Malcev algebra A. Then, over a field of characteristic zero, the radicals of A and T_A coincide, and any characteristic ideal of T_A is a characteristic ideal of A (see [4, Lemma 2, and Theorem 2]). Further, any Lie triple system T over a field of characteristic zero is a direct sum of its radical R and a semisimple subsystem T' of T (see [3, Theorem 2.21]).

PROOF OF LEMMA 3. The only implication which is neither known nor is trivial is (i)$\Rightarrow$(iii). We prove this as follows. If the radical R of A is contained in the center of A, it is the center itself. R is the radical of T_A. Now, $T_A = R \oplus T'$, where T' is a semisimple subsystem of T_A. Since $xz = 0$ for all x in R and z in A, $[x T_A T_A] = 0 = [T_A x T_A] = [T_A T_A x]$. As a consequence we have $[T' T_A T_A] = [T'R + T'R + T'] = [T' T' T'] \subseteq T'$. Thus T' is an ideal of T_A. Further, T' being semisimple, $[T'T'T']=T'$, so that T' is a characteristic ideal of T_A and therefore also of A. In particular, T' is an ideal of A. But, considered as a Malcev algebra, T' is semisimple. We thus have $A=R\oplus T'$ where R is the radical (=center) of A and T' is an ideal of A, which is a semisimple Malcev algebra. This immediately shows that A has property (iii).

REMARK. We incidentally note that any Malcev algebra A over a field of characteristic zero such that $\{R_x\}_{x\in A}$ does not contain any nilpotent element other than 0 is reductive (cf. [7, Proposition 7]).

2. We now obtain a characterization of Malcev (A_k)-algebras $(k\geqslant 3)$ over a field of characteristic zero.

THEOREM 4 (cf. [1, Theorem 3]). *A Malcev (A_k)-algebra $(k\geqslant 3)$ A over a field of characteristic zero is either abelian or reductive and is such that there exists no nonzero element x in A^2 with $R_x^k=0$, for the right multiplication R_x in A.*

PROOF. Let R be the radical of A. Then, by Corollary 2, R is the center of A. As an abelian algebra is an (A_k)-algebra, we assume A to be non-abelian, so that $R\neq A$. By Lemma 3, A is then reductive. Further, A^2 is an (A_k)-algebra (see [5, Proposition 1.4]), and A^2 being semisimple, its center is the zero ideal. Equivalently, there exists no non-zero element x in A^2 such that $R_x^k=0$.

Let us now consider a Malcev (A_3)-algebra A over an algebraically closed field of characteristic zero. If A is not abelian, from Theorem 4 it follows that A is the direct sum of its center and an ideal B, which is also a semisimple Malcev algebra. B is then a direct sum of simple ideals B_i (of B), which are also simple ideals of A [4, Corollary 1]. B_i are thus (A_3)-algebras (see [5, Proposition 1.4]). All or some of the B_i may be Lie algebras in which case, by a result of Jôichi ([1, Theorem 4]), such B_i are abelian. This would mean that these B_i are contained in the center of A, a contradiction. Thus all the B_i are simple non-Lie Malcev algebras. But, a simple non-Lie Malcev algebra over an algebraically closed field of characteristic zero is precisely the 7-dimensional algebra of Sagle (see [4, Theorem B] and [6, Example 3.2]), which is not, however, an (A_3)-algebra as can be easily seen. Thus we are led to

THEOREM 5. *A Malcev (A_3)-algebra over an algebraically closed field of characteristic zero is abelian.*

Theorem 5 extends the corresponding result of Jôichi (loc. cit.) used in its proof.

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