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Rings Satisfying the Three Noether Axioms

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1. Introduction

This paper is concerned with the ideal theory of a commutative ring R (which may not have an identity). We say that R is *integrally closed in its total quotient ring* T (or, simply, *integrally closed*) provided R contains every element $\alpha \in T$ such that α is integral over R (i, e., $\alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0$ for some $r_1, \cdots, r_n$ in R). A ring R is *n-dimensional* (n a non-negative integer), or has dimension n ($\dim R = n$), provided there exists a chain $P_0 < P_1 < \cdots < P_n < R$ of prime ideals in R and there is no such chain of prime ideals with greater length. If R has no prime ideals except R, then we say that $\dim R = -1$.

A ring is said to have property (N) provided the following three conditions are satisfied:

(1) The ascending chain condition on ideals of R (*a.c.c.*)

(2) Proper prime ideals (i. e. $\neq R$, (0)) of R are maximal.

(3) The ring R is integrally closed;

and R has property (ν) provided (1), (3) and

(2′) $\dim R \leq 1$

hold in R. Properties (N) and (ν) are not equivalent even in a domain, but (N) always implies (ν). We say that R has property (π) provided every ideal of R is a product of prime ideals of R (rings with this property are called general *Z. P. I.* rings). It is well known that if R is a domain with an identity then R has property (N) if and only if R has property (π). For a brief history see [3; 32], and in addition see [15; 53], [16; 2.75], [8; 80], [10], [14], and [12]. Rings having property (π) have been studied extensively—for example, see [12], [6; 579], [7] and [2]. In [6] Gilmer studied domains without an identity which have property (π). In general (N) and (π) are not equivalent in a commutative ring—in fact the ring of even integers has property (N) and does not have property (π).

The purpose of this paper is to investigate commutative rings having

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property (N), (or property (ν)) and such rings will be called N-rings (ν-rings). In the case when R is a domain they will be called N-domains (ν-domains).

If R is a ring and S is a ring with an identity e containing R as a subring, then we denote $\{r+ne \mid r \in R,\ n \text{ an integer}\}$ by $R^*(S)$. In case D is a domain D^* will mean $D^*(K)$ where K is the quotient field of D unless stated otherwise.

We show that a domain D is an N-domain if and only if D is a product of distinct prime ideals in a Dedekind domain $\bar{D}$ which is a finite D^*-module. In order to prove the above theorem, we first obtain a generalization of a theorem of Akizuki [13; 25] which states that an integral domain D with an identity has the restricted minimum (RM) condition if and only if D satisfies axioms (1) and (2) above. See Theorem 3 its corollaries and Theorem 25 for this result. A ring R is said to have the (RM) *condition*, (or be an RM-ring), provided R/A has the descending chain condition (*d. c. c.*) on ideals, for all ideals $A \neq (0)$. In addition, some results are obtained concerning N-rings (ν-rings) with zero divisors. In particular, if (1) and (2) hold in a ring R which is not a domain, then (3) is valid in R. Finally we investigate rings with the property that every proper residue class ring is an N-ring.

In the last section we consider an alternative to our definition of N-domain. Condition (3) is replaced by:

(3′) The ring R is *integrally closed as an ideal* (i. e. R contains all elements α of T for which there exist elements $r_i \in R^i$ for $i=1, \cdots, n$ such that $\alpha^n + r_1\alpha^{n-1} + \cdots + r_n = 0$).

A ring has property (N') provided (1), (2) and (3′) hold in R. We show that a domain D has property (N') if and only if D is an ideal in a Dedekind domain $\bar{D}$ such that $\bar{D}$ is a finite D^*-module.

The notation and terminology are those of Zariski and Samuel, *Commutative Algebra* with the following exceptions—we do not require that a noetherian ring have an identity element and we do not require that a domain have an identity element. In particular we use $\subset$ to denote containment and $<$ to denote proper containment. An ideal A in a ring R is proper provided $(0) < A < R$. The ring of integers will be denoted by Z and all rings considered are assumed to be commutative and have more than one element.

In addition, we use the term *semi-prime ideal* A to mean $A = \sqrt{A}$. Also we use the term *special primary* ring to mean a ring R with identity in which the only ideals are R, M, and powers of M, where M is the unique maximal ideal of R and $M^i = (0)$ for some $i \in Z$. An ideal A is called *regular* provided it contains a regular element of the ring.

2. Restricted minimum condition in domains without identity

In this section, we study the relationship between the *a. c. c.* and the

(RM) condition in domains without identity. We first prove four lemmas which will be used in the main theorem.

LEMMA 1. *Let S be a ring with identity e containing R as a subring and let $R^*=R^*(S)$. Then every ideal of R is an ideal of R^* and $R^*/R \cong Z/(n)$ for some non-negative integer n; hence, if P is a proper prime ideal of R^* such that $R^*>P>R$, then P is maximal.*

PROOF. It is easy to check that every ideal of R is an ideal of R^*. The function $f: Z \to R^*/R$, defined by $f(m)=me+R$ for $m \in Z$, is a homomorphism from Z onto R^*/R; hence $R^*/R \cong Z/(n)$ for some non-negative integer n. If P is a prime ideal in R^* such that $R^*>P>R$, then P is maximal in R^* since proper prime ideals are maximal in $Z/(n)$.

LEMMA 2. *Let R, S, and R^* be as in Lemma 1. If $P_1<P_2<\cdots<P_n<R$ is a chain of prime ideals in R, then there exists a chain $P_1^*<P_2^*<\cdots<P_n^*$ of prime ideals in R^* such that $P_i^* \cap R=P_i$ for $i=1, \cdots, n$.*

PROOF. We first prove the lemma in the case that S is a domain. For $i=1, \cdots, n$ set $P_i^*=P_iR_{P_n} \cap R^*$, where R_{P_n} is the quotient ring of R with respect to the prime ideal P_n. Since $R_{P_n} \supset R^* \supset R$ and $P_iR_{P_n} \cap R=P_i$, we have $P_i^* \cap R=P_i$ for $i=1, \cdots, n$. We now consider the case in which S is a ring. Denote by $\mathcal{S}$ the set of ideals A^* and R^* such that $A^* \cap R=P_1$. Since $P_1 \in \mathcal{S}$, $\mathcal{S}$ is nonempty and there exists a maximal element $P_1^* \in \mathcal{S}$ by Zorn's lemma. A standard argument shows that P_1^* is prime in R^*, and the proof is completed by applying the domain case to the domains $R/P_1 \subset R^*/P_1^*$.

LEMMA 3. *Let R, S, and R^* be as in Lemma 1. If $P \neq R$ is a prime ideal of R and P^* is a prime ideal of R^* such that $P^* \cap R=P$, then P is maximal in R if and only if P^* is maximal in R^*.*

PROOF. We have $R/P \subset R^*/P^*$ (to within isomorphism) and R/P is a nonzero ideal of R^*/P^*.

If P^* is maximal in R^*, then R/P is a nonzero ideal in the field R^*/P^* so $R/P=R^*/P^*$ and P is maximal in R.

If P is maximal in R, then R/P is a field with identity $\bar{f}$. It follows easily that $\bar{f}$ is the identity of R^*/P^*, and since R/P is an ideal of R^*/P^* containing the identity then $R/P=R^*/P^*$ is a field and P^* is maximal in R^*.

LEMMA 4. *Let R be a subring of a ring S with identity and let $R^*=R^*(S)$. If P and Q are prime ideals in R^* such that $R^*>P>Q$ and $P \not\supset R$, then $R>P \cap R>Q \cap R$.*

PROOF. It is clear that $R>R \cap P \supset R \cap Q$. Now suppose that $P \cap R=Q \cap R$, and choose $r \in R-(P \cap R)=R-(Q \cap R)$ and $p \in P-Q$; then $rp \in P \cap R=Q \cap R$ which implies $rp \in Q$ and hence $r \in Q$. But this contradicts our choice of

$r \in R-(Q \cap R)$, so $P \cap R > Q \cap R$.

COROLLARY 5. *Let R, S and R^* be as in Lemma 1 then* $\dim R \leq \dim R^* \leq \dim R+2$.

PROOF. The proof follows directly from Lemmas 1, 2, and 4.

THEOREM 6. *A domain D has properties (1) and (2) if and only if D^* has properties (1) and (2).*

PROOF. If D has an identity, then $D=D^*$ and the theorem is valid. Suppose D does not have an identity and that properties (1) and (2) hold in D; then D^* is noetherian [5; 184]. If P is a proper prime ideal in D^* such that $P>D$, then P is maximal by Lemma 1. If P is a proper prime in D^* such that $P \not\supset D$, then $D \neq P \cap D \supset PD \neq (0)$ and by Lemma 3 we see that P is maximal. Finally we will show that if D is prime in D^* then D is maximal. If D is prime in D^* and not maximal, then there exists a maximal ideal M of D^* such that $D^*>M>D>(0)$. By [16; 240] there exists a chain $D^*>M>P>(0)$ of prime ideals in D^* such that $P \not\subset D$. But we have just shown that all prime ideals of D^* different from D are maximal and we have a contradiction. Therefore, all proper prime ideals in D^* are maximal. Conversely, if D^* has properties (1) and (2), then clearly D has property (1) since ideals of D are ideals of D^*. By the theorem of Akizuki [13; 25] D^* has the (RM) condition, and consequently D has the (RM) condition since ideals of D are ideals of D^*. Let P be a proper prime ideal of D; then D/P is a domain with the *d. c. c.* (and hence is a field) so P is maximal.

COROLLARY 7. *The (RM) condition holds in a domain D if and only if conditions (1) and (2) hold in D.*

PROOF. In [1; 342] Akizuki proved that a regular RM-ring has the *a. c. c.* In any ring with the (RM) condition proper prime ideals are maximal, so conditions (1) and (2) hold. Conversely, if conditions (1) and (2) hold in D, then they hold in D^* by Theorem 6. By [3; 29] D^* is therefore an RM-domain and D is an RM-domain.

COROLLARY 8. *A domain D is an RM-domain if and only if D^* is an RM-domain.*

3. Characterization of regular ν-rings

THEOREM 9. *If R is a ring with an identity and A is a regular ideal of R, then A is a noetherian ring if and only if R is noetherian and R is a finite $A^*=A^*(R)$ module.*

PROOF. If A is noetherian, then A^* is noetherian by [5; 184]. Since A is an ideal in R and in A^*, A is contained in the conductor of R over A^*. Let

$\partial \in R$ and let r be an element of A regular in R; then $\partial \cdot r \in A \subset A^*$ which implies that $\partial \in r^{-1}A^*$. Since A^* is noetherian and $r^{-1}A^*$ is finite over A^*, we see that $r^{-1}A^*$ is a noetherian A^*-module. But $R \subset r^{-1}A^*$, so R is a noetherian A^*-module and hence R is a noetherian ring. Conversely, suppose R is noetherian and R is a finite A^* module; then by Eakin [4] A^* is noetherian and by [5; 184] A is noetherian. Note that we did not use the hypothesis that A is a regular ideal in the proof of the converse.

Lemma 10. *If A is a regular ideal of a ring R, then the total quotient ring of A is equal to the total quotient ring of R.*

Proof. Let r be an element of A which is regular in R, and let a be a regular element of the ring A. If $ax=0$ for $x \in R$, then $a(rx)=0$ implies that $rx=0$ and $x=0$. Hence a is regular in R.

Theorem 11. *If A is a regular ideal of an integrally closed ring R, then A is integrally closed if and only if $A=\sqrt{A}$ in R.*

Proof. Suppose A is integrally closed. If $x \in \sqrt{A}$ then $x^n \in A$ which implies $x \in A$ since A is integrally closed, and therefore $A=\sqrt{A}$ in R. Conversely, suppose $A=\sqrt{A}$ in R and let x be an element of the total quotient ring of A which is integral over A. Since R is integrally closed, it follows from Lemma 10 that $x \in R$. Furthermore, we have $x^{n+1}+a_n x^n+\cdots+a_0=0$ with $a_i \in A$ for $i=0, \cdots, n$. This implies that $x^{n+1} \in A$ since $x \in R$ and A is an ideal of R. Hence $x \in \sqrt{A}=A$ and A is integrally closed.

Theorem 12. *If R is a regular ring with total quotient ring T, then R is a regular ν-ring if and only if all of the following hold:*

(a) *R is a semi-prime ideal in a noetherian, integrally closed ring S with identity;*

(b) *$R^*(T)=R^* \subset S \subset T$, S is a finite R^*-module, and* $\dim S \leq 2$;

(c) *If P is a prime ideal of S such that $P \not\supset R$, then height $P \leq 1$* [17; 240].

Proof. Suppose that R is a regular ν-ring and let S be the integral closure of R^* in T. If $\alpha \in S$ and $d \in R$, then $d\alpha$ is integral over R and hence $d\alpha \in R$; so R is an ideal of S. Since R is noetherian, it follows that S is noetherian and S is a finite R^*-module by Theorem 9. Theorem 11 gives us $\sqrt{R}=R$ in S and R is a semi-prime ideal of S.

To establish that $\dim S \leq 2$, let $R^* > P_1^* > P_2^* > P_3^* > P_4^*$ be a chain of prime ideals in R^*. If $P_1^* \supset R$ and $R_2^* \neq R$, it follows from Lemma 1 that $P_2^* \not\supset R$ and applying Lemma 4, we have $R \cap P_2^* > R \cap P_3^* > R \cap P_4^*$, contradicting $\dim R \leq 1$. If $P_1^* \supset R$ and $P_2^*=R$, then there exists a prime ideal $\bar{P}_2^*$ in R^* such that $P_1^* > \bar{P}_2^* > P_3^*$ and $\bar{P}_2^* \neq P_2^*$ since R^* is noetherian [17; 240], and Lemma 1 yields $\bar{P}_2^* \not\supset R$; again we contradict $\dim R \leq 1$. If $P_1^* \not\supset R$, it is clear that we have a contradiction by Lemma 4; hence $\dim R^* \leq 2$. Since S is integral over R^*, it follows from the lying over theorem [17; 259] that $\dim R^* = \dim S \leq 2$.

If P is a prime ideal of S such that $P \not\supset R$, then $P^* = P \cap R^*$ is a prime ideal of R^* such that $P^* \not\supset R$; applying the lying over theorem and Lemma 4, it follows from $\dim R \leq 1$ that height $P \leq 1$.

Conversely, suppose (a), (b) and (c) hold. Then R is noetherian and integrally closed by Theorems 9 and 11. Since S is a finite R^*-module, then S is integral over R^* [17; 254] and $\dim R^* = \dim S \leq 2$ by the lying over theorem [17; 259]. Now we wish to show that $\dim R \leq 1$. Suppose $P_1 < P_2 < P_3 < R$ is a chain of prime ideals of R, then by Lemma 2 there exists a chain $P_1^* < P_2^* < P_3^*$ of prime ideals of R^* such that $P_i^* \cap R = P_i$. Now $P_3^* \supset R$ since height $P_3^* = 2$ so that $P_3 = R$ which also yields a contradiction so $\dim R \leq 1$.

By modifying the proof of Theorem 12 slightly, we can establish the following result.

THEOREM 13: *Let R be a regular ring with total quotient ring T and let n be a non-negative integer. Then R is noetherian, integrally closed, and* $\dim R \leq n$ *if and only if all of the following hold*:

(a) *R is a semi-prime ideal in a noetherian, integrally closed ring S with identity*;

(b) $R^*(T) = R^* \subset S \subset T$, *$S$ is a finite R^*-module, and* $\dim S \leq n+1$;

(c) *If P is a prime ideal of S such that $P \not\supset R$, then height $P \leq n$.*

We remark that $\dim R \geq 0$ in Theorem 13 since R is a regular ring (the powers of a regular element form a multiplicative system S, and there exists a prime ideal P such that $P \cap S$ is empty). However, it can happen that R is noetherian, integrally closed, and $\dim R = -1$ while R^* is noetherian, integrally closed, and $\dim R^* = 1$ (e. g. the ring D/D^2 in Example 15).

THEOREM 14. *A domain D is an N-domain if and only if D is a product of distinct prime ideals in a Dedekind domain $\bar{D}$ such that $\bar{D}$ is a finite D^*-module.*

PROOF. Let D be an N-domain with quotient field K and let $\bar{D}$ be the integral closure of D^* in K. Conditions (1) and (2) hold in D^* by Theorem 6, so that $\dim \bar{D} = \dim D^* = 1$. As in the proof of Theorem 12, D is an ideal in $\bar{D}$, $\bar{D}$ is noetherian, integrally closed, and a finite D^*-module. Hence $\bar{D}$ is a Dedekind domain, and D is a product of distinct prime ideals since $\sqrt{D} = D$ in $\bar{D}$.

Conversely, D is noetherian by Theorem 9 and therefore D^* is noetherian. Since $\bar{D}$ is a finite D^*-module, then $\dim D^* = \dim \bar{D} = 1$. Hence conditions (1) and (2) hold in D by Theorem 6, and D is an N-domain by Theorem 11.

It is clear that an N-domain is a ν-domain, but the converse is false as is shown by the following example.

EXAMPLE 15. Denote by $Z[x]$ the ring of polynomials with integer coefficients and let $S' = \bigcup_p (p, x)$, i. e. S' is the union of all of the maximal ideals of $Z[x]$ of the form (p, x) where p is a prime number. Set $S = Z[x] \setminus S'$ and $J = Z[x]_S$, i. e. the quotient ring of $Z[x]$ with respect to the multiplicative system S. Let $D = xJ$. It follows directly that $J = D^*$, J is noetherian, integrally closed, and 2-dimensional. Furthermore, the maximal ideals of J are exactly the ideals of the form $(p, x)J$, where p is a prime number, i. e. all of the maximal ideals of J contain D. There are infinitely many non-maximal prime ideals of J[16; 240], the only prime ideals of $J = D^*$ which contain D are maximal by Lemma 1, and D is prime in J. If $P^* \neq (0)$ is a non-maximal prime ideal of J, then $D \cap P^*$ is a proper prime ideal of D; hence D has proper prime ideals. If P is a proper prime of D, Lemma 2 implies that there exists a prime ideal P^* of J such that $P^* \cap D = P$; furthermore, P is maximal if and only if P^* is maximal by Lemma 3. It follows from Lemma 2 that D is 1-dimensional; however, no proper prime ideal of D is maximal. Consequently, D is not an N-domain; however, Theorem 12 implies that D is a ν-domain.

THEOREM 16. *If A is a product of distinct prime ideals in a general Z. P. I. ring R with an identity and R is a finite $A^* = A^*(R)$ module, then A is a ν-ring.*

PROOF. Since R is a general *Z. P. I.* ring, we have $R = R_1 \oplus \cdots \oplus R_n$ where R_i is either a Dedekind domain or a special primary ring for $i = 1, \cdots, n$ [2; 89]. Set $A_i = AR_i$, $A_i^* = A_i^*(R_i)$, and note that A_i is a product of distinct prime ideals in R_i (including R_i) for $i = 1, \cdots, n$. Since R is a finite A^*-module, we have $R = \sum_1^t s_i A^*$ where $s_i \in R$ for $i = 1, \cdots, t$. Now, $s_i = \sum_{j=1}^n r_{i_j}$ with $r_{i_j} \in R_j$ for $i = 1, \cdots, t$ and it follows readily that $R_j = \sum_{i=1}^t r_{i_j} A_j^*$ and R_j is a finite A_j^* module for $j = 1, \cdots n$. If R_j is a Dedekind domain, then $A_j = (0)$ or A_j is a ν-ring by Theorem 12. If R_j is a special primary ring, then A_j is the maximal ideal in R_j (or, $A_j = R_j$ and A_j is a ν-ring). Since A_j is a nilpotent ring, we have $\dim A_j = -1$ or $A_j = (0)$. Furthermore, R_j is noetherian, which implies that A_j^* is noetherian [4], hence A_j is noetherian [5; 184]. Since A_j is integrally closed (trivially) then A_j is a ν-ring. Finally, A is a ν-ring since a finite direct sum of ν-rings is a ν-ring.

The converse to Theorem 16 is false; in fact, if A is a ring with an identity then A is an ideal in a general *Z. P. I.* ring if and only if A is a general *Z. P. I.* ring (as we will presently show), and in Example 19 we exhibit a ν-ring with an identity which is not a general *Z. P. I.* ring.

PROPOSITION 17. *If R is a ring and A is a finitely generated ideal of R such that $A = A^2$, then $R = A \oplus R_1$.*

PROOF. If R does not have an identity, let S be a ring with identity containing R as a subring [11; 87] and set $R^*=R^*(S)$. If R has an identity, set $R=R^*$. In either case, A is an ideal of R^*. Since $A=A^2$ there exists an $e \in A$ such that $ea=a$ for all $a \in A$ [5; 185]. If e^* is the identity of R^*, then e and e^*-e are orthogonal idempotents and $R^*=eR^* \oplus (e^*-e)R^*$. If follows that $R=eR \oplus (e^*-e)R$, $eR=A$, and $R=A \oplus R_1$.

COROLLARY 18. *If $(0) \neq A=A^2$ is an ideal in a general Z. P. I. ring R, then A is a general Z. P. I. ring.*

PROOF. Since R is noetherian [12; 125], it follows by Proposition 14 that $R=A \oplus R_1$ and $A \cong R/R_1$ is a general *Z. P.I.* ring.

EXAMPLE 19. Let x and y be indeterminates over a field F and set $R=F[x, y]/(x, y)^2$. The ring R has exactly one proper prime ideal $P=(x, y)/(x, y)^2$ and consequently R is its own total quotient ring and is integrally closed. It is clear that R is noetherian and $\dim R=0$, hence R is an N-ring. Obviously R is not a general *Z. P. I.* ring since $P^2=(0)$.

It follows from Theorem 14 that an N-domain can be imbedded as an ideal in a Dedekind domain (i. e. *Z. P. I.* domain with identity) in a special way. However, Corollary 18 and Example 19 show that in general a ν-ring cannot be imbedded as an ideal in a general *Z. P. I.* ring.

We complete this section with a sufficient condition that D^* be a Dedekind domain when D is an N-domain, and give two examples.

THEOREM 20: *If there exists $d \in D$ such that $D=dD+dZ$ and D is an N-domain, then D^* is a Dedekind domain.*

PROOF. It suffices to prove that D^* is integrally closed since D^* has properties (1) and (2) by Theorem 6. Let α be an element of the quotient field of D^* which is integral over D^*; then $\alpha=a/b$ with a and $b \in D$ and there exist $d_i^* \in D^*$, $i=0, \cdots, n-1$, such that $\alpha^n+d_{n-1}^*\alpha^{n-1}+\cdots+d_0^*=0$. Hence $(d\alpha)^n+dd_{n-1}^*(d\alpha)^{n-1}+\cdots+d_0^*d^n=0$ and $d\alpha$ is integral over D, which implies $d\alpha \in D$ since D is integrally closed. Therefore $d\alpha=d(a/b)=kd+nd$ where $k \in D$ and $n \in Z$ and consequently $\alpha=a/b=(kd+nd)/d=k+n \in D^*$ and D^* is integrally closed.

EXAMPLE 21: This example shows that the domain D^* of Theorem 14 may not be a Dedekind domain (i. e. $\bar{D} > D^*$). Let $\omega=(1+\sqrt{5})/2$, $S=\{a+b\omega \mid a, b \in Z\}$, $2S=(2)$, and $(2)^*=\{n+2a+2b\omega \mid a, b, n \in Z\}$. Then (2) is a prime ideal in the Dedekind domain S [9; 33, 66], $S=(2)^*+\omega(2)^*$ is a finite $(2)^*$-module, and $S \neq (2)^*$ since $\omega \notin (2)^*$. It follows from Theorem 14 that (2) is an N-domain, but $(2)^*$ is not a Dedekind domain since the integral closure of $(2)^*$ is S (however, $(2)^*$ is an RM-domain).

EXAMPLE 22. In this example, we show that a prime ideal in a Dedekind domain need not be an N-domain (in fact, need not be noetherian). Denote by Q the field of rational numbers, let x be an indeterminate over Q, and set $\bar{D}=Q[x]_{(x)}$ (i. e., the quotient ring of $Q[x]$ with respect to the prime ideal (x)). The ideal $D=x\bar{D}$ is a prime in $\bar{D}$, and we will show that D is not noetherian by showing that D^* is not noetherian. If p_n denotes the n^{th} prime number and $A_1=(x/2)D^*$, then define A_n for $n>1$ by $A_n=A_{n-1}+(x/p_n)D^*$. It follows easily that x/p_{n+1} does not belong to A_n for $n\geq 1$, and therefore the sequence $A_1<A_2<\cdots$ is strictly increasing—which implies that D^* (and hence D) is not noetherian.

4. Characterization of N-rings with proper zero divisors

We state without proof the following theorem, which is an easy consequence of Theorem 4 of [1; 339].

THEOREM 23. *Let R be a ring and let $P_1, \cdots, P_r$ be ideals of R such that R/P_i is a field for $i=1, \cdots, r$ and such that $(0)=\prod_{i=1}^{r} P_i^{m_i}$. Then there exists a positive integer n such that $R=R^n \oplus N$ where $R^n=R^{n+1}$ has an identity, N is nilpotent, and in R^n, $(0)=\prod_{i=1}^{r}\bar{P}_i^{m_i}$ where $\bar{P}_i=P_i\cap R^n$ and $R^n/\bar{P}_i$ is a field for $i=1, \cdots, r$.*

COROLLARY 24. *Let R be a regular ring in which* (0) *is not a prime ideal. If conditions* (1) *and* (2) *hold in R, then R has an identity.*

PROOF. Since R is noetherian every ideal of R contains a product of prime ideals, hence $(0)=\prod_{i=1}^{k} P_i$. The P_i are maximal by (2) and we apply Theorem 23 to R and see that $N=(0)$ since R is regular.

THEOREM 25. *If R is a ring with a regular element, then R is an RM-ring if and only if conditions* (1) *and* (2) *hold in R.*

PROOF. The result follows from Corollary 7 in case R is a domain, so we may assume that (0) is not prime in R. If conditions (1) and (2) hold, then Corollary 24 applies and R has an identity, and hence R is an RM-ring [3; 29]. Conversely, if R is an RM-ring with a regular element then the *a. c. c.* is valid in R [1; 342] and since property (2) holds in any RM-ring, the proof is complete.

REMARK 26. We note that it follows from the proof of Theorem 25 that a regular RM-ring, in which (0) is not a prime ideal, has an identity. However, an RM-domain need not have an identity (e. g. the even integers).

LEMMA 27. *If R has the d. c. c., then R is equal to its total quotient ring T (and R is integrally closed).*

PROOF. If there are no regular elements in R, then $R=T$. If r is regular in R, then $(r)^n=(r)^{n+1}$ for some integer n. Hence $r^n=sr^{n+1}+mr^{n+1}$ with $s\in R$, $m\in Z$ so that $r=r(sr+mr)$ and $e=sr+mr$ is an identity for R. It follows easily that every regular element of R has an inverse in R and $R=T$.

PROPOSITION 28. *Let R be a ring with a regular element which is not a domain. Then R is an N-ring if and only if R is a ring with identity in which the d. c. c. holds.*

PROOF. Suppose R is an N-ring. It follows from Theorem 25 and Remark 26 that R has an identity. Now since R is a noetherian ring with an identity and every prime ideal different from R is maximal, R has the *d. c. c.* [3, 28].

Conversely, by [3, 28] R is noetherian and every prime ideal $\neq R$ is maximal. By Lemma 27, R is integrally closed and therefore R is an N-ring.

COROLLARY 29. *If R is a ring in which* (0) *is not prime, then R is an N-ring if and only if conditions* (1) *and* (2) *hold in R.*

PROOF. Suppose (1) and (2) hold in R. If R has a regular element then Corollary 24 implies that R has an identity, and therefore R has the *d. c. c.* by [3; 28]. It follows from Lemma 27 that R is an N-ring. If R has no regular elements then $R=T$, its total quotient ring, and R is an N-ring.

THEOREM 30. *Let R be a ring in which* (0) *is not prime. Then R is an N-ring if and only if $R\cong R_1\oplus\cdots\oplus R_k\oplus N$ where each R_i is a noetherian primary ring with identity and N is a noetherian nilpotent ring.*

PROOF. Suppose R is an N-ring. If R has a proper prime ideal P, then $(0)=\prod_{i=1}^{k} P_i^{e_i}$ where R/P_i is a field for $i=1,\cdots,k$ because every ideal in a noetherian ring contains a product of prime ideals. (If $(0)=R^s P_2^{e_2}\cdots P_k^{e_k}$ then $(0)\supset P^s P_2^{e_2}\cdots P_k^{e_k}$ where P is a proper prime, hence $(0)=P^s P_2^{e_2}\cdots P_k^{e_k}$). By Theorem 23, $R=R^n\oplus N$ where R^n has an identity, and $(0)=\bar{P}_1^{e_1}\cdots\bar{P}_k^{e_k}$ in R^n where the $R^n/\bar{P}_i$ are fields. Therefore $R^n\cong R^n/\bar{P}_1^{e_1}\oplus\cdots\oplus R^n/\bar{P}_k^{e_k}$ by [16; 176] and each $R^n/\bar{P}_i^{e_i}=R_i$ is a noetherian primary ring with identity. If R has no proper prime ideals, then $\sqrt{(0)}=R$ and $R^n=(0)$ so $R=N$.

Conversely, if $R\cong R_1\oplus\cdots\oplus R_k\oplus N$, where each R_i is a noetherian primary ring with identity, then it is clear that properties (1) and (2) hold in R. We consider two cases. If $N=(0)$, then R satisfies the *d. c. c.* and R is an N-ring by Proposition 25. Second, if $N\neq(0)$, then there are no regular elements and R is an N-ring since R is integrally closed.

THEOREM 31. *Let R be a general Z.P.I. ring with identity and suppose that R is a finite $A^*=A^*(R)$ module where A is an ideal of R. Then A is an N-ring if and only if one of the following holds*:

(a) *Either A is a Dedekind domain D or a product of distinct prime ideals in a Dedekind domain D such that D is a finite $A^*\cap D$ module.*

(b) *The ideal A is a product of prime ideals in a general Z. P. I. ring R_1 such that primes different from R_1 are maximal in R_1, and $R=R_1\oplus R_2$.*

PROOF. Let $R=D_1\oplus\cdots\oplus D_t\oplus S_1\oplus\cdots\oplus S_u$ where the D_i are Dedekind domains (not fields) and the S_i are special primary rings (possibly fields) [2; 84]. If A is a domain then $A\subset D_i$ for some i say $i=1$ or A is a field. By Theorem 14, we see that (a) holds.

If A is not a domain, then by Theorem 30 some power of A is idempotent and consequently AD_i is either (0) or D_i for $i=1, \cdots, t$. But AD_i cannot be D_i because proper prime ideals are maximal in A and $A\not\subset D_i$, therefore (b) holds. Note that there are only a finite number of such ideals in a given ring.

If (a) holds, then A is an N-ring by Theorem 14. If (b) holds, then A is noetherian by [4] and A^* has property (2) by [16; 259], so proper prime ideals are maximal in A by Lemmas 2 and 3. Since any ideal in a special primary ring is integrally closed, we see that A is an N-ring.

5. Characterization of almost *N*-rings

In this section, we investigate rings with the property that every proper homomorphic image is an N-ring.

THEOREM 29. *A ring R has the property that R/A is an N-ring for every ideal $A\neq(0)$ if and only if R is one of the following types of rings*:

(a) *R is a one dimensional noetherian ring with a non-maximal prime ideal $P\neq(0)$ such that $P^2=(0)$, there are no ideals between P and (0), and R/P is an N-domain.*

(b) *$R=D\oplus K$ where D is an N-domain and K is a field.*

(c) *$R=R_1\oplus\cdots\oplus R_k\oplus N$ where each R_i is a noetherian primary ring with identity and N is a nilpotent ring with the a. c. c.*

(d) *R is an RM-domain.*

PROOF. The ring R is noetherian since R/A is noetherian for all $A\neq(0)$.

Case 1. R is a domain. If R has a proper prime, then let P denote one such prime. If $0\neq x\in P$, then $R/(x^2)$ is an N-ring. Let $\phi\colon R\to R/(x^2)$ be the natural map. Then $\phi(P)$ is maximal and P is maximal since $P\supset(x^2)$. Therefore, R is an RM-domain by Corollary 7. If R has no proper prime ideals then R is again an RM-domain by Corollary 7.

Case 2. R is not a domain and R has at least one proper prime ideal P which is not maximal. Let $P_1 \neq R$ be a prime of R. Then R/P_1 is an N-domain and $\dim R/P_1 \leq 1$. There are no ideals between P and (0) because $P > A > (0)$ implies P/A is maximal in R/A which is a contradiction. Therefore either $P = P^2$ or $P^2 = (0)$. If $P^2 = (0)$, then R is a ring of type (a). If $P = P^2$, then by Proposition 17, $R = P \oplus R(1-e)$. Since there are no ideals between P and (0) and P has an identity, P must be a field since any ideal of P is an ideal of R. Therefore $R \cong K \oplus D$, where K is a field and $D \cong R/P$ is an N-domain, and R is of type (b).

Case 3. R is not a domain and every prime ideal of R except R is maximal. If R has no proper primes then $\sqrt{(0)} = R$ which implies that $R^k = (0)$ since R is noetherian, and R is a ring of type (c). If R has at least one proper prime ideal then $(0) = P^{e_1} \cdots P_k^{e_k}$ where the P_i are maximal and prime. By Theorem 20 $R = R^n \oplus N$ where R^n has an identity and N is nilpotent. In $R^n = \bar{R}$, $(0) = \bar{P}_1^{e_1} \cdots \bar{P}_k^{e_k}$ such that $\bar{R}/\bar{P}_i$ is a field. Therefore $R \cong \bar{R}/\bar{P}_1^{e_1} \oplus \cdots \oplus R/P_k^{e_k}$ by [16; 178] and $R \cong R_1 \oplus \cdots \oplus R_k \oplus N$ where $R_i = \bar{R}/\bar{P}_i^{e_i}$ is a noetherian primary ring with identity for each i and N is a noetherian nilpotent ring.

Conversely, suppose R is a ring of type (a) and let $B \neq (0)$ be an ideal of R. If $B = P$, then R/B is an N-ring by hypothesis. If $B \neq P$, then all proper primes of R/B are maximal and R/B is noetherian. Hence R/B is an N-ring by Corollary 26, or R/B is a field which is an N-ring. Suppose $R \cong D \oplus K$, where D is an N-domain and K is a field. If $B \neq (0)$ is an ideal of R, then $B = B_1 + B_2$ and $R/B \cong D/B_1 \oplus K/B_2$. By considering the cases $B_2 = (0)$ and $B_2 = K$, it follows easily (see Corollary 26) that R/B is an N-ring. Similarly, if R is of type (c) or (d) then it follows readily that R/B is an N-ring for each $B \neq (0)$ in R.

6. An alternate definition of N-domains

In this section, we consider a variation of the concept of N-domains obtained by replacing (1), (2), (3) by (1), (2) and

(3′) D contains every element α of K (the quotient field of D) for which there exist elements $d_i \in D^i$ for $i = 1, \cdots, n$ such that $a^n + d_1 a^{n-1} + \cdots + d_n = 0$.

Since D is an ideal in D^*, (3′) simply states that D is integrally closed as an ideal of D^* in the sense of [17; 349]. It is shown in [17] that an ideal D of R has property (3′) if and only if D is complete (i. e., $D = \bigcap_{v \in S} DR_v$, where S is the set of all valuations v of K non-negative on R and R_v is the valuation ring corresponding to the valuation v).

THEOREM 30. *A domain D with quotient field K is complete (integrally closed as an ideal in D^*) and has properties (1) and (2) if and only if D is an ideal in a Dedekind domain $\bar{D}$ such that $\bar{D}$ is a finite D^*-module.*

PROOF. Suppose D satisfies conditions (1), (2) and (3′). If D has an identity then D is a Dedekind domain. If D does not have an identity, then let $\bar{D}$ be the integral closure of D^* in K. We will show that D is an ideal in $\bar{D}$; $D=D'=(D\bar{D})'\supset D\bar{D}$ where A' denotes the completion of A [17; 347, 348], therefore D is an ideal of $\bar{D}$. Since D is an ideal of both D^* and $\bar{D}$, D is contained in the conductor of $\bar{D}$ over D^*. Fix $0\neq d\in D$ and let $\bar{d}\in\bar{D}$; then $d\bar{d}\in D$, which implies that $\bar{d}\in Dd^{-1}\subset D^*d^{-1}$ and $\bar{D}\subset D^*d^{-1}$. Now D is noetherian so D^* is noetherian [5; 184] and D^*d^{-1} is a noetherian D^*-module since it is finite over D^* [16; 158]. Hence $\bar{D}$ is a noetherian D^*-module [16; 156] so $\bar{D}$ is a noetherian ring since any ideal of $\bar{D}$ is a D^*-submodule. By Theorem 6, D^* has properties (1) and (2) so $\bar{D}$ is an RM-domain [3; 29], and consequently $\bar{D}$ is a Dedekind domain.

Conversely, if D is an ideal of a Dedekind domain $\bar{D}$ such that $\bar{D}$ is a finite D^*-module, then D^* is noetherian by [4]. By the lying over theorem [16; 259] $\dim D^*=\dim\bar{D}\leq 1$ so $D^*$is an RM-domain [16; 203] and consequently D has properties (1) and (2). Any ideal A in a Dedekind domain is complete because $\bigcap_{v\in S} AR_v=\bigcap_{p} A\bar{D}_p=A$ [17; 84], so D is complete.

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