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Explicit and Implicit Difference Formulas of Higher Order Accuracy for One-dimensional Heat Equation

Hisayoshi Shintani

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1. Introduction

In this paper we are concerned with the first and the second boundary value problems for the one-dimensional heat equation

$$u_t(x, t)=u_{xx}(x, t) \qquad (0\leqq x\leqq 1,\ 0\leqq t) \tag{1.1}$$

with the initial condition

$$u(x, 0)=\varphi(x) \qquad (0\leqq x\leqq 1). \tag{1.2}$$

For the numerical solution of this problem by the finite-difference methods, there are known the two-level explicit formula with the truncation error of order h^2, Crank-Nicolson's method [16][1], Douglas' high order correct method [4], three-level difference formulas [6], and so on.

The object of this paper is to construct two-level explicit formulas with truncation errors of orders h^4 and h^6, to determine their ranges of stability, and to derive the unconditionally stable two-level implicit formulas of higher order accuracy. Although the formulas obtained here are not all new, the stability conditions are considered in a somewhat unified form. These formulas will be useful not only for the direct use but also for the approximation of the truncation errors of the formulas of the lower order accuracy.

2. Preliminaries

2.1 Difference formulas

Let h and k be the mesh-sizes in the x- and t-directions respectively and put $r=k/h^2$. Then, for the function $u(x, t)$ which is sufficiently smooth and satisfies the equation (1.1), using the relations

$$\frac{\partial^{2n}u}{\partial x^{2n}}=\frac{\partial^n u}{\partial t^n} \qquad (n=1, 2, \dots), \tag{2.1}$$

1) Numbers in square brackets refer to the references listed at the end of this paper.

we have the following results:

(2.2) $$\Delta_t u(x, t) = u(x, t+k) - u(x, t) = rh^2 u_t(x, t) + \frac{1}{2} r^2 h^4 u_{tt}(x, t) + \frac{1}{3!} r^3 h^6 u_{ttt}(x, t) + \frac{1}{4!} r^4 h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.3) $$\delta^2 u(x, t) = u(x+h, t) - 2u(x, t) + u(x-h, t) = h^2 u_t(x, t) + \frac{2}{4!} h^4 u_{tt}(x, t) + \frac{2}{6!} h^6 u_{ttt}(x, t) + \frac{2}{8!} h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.4) $$\delta^4 u(x, t) = h^4 u_{tt}(x, t) + \frac{1}{6} h^6 u_{ttt}(x, t) + \frac{1}{80} h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.5) $$\delta^6 u(x, t) = h^6 u_{ttt}(x, t) + \frac{1}{4} h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.6) $$\delta^2 u(x, t+k) - \delta^2 u(x, t) = rh^4 u_{tt}(x, t) + r\left(\frac{r}{2} + \frac{1}{12}\right) h^6 u_{ttt}(x, t) + \frac{r}{6}\left(r^2 + \frac{r}{4} + \frac{1}{60}\right) h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.7) $$\delta^4 u(x, t+k) - \delta^4 u(x, t) = rh^6 u_{ttt}(x, t) + r\left(\frac{r}{2} + \frac{1}{6}\right) h^8 u_{tttt}(x, t) + O(h^{10}),$$

(2.8) $$\delta^2 u(x, t+k) - \delta^2 u(x, \mathrm{t}) - \left(\frac{r}{2} - \frac{1}{12}\right)\delta^4 u(x, t+k) - \left(\frac{r}{2} + \frac{1}{12}\right)\delta^4 u(x, t) = \frac{r}{12}\left(\frac{1}{20} - r^2\right) h^8 u_{tttt}(x, t) + O(h^{10}),$$

where Δ_t is a forward difference operator and δ is a central differecne operator.

From these we obtain the following formulas:

(2.9) $$\Delta_t u(x, t) - r\delta^2 u(x, t) - r\left(\frac{r}{2} - \frac{1}{12}\right)\delta^4 u(x, t) = \frac{r}{6}\left(r^2 - \frac{r}{2} + \frac{1}{15}\right) h^6 u_{ttt}(x, t) + O(h^8),$$

(2.10) $$\Delta_t u(x, t) - r\delta^2 u(x, t) - r\left(\frac{r}{2} - \frac{1}{12}\right)\delta^4 u(x, t)$$

$$-\frac{r}{6}\Big(r^2-\frac{r}{2}+\frac{1}{15}\Big)\delta^6 u(x,\ t)$$

$$=\frac{r}{24}\Big(r^3-r^2+\frac{7}{20}r-\frac{3}{70}\Big)h^8 u_{tttt}(x,\ t)+O(h^{10}),$$

(2.11)
$$\varDelta_t u(x,\ t)-\Big(\frac{r}{2}-\frac{1}{12}\Big)\delta^2 u(x,\ t+k)-\Big(\frac{r}{2}+\frac{1}{12}\Big)\delta^2 u(x,\ t)$$

$$=\frac{r}{12}\Big(\frac{1}{20}-r^2\Big)h^6 u_{ttt}(x,\ t)-\frac{r}{24}\Big(r^3+\frac{1}{6}r^2-\frac{r}{20}$$

$$-\frac{22}{7!}\Big)h^8 u_{tttt}(x,\ t)+O(h^{10}),$$

(2.12)
$$\varDelta_t u(x,\ t)-r\delta^2 u(x,\ t+k)+\frac{1}{3}\Big(r^2-\frac{1}{30}\Big)\delta^4 u(x,\ t+k)$$

$$+\frac{r}{6}\Big(r^2+\frac{r}{2}+\frac{1}{15}\Big)\delta^4 u(x,\ t)$$

$$=\frac{r}{24}\Big(r^3+\frac{1}{3}r^2-\frac{r}{20}-\frac{13}{630}\Big)h^8 u_{tttt}(x,\ t)+O(h^{10}),$$

(2.13)
$$\varDelta_t u(x,\ t)-r\delta^2 u(x,\ t+k)+\frac{r}{2}\Big(r+\frac{1}{6}\Big)\delta^4 u(x,\ t+k)$$

$$=\frac{r}{6}\Big(r^2+\frac{r}{2}+\frac{1}{15}\Big)h^6 u_{ttt}(x,\ t)+O(h^8).$$

2.2 Boundary conditions

In the sequel, we are concerned with the following three cases of boundary conditions:

Case 1. case where $u(0,\ t)$ and $u(1,\ t)$ are given;

Case 2. case where $u_x(0,\ t)$ and $u_x(1,\ t)$ are given;

Case 3. case where $u_x(0,\ t)$ and $u(1,\ t)$ are given.

We assume that the initial and boundary data are sufficiently smooth.

Corresponding to the above three cases, we choose the mesh-size as $h=1/(N+1)$, $h=1/(N-1)$ and $h=1/N$ respectively, and replace $u(-ph,\ t)$ and $u(1+ph,\ t)$ $(p=1,\ 2,\ 3)$ by the following formulas:

(2.14)
$$u(-ph,\ t)=2u(0,\ t)-u(ph,\ t)+p^2h^2 u_t(0,\ t)+\frac{1}{12}p^4h^4 u_{tt}(0,\ t)$$

$$+\frac{2}{6!}p^6h^6 u_{ttt}(0,\ t)+\frac{2}{8!}p^8h^8 u_{tttt}(0,\ t)+O(h^{10}),$$

$$u(1+ph,\ t)=2u(1,\ t)-u(1-ph,\ t)+p^2h^2u_t(1,\ t)+\frac{1}{12}p^4h^4u_{tt}(1,\ t)$$
$$+\frac{2}{6!}p^6h^6u_{ttt}(1,\ t)+\frac{2}{8!}p^8h^8u_{tttt}(1,\ t)+O(h^{10}), \tag{2.15}$$

$$u(-ph,\ t)=u(ph,\ t)-2ph\Big[u_x(0,\ t)+\frac{1}{3!}p^2h^2u_{xt}(0,\ t)$$
$$+\frac{1}{5!}p^4h^4u_{xtt}(0,\ t)+\frac{1}{7!}p^6h^6u_{xttt}(0,\ t)\Big]+O(h^9), \tag{2.16}$$

$$u(1+ph,\ t)=u(1-ph,\ t)+2ph\Big[u_x(1,\ t)+\frac{1}{3!}p^2h^2u_{xt}(1,\ t)$$
$$+\frac{1}{5!}p^4h^4u_{xtt}(1,\ t)+\frac{1}{7!}p^6h^6u_{xttt}(1,\ \mathrm{t})\Big]+O(h^9). \tag{2.17}$$

Then we obtain the systems of linear equations in N unknowns of the form

$$\boldsymbol{x}_{n+1}=M_i\boldsymbol{x}_n+\boldsymbol{f}_n \qquad (n=0,\ 1,\cdots;\ i=1,\ 2,\ 3) \tag{2.18}$$

in the case of explicit formulas, and those of the form

$$P_i\boldsymbol{x}_{n+1}=Q_i\boldsymbol{x}_n+\boldsymbol{f}_n \qquad (n=0,\ 1,\cdots;\ i=1,\ 2,\ 3) \tag{2.19}$$

in the case of implicit formulas, where M_i, P_i and Q_i are $N\times N$ matrices, and $\boldsymbol{x}_j$ and $\boldsymbol{f}_j$ $(j=0,\ 1,\ \ldots)$ are N-vectors.

2.3 Special matrices

Let L_i $(i=1,\ 2,\ 3)$ be the $N\times N$ matrices such that

$$L_1=\begin{pmatrix} 0, & 1 & & & \\ 1, & 0, & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1, & 0, & 1 \\ & & & 1, & 0 \end{pmatrix},\quad L_2=\begin{pmatrix} 0, & 2 & & & \\ 1, & 0, & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1, & 0, & 1 \\ & & & 2, & 0 \end{pmatrix},\quad L_3=\begin{pmatrix} 0, & 2 & & & \\ 1, & 0, & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1, & 0, & 1 \\ & & & 1, & 0 \end{pmatrix}. \tag{2.20}$$

Then, as is easily checked, the following relations are valid:

$$(L_1-2I)^2=\begin{pmatrix} 5, & -4, & 1 & & & & \\ -4, & 6, & -4, & 1 & & & \\ 1, & -4, & 6, & -4, & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1, & -4, & 6, & -4, & 1 \\ & & & 1, & -4, & 6, & -4 \\ & & & & 1, & -4, & 5 \end{pmatrix}, \tag{2.21}$$

$$(L_2-2I)^2=\begin{pmatrix} 6, & -8, & 2 & & & & \\ -4, & 7, & -4, & 1 & & & \\ 1, & -4, & 6, & -4, & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1, & -4, & 6, & -4, & 1 & \\ & & 1, & -4, & 7, & -4 & \\ & & & 2, & -8, & 6 & \end{pmatrix},$$

$$(L_3-2I)^2=\begin{pmatrix} 6, & -8, & 2 & & & & \\ -4, & 7, & -4, & 1 & & & \\ 1, & -4, & 6, & -4, & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1, & -4, & 6, & -4, & 1 & \\ & & 1, & -4, & 6, & -4 & \\ & & & 1, & -4, & 5 & \end{pmatrix},$$

$$(2.22)\qquad (L_1-2I)^3=\begin{pmatrix} -14, & 14, & -6, & 1 & & & & & \\ 14, & -20, & 15, & -6, & 1 & & & & \\ -6, & 15, & -20, & 15, & -6, & 1 & & & \\ 1, & -6, & 15, & -20, & 15, & -6, & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1, & -6, & 15, & -20, & 15, & -6, & 1 & \\ & & 1, & -6, & 15, & -20, & 15, & -6 & \\ & & & 1, & -6, & 15, & -20, & 14 & \\ & & & & 1, & -6, & 14, & -14 & \end{pmatrix},$$

$$(2.23)\qquad (L_2-2I)^3=\begin{pmatrix} -20, & 30, & -12, & 2 & & & & & \\ 15, & -26, & 16, & -6, & 1 & & & & \\ -6, & 16, & -20, & 15, & -6, & 1 & & & \\ 1, & -6, & 15, & -20, & 15, & -6, & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1, & -6, & 15, & -20, & 15, & -6, & 1 & \\ & & 1, & -6, & 15, & -20, & 16, & -6 & \\ & & & 1, & -6, & 16, & -26, & 15 & \\ & & & & 2, & -12, & 30, & -20 & \end{pmatrix},$$

$$(2.24)\qquad (L_3-2I)^3=\begin{pmatrix} -20, & 30, & -12, & 2 & & & & & & \\ 15, & -26, & 16, & -6, & 1 & & & & & \\ -6, & 16, & -20, & 15, & -6, & 1 & & & & \\ 1, & -6, & 15, & -20, & 15, & -6, & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1, & -6, & 15, & -20, & 15, & -6, & 1 & \\ & & & 1, & -6, & 15, & -20, & 15, & -6 & \\ & & & & 1, & -6, & 15, & -20, & 14 & \\ & & & & & 1, & -6, & 14, & -14 & \end{pmatrix}.$$

From these results it can readily be seen that, corresponding to the differences $\delta^{2p}u(x, t)$ $(p=1, 2, 3)$ and the boundary conditions of the Case i $(i=1, 2, 3)$, we obtain the matrices $(L_i-2I)^p$ $(i, p=1, 2, 3)$.

Put

$$(2.25)\qquad G_i=\mathrm{diag}\,(2\cos\theta_{i1}, 2\cos\theta_{i2}, \cdots, 2\cos\theta_{iN})\qquad (i=1, 2, 3),$$

$$(2.26)\qquad H_i=\mathrm{diag}\,(\omega_{i1}, \omega_{i2}, \cdots, \omega_{iN})\qquad (i=1, 2, 3),$$

$$(2.27)\qquad R_1=(\sin(i\theta_{1j})),\quad R_2=(\cos(i-1)\theta_{2j}),\quad R_3=(\cos(i-1)\theta_{3j})\qquad (i, j=1, 2, \cdots, N),$$

where

$$(2.28)\qquad \theta_{1j}=\frac{j\pi}{N+1},\qquad \theta_{2j}=\frac{(j-1)\pi}{N-1},\qquad \theta_{3j}=\frac{(2j-1)\pi}{2N},\qquad \omega_{ij}=\sin^2\frac{\theta_{ij}}{2}.$$

Then it is evident that

$$(2.29)\qquad G_i=2I-4H_i\qquad (i=1, 2, 3).$$

In our previous paper [15], we have shown the following

LEMMA. *There hold the relations*

$$(2.30)\qquad L_i=R_iG_iR_i^{-1}\qquad (i=1, 2, 3)$$

and R_i^{-1} $(i=1, 2, 3)$ *can be represented as follows:*

$$(2.31)\qquad R_1^{-1}=\frac{2}{N+1}R_1,\qquad R_2^{-1}=\frac{2}{N-1}R_2^TD_2,\qquad R_3^{-1}=\frac{2}{N}R_3^TD_3,$$

where

$$(2.32)\qquad D_2=\mathrm{diag}(1/2, 1, \cdots, 1, 1/2),\quad D_3=\mathrm{diag}(1/2, 1, \cdots, 1, 1).$$

By this lemma we can express the matrices $(L_i-2I)^p$ as follows:

$$(L_i-2I)^p=R_i(-4\mathrm{H}_i)^pR_i^{-1} \qquad (i,\ p=1,\ 2,\ 3). \tag{2.33}$$

Hence we can directly find out the eigenvalues and eigenvectors of the matrices M_i, P_i and Q_i $(i=1,\ 2,\ 3)$, so that the stability conditions can be obtained easily.

The elements of the matrices R_i $(i=1,\ 2,\ 3)$ need not be stored but can be generated through recurrence formulas [14], so that the systems (2.19) can be solved without the direct inversion of matrices. Needless to say, they can also be solved by the Gaussian elimination method with interchanges or by the LR-decomposition method.

3. Explicit formulas

3.1 Formula with truncation error of order h^4

From (2.9) we have the following formula [13] and matrices:

$$\begin{aligned}u(x,\ t+k)=&a[u(x+2h,\ t)+u(x-2h,\ t)]\\&+b[u(x+h,\ t)+u(x-h,\ t)]+cu(x,\ t)+T(x,\ t),\end{aligned} \tag{3.1}$$

$$M_i=a(L_i-2I)^2+r(L_i-2I)+I, \tag{3.2}$$

where

$$a=\frac{r}{2}\left(r-\frac{1}{6}\right),\qquad b=2r\left(\frac{2}{3}-r\right),\qquad c=1-\frac{5}{2}r+3r^2, \tag{3.3}$$

$$T(x,\ t)=\frac{r}{6}\left(r^2-\frac{r}{2}+\frac{1}{15}\right)h^6u_{ttt}(x,\ t)+O(h^8). \tag{3.4}$$

Let λ_{ij} $(j=1,\ 2,\ \cdots,\ N)$ be the eigenvalues of M_i $(i=1,\ 2,\ 3)$. Then, since by the lemma

$$M_i=R_i[16aH_i^2-4rH_i+I]R_i^{-1}, \tag{3.5}$$

it follows that

$$\lambda_{ij}=1-4r\omega_{ij}+8r\left(r-\frac{1}{6}\right)\omega_{ij}^2. \tag{3.6}$$

From this we find that $\lambda_{ij}>-1$ $(j=1,\ 2,\ \cdots,\ N;\ i=1,\ 2,\ 3)$, because $0\leqq\omega_{ij}\leqq1$ and

$$(\lambda_{ij}+1)/2=(2r\omega_{ij}-\sigma_{ij})^2+1-\sigma_{ij}^2>0, \tag{3.7}$$

where

$$\sigma_{ij}=\frac{1}{2}+\frac{1}{6}\omega_{ij}\leqq\frac{2}{3}.$$

On the other hand, the inequalities $\lambda_{ij} \leqq 1$ ($j=1, 2, \cdots, N$; $i=1, 2, 3$) are valid, if $r \leqq \frac{2}{3}$. The sign of equality holds when $i=2$, $j=1$ and when $i=2$, $j=N$ and $r=\frac{2}{3}$, but then $\lambda_{2,1}$ and $\lambda_{2,N}$ are eigenvalues corresponding to linear elementary divisors because the matrix M_2 is similar to a diagonal matrix. Thus the difference scheme connected with (3.1) is stable if $r \leqq \frac{2}{3}$.

3.2 Formula with truncation error of order h^6

From (2.10) we have the following results:

$$u(x, t+k)=a[u(x+3h, t)+u(x-3h, t)]+b[u(x+2h, t)+u(x-2h, t)]+c[u(x+h, t)+u(x-h, t)]+du(x, t)+T(x, t), \tag{3.8}$$

$$M_i=a(L_i-2I)^3+(b+6a)(L_i-2I)^2+r(L_i-2I)+I, \tag{3.9}$$

where

$$a=\frac{r}{6}\left(r^2-\frac{r}{2}+\frac{1}{15}\right), \qquad b=-r\left(r^2-r+\frac{3}{20}\right), \tag{3.10}$$

$$c=\frac{r}{2}\left(5r^2-\frac{13r}{2}+3\right), \qquad d=1-\frac{r}{3}\left(10r^2-14r+\frac{49}{6}\right),$$

$$T(x, t)=\frac{r}{24}\left(r^3-r^2+\frac{7}{20}r-\frac{3}{70}\right)h^8 u_{tttt}(x, t)+O(h^{10}). \tag{3.11}$$

Since

$$M_i=R_i[-8\alpha H_i^3+2\beta H_i^2-4rH_i+I]R_i^{-1}, \tag{3.12}$$

it follows that

$$\lambda_{ij}=1-4r\omega_{ij}-2\beta\omega_{ij}^2-8\alpha\omega_{ij}^3 \qquad (j=1, 2, \dots, N;\ i=1, 2, 3), \tag{3.13}$$

where

$$\beta=2r\left(\frac{1}{3}-2r\right), \qquad \alpha=8a=\frac{4r}{3}\left(r-\frac{1}{4}\right)^2+\frac{r}{180}>0. \tag{3.14}$$

It is easily seen that $\lambda_{ij} \leqq 1$ ($j=1, 2, \dots, N$; $i=1, 2, 3$), because $0 \leqq \omega_{ij} \leqq 1$ and

$$1-\lambda_{ij}=4r\omega_{ij}\left[\left(1+\left(\frac{1}{6}-r\right)\omega_{ij}\right)^2+\frac{5}{3}\left(r-\frac{3}{10}\right)^2\omega_{ij}^2\right] \geqq 0. \tag{3.15}$$

The equal sign is valid only when $i=2$ and $j=1$.

Now we seek for the condition under which $\lambda_{ij} \geqq -1$. From (3.13) it follows that

(3.16) $$(\lambda_{ij}+1)/2 = 1-2r\omega_{ij}-\beta\omega_{ij}^2-4\alpha\omega_{ij}^3.$$

Corresponding to (3.16) we put

(3.17) $$f(x)=1-2rx-\beta x^2-4\alpha x^3 \qquad (0 \leqq x \leqq 1),$$

and transform (3.17) as follows:

(3.18) $$g(y)=y^3 f\left(\frac{1}{y}\right)=y^3-2ry^2-\beta y-4\alpha,$$

and

(3.19) $$h(z)=g(z+1)=z^3+c_1z^2+c_2z+c_3,$$

where

(3.20) $$c_1=3-2r, \quad c_2=4r^2-\frac{14}{3}r+3, \quad c_3=1-\frac{136}{45}r+\frac{20}{3}r^2-\frac{16}{3}r^3.$$

It can be shown by means of the discriminant that the cubic equation $c_3=0$ has one and only one real root r_0, which is given numerically as follows:

(3.21) $$r_0=0.8413602280\cdots.$$

Then, since $c_1>0$ for $r \leqq r_0$, $c_2>0$ and $c_3>0$ for $r<r_0$, it follows that $f(x) \geqq 0$ for $r \leqq r_0$, and the sign of equality holds only when $x=1$ and $r=r_0$. Hence $\lambda_{ij} \geqq -1$ for $r \leqq r_0$ and the equal sign is valid only when $i=2$, $j=N$ and $r=r_0$.

Thus the difference scheme corresponding to (3.8) is stable if $r \leqq r_0$.

4. Implicit formulas

4.1 Formula with truncation error of order h^4

From (2.11) we have the following formula [4] and matrices:

(4.1) $$a[u(x+h,\, t+k)+u(x-h,\, t+k)]+bu(x,\, t+k)$$
$$=\alpha[u(x+h,\, t)+u(x-h,\, t)]+\beta u(x,\, t)+T(x,\, t),$$

(4.2) $$P_i=a(L_i-2I)+12I, \qquad Q_i=\alpha(L_i-2I)+12I,$$

where

(4.3) $$a=1-6r, \quad b=10+12r, \quad \alpha=1+6r, \quad \beta=10-12r,$$

(4.4) $$T(x,\, t)=r\left(\frac{1}{20}-r^2\right)h^6u_{ttt}(x,\, t)+O(h^8).$$

Let μ_{ij}, ρ_{ij} and λ_{ij} ($j=1, 2, \cdots, N$) be the eigenvalues of P_i, Q_i and $P_i^{-1}Q_i$ ($i=1, 2, 3$) respectively. Then, since

$$P_i=R_i[4(6r-1)H_i+12I]R_i^{-1},\quad Q_i=R_i[-4(6r+1)H_i+12I]R_i^{-1}, \tag{4.5}$$

it follows that

$$\mu_{ij}=12+4(6r-1)\omega_{ij},\qquad \rho_{ij}=12-4(6r+1)\omega_{ij},\qquad \lambda_{ij}=\rho_{ij}/\mu_{ij}. \tag{4.6}$$

it is easily seen that $1\geqq\lambda_{ij}>-1$ and that the sign of equality holds only when $i=2$ and $j=1$.

Thus the difference scheme corresponding to (4.1) is unconditionally stable.

4.2 Formula with truncation error of order h^6

From (2.12) we have the following results:

$$\begin{aligned} &a[u(x+2h,\ t+k)+u(x-2h,\ t+k)]+b[u(x+h,\ t+k)\\ &\quad+u(x-h,\ t+k)]+cu(x,\ t+k)=\alpha[u(x+2h,\ t)\\ &\quad+u(x-2h,\ t)]+\beta[u(x+h,\ t)+u(x-h,\ t)]+\gamma u(x,\ t)+T(x,\ t), \end{aligned} \tag{4.7}$$

$$P_i=a(L_i-2I)^2+(b+4a)\ (L_i-2I)+90I,\quad Q_i=\alpha(L_i-2I)^2+90I, \tag{4.8}$$

where

$$\begin{aligned} &a=30r^2-1, &&b=-120r^2-90r+4, &&c=180r^2+180r+84,\\ &\alpha=-\beta/4, &&\beta=60r^2+30r+4, &&\gamma=-90r^2-45r+84, \end{aligned} \tag{4.9}$$

$$T(x,\ t)=\frac{r}{4}\left(15r^3+5r^2-\frac{3}{4}r-\frac{13}{42}\right)h^8u_{tttt}(x,\ t)+O(h^{10}). \tag{4.10}$$

Since

$$P_i=R_i[16aH_i^2+360rH_i+90I]R_i^{-1},\quad Q_i=R_i[16\alpha H_i^2+90I]R_i^{-1}, \tag{4.11}$$

it follows that

$$\mu_{ij}=90+360r\omega_{ij}+16(30r^2-1)\omega_{ij}^2>0 \tag{4.12}$$

and

$$\rho_{ij}=90-(240r^2+120r+16)\omega_{ij}^2. \tag{4.13}$$

It can be seen that $1\geqq\lambda_{ij}>-1$ and that the equal sign is valid only when $i=2$ and $j=1$, because

$$\rho_{ij}+\mu_{ij}=148+240r\omega_{ij}(1+r\omega_{ij})+(1-\omega_{ij})\ (32+32\omega_{ij}+120r\omega_{ij})>0$$

and

$$\mu_{ij}-\rho_{ij}=120r\omega_{ij}[3+(1+6r)\omega_{ij}]\geqq 0.$$

Thus the difference scheme connected with (4.7) is unconditionally stable.

REMARK. For the boundary value problem of the *Case* 2, the matrices M_2 and $P_2^{-1}Q_2$ have eigenvalues equal to one in modulus, so that the persisting errors [11, 7] will be observed.

5. Numerical example

We consider the problem (1.1) with the following conditions:

$$u(x,\ 0)=\sin\pi x, \qquad u(0,\ t)=u(1,\ t)=0. \tag{5.1}$$

Its exact solution is given by

$$u(x,\ t)=\exp(-\pi^2 t)\sin\pi x. \tag{5.2}$$

This problem is solved numerically first by the well-known formula

$$U(x,\ t+k)=r[U(x+h,\ t)+U(x-h,\ t)]+(1-2r)U(x,\ t), \tag{5.3}$$

and then by the formulas corresponding to (3.1), (3.8), (4.1) and (4.7), with the uniform mesh-sizes $h=\frac{1}{8}$ and $r=\frac{1}{4}$. The approximate values of $u\left(\frac{1}{2},\ t\right)$ are given in Table 1.

Table 1.

formula \ t	0. 25	0. 5	0. 75	1. 0
(5. 3)	0. 83457281847-01	0. 69651178933-02	0. 58128980711-03	0. 48512867266-04
(3. 1)	0. 84808500771-01	0. 71924818031-02	0. 60998359853-03	0. 51731794488-04
(3. 8)	0. 84805045131-01	0. 71918956799-02	0. 60990903771-03	0. 51723363470-04
(4. 1)	0. 84799916378-01	0. 71910258176-02	0. 60979838803-03	0. 51710852311-04
(4. 7)	0. 84805117504-01	0. 71919060356-02	0. 60991027361-03	0. 51723498613-04
(5. 2)	0. 84804972470-01	0. 71918833555-02	0. 60990746996-03	0. 51723186198-04

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Department of Mathematics
Faculty of Science
Hiroshima University