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Principal Oriented Bordism Modules of Generalized Quaternion Groups

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Introduction

The principal oriented bordism module $\Omega_*(G)$ of a group G is defined to be the module of all equivariant bordism classes of closed principal oriented (smooth) G-manifolds. $\Omega_*(G)$ is a module over the oriented bordism ring Ω_*, and this module $\Omega_*(G)$ and the unoriented one $\mathfrak{N}_*(G)$ are studied by several authors.

The purpose of this paper is to determine the Ω_*-module structure of $\Omega_*(H_m)$, $m \geqq 2$, where H_m is the generalized quaternion group generated by two elements x and y with two relations

$$x^{2^{m-1}} = y^2 \quad \text{and} \quad xyx = y,$$

that is, the subgroup of the unit sphere S^3 in the quaternion field $\boldsymbol{H}$ generated by $x = \exp(\pi i/2^{m-1})$ and $y = j$.

The group H_m acts freely on the unit sphere S^{4n+3} in the quaternion $(n+1)$-space $\boldsymbol{H}^{n+1}$ by the diagonal action $\alpha_m(q, (q_0, \dots, q_n)) = (qq_0, \dots, qq_n)$ $(q, q_i \in \boldsymbol{H})$, and we obtain the principal oriented H_m-manifold

$$(0.1) \qquad (\alpha_m, S^{4n+3}) \qquad (n \geqq 0).$$

Also, the element $x = \exp(\pi i/2^{m-1})$ generates the cyclic subgroup Z_{2^m} of order 2^m, and this group acts on the unit sphere S^{2n+1} in the complex $(n+1)$-space $\boldsymbol{C}^{n+1}$ by the diagonal action $x(z_0, \dots, z_n) = (xz_0, \dots, xz_n)$ $(z_i \in \boldsymbol{C})$. We denote this Z_{2^m}-manifold by (T_m, S^{2n+1}). Hence we obtain the extension

$$(0.2) \qquad i_m(T_m, S^{4n+1}) \qquad (n \geqq 0),$$

by the inclusion $i_m \colon Z_{2^m} \subset H_m$, which is the disjoint union $Z_2 \times S^{4n+1}$ with the H_m-action given by

$$x(\varepsilon, z) = (\varepsilon, x^{\varepsilon} z), \quad y(\varepsilon, z) = (-\varepsilon, \varepsilon z) \qquad (\varepsilon = \pm 1,\ z \in S^{4n+1}).$$

Let π be the set of partitions $\omega = (a_1, \dots, a_r)$ with unequal parts a_j, none of which is a power of 2. By the consideration of K. Kawakubo [6], there is a Z_2-manifold

$$(T, N(\omega)) \quad \text{for} \quad \omega \in \pi$$

such that $N(\omega)$ represents Wall's generator $g(\omega)$ of the oriented bordism ring Ω_* and $N(\omega)$ admits an orientation reversing involution T (cf. § 5). By using this manifold and S^{2n+1}, we can consider the principal oriented H_m-manifold

$$(0.3) \qquad \bar{\beta}_m(n, \omega) = (\bar{\beta}_m, S^{2n+1} \times S^{2n+1} \times N(\omega)) \qquad (n \geqq 0, \omega \in \pi),$$

whose action $\bar{\beta}_m$ of H_m is given by

$$\bar{\beta}_m(x, (z, z', u)) = (xz, x^{-1}z', u), \quad \bar{\beta}_m(y, (z, z', u)) = (-z', z, Tu),$$

for $z, z' \in S^{2n+1}$ and $u \in N(\omega)$. (This definition is suggested by K. Shibata.)

For any H_2-manifold M, we denote by γM the manifold M with the new H_2-action $x*m = ym$, $y*m = xm$ $(m \in M)$, and we can consider its extension $k_m\gamma M$ by the inclusion $k_m: H_2 \subset H_m$, $k_m(x) = x^{2^{m-2}}$, $k_m(y) = y$. Therefore, we obtain the principal oriented H_m-manifolds

$$(0.4) \qquad k_m\gamma i_2(T_2, S^{4n+1}) \qquad (n \geqq 0),$$

$$(0.5) \qquad k_m\gamma\bar{\beta}_2(n, \omega) \qquad (n \geqq 0, \omega \in \pi),$$

from the H_2-manifolds of (0.2) and (0.3).

Finally, we consider the extension

$$(0.6) \qquad iE^{4n+3}W(\omega) \qquad (n \geqq 0, \omega \in \pi)$$

of the bordism class $E^{4n+3}W(\omega) \in \tilde{\Omega}_*(Z_2)$, due to K. Shibata [7], by the inclusion $i: Z_2 \subset H_m$, $i(-1) = x^{2^{m-1}} = y^2$.

Then, we have the following

THEOREM 7.5. *The principal oriented bordism module $\tilde{\Omega}_*(H_m)$ of the generalized quaternion group H_m $(m \geqq 2)$ is the direct sum*

$$\mathfrak{L}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \oplus \mathfrak{W}_m,$$

where $\mathfrak{L}_m$, $\mathfrak{Q}_m$, $\mathfrak{Q}'_m$ and $\mathfrak{W}_m$ are the Ω_*-submodules of $\tilde{\Omega}_*(H_m)$ generated by the bordism classes (represented by the H_m-manifolds) of (0.1), (0.2–3), (0.4–5) *and* (0.6), *for $n \geqq 0$ and $\omega \in \pi$, respectively.*

Furthermore, we study the relations among the generators in these Ω_*-submodules and obtain the Ω_*-module structure of $\tilde{\Omega}_*(H_m)$ in Theorem 8.12.

We prepare in § 1 some results for the homology of H_m. In § 2, we study the unoriented bordism module $\mathfrak{N}_*(H_m)$ by using the isomorphism

$$\mathfrak{N}_*(H_m) \cong \mathfrak{N}_* \otimes H_*(H_m; Z_2) \qquad \text{(cf. [3, (19.3)])},$$

and determine the free $\mathfrak{N}_*$-module structure of $\mathfrak{N}_*(H_m)$ (Theorem 2.13). Also, we study the module $\mathfrak{N}_*(NS^1)$ of the normalizer NS^1 of S^1 in S^3 (Proposition 2.19).

We recall in §3 the results of [5] for the oriented bordism module $\Omega_*(Z_{2^m})$. By using these results and the isomorphism

$$\tilde{\Omega}_*(H_m) \cong \sum_{p+q=n}\tilde{H}_p(H_m; \Omega_q) \qquad \text{(cf. [3, Th. 14.2])},$$

we study in §4 the Ω_*-submodules $\mathfrak{L}_m$ and $\mathfrak{W}_m$ in Theorem 7.5 and the submodules $\mathfrak{I}_m$ and $\mathfrak{J}_m$ generated by the bordism classes of (0.2) and (0.4), respectively. We define the Z_2-manifold $(T, N(\omega))$ in §5 and the H_m-manifold $\bar{\beta}_m(n, \omega)$ of (0.3) in §6, and prove our main results in §§7–8.

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§1. The homology of H_m

The generalized quaternion group H_m ($m \geqq 2$), generated by

$$x = \exp(\pi i/2^{m-1}) \quad \text{and} \quad y = j,$$

acts freely on the infinite dimensional sphere $S^\infty = \cup_n S^{4n+3}$ by the action of (0.1), and an H_m-equivariant CW-decomposition of S^∞ is given in [4, §2] as follows:

$$(1.1)\quad \begin{aligned} S^\infty &= \{qe^{4l+s},\ qe_\varepsilon^{4l+t} \mid s = 0, 3;\ t = 1, 2;\ \varepsilon = 1, 2;\ q \in H_m\},\\ \partial e^{4l} &= \textstyle\sum_{q\in H_m} qe^{4l-1},\\ \partial e_1^{4l+1} &= (x-1)e^{4l}, \quad \partial e_2^{4l+1} = (y-1)e^{4l},\\ \partial e_1^{4l+2} &= \textstyle\sum_{i=0}^{2^{m-1}-1} x^i e_1^{4l+1} - (y+1)e_2^{4l+1},\\ \partial e_2^{4l+2} &= (xy+1)e_1^{4l+1} + (x-1)e_2^{4l+1},\\ \partial e^{4l+3} &= (x-1)e_1^{4l+2} - (xy-1)e_2^{4l+2}. \end{aligned}$$

This induces a CW-decomposition of the classifying space S^∞/H_m of H_m:

$$(1.2)\quad \begin{aligned} S^\infty/H_m &= \{e_m^{4l+s},\ e_{\varepsilon,m}^{4l+t} \mid s = 0, 3;\ t = 1, 2;\ \varepsilon = 1, 2\},\\ \partial e_m^{4l} &= 2^{m+1} e_m^{4l-1}, \quad \partial e_{1,m}^{4l+1} = \partial e_{2,m}^{4l+1} = 0,\\ \partial e_{1,m}^{4l+2} &= 2^{m-1} e_{1,m}^{4l+1} - 2e_{2,m}^{4l+1}, \quad \partial e_{2,m}^{4l+2} = 2e_{1,m}^{4l+1}, \quad \partial e_m^{4l+3} = 0. \end{aligned}$$

Therefore, we have

LEMMA 1.3. ([2, Ch. XII, §7], [4, §2]) *The homology groups of the generalized quaternion group H_m are given by*

$$\tilde{H}_k(H_m; Z) = \begin{cases} Z_2[e_{1,m}^{4l+1}] \oplus Z_2[e_{2,m}^{4l+1}] & \textit{for} \quad k = 4l+1, \\ Z_{2^{m+1}}[e_m^{4l+3}] & \textit{for} \quad k = 4l+3, \\ 0 & \textit{otherwise}, \end{cases}$$

$$H_k(H_m; Z_2) = \begin{cases} Z_2[e_m^k] & \textit{for} \quad k = 4l, 4l+3, \\ Z_2[e_{1,m}^k] \oplus Z_2[e_{2,m}^k] & \textit{for} \quad k = 4l+1, 4l+2, \end{cases}$$

where $Z_t[e]$ means the cyclic group of order t generated by the homology class of the cell e.

Now, consider the cyclic subgroup Z_{2^m} generated by the element x. Then, (1.1) induces the CW-decomposition

$$S^\infty/Z_{2^m} = \{e^{4l+s}, ye^{4l+s}, e_\varepsilon^{4l+t} | s = 0, 3; t = 1, 2; \varepsilon = 1, 2\}$$

with the boundary formulae obtained by setting $x=1$ in those of (1.1). Therefore, we see easily that the generators of the homology groups

(1.4) $$H_{2l+1}(Z_{2^m}; Z) = Z_{2^m}[z_{2l+1}], \qquad H_k(Z_{2^m}; Z_2) = Z_2[z_k]$$

are given by

$$z_{4l} = e^{4l}, \quad z_{4l+1} = e_1^{4l+1}, \quad z_{4l+2} = (1+y)e_1^{4l+2}, \quad z_{4l+3} = (1+y)e^{4l+3},$$

and that the extension and the transfer homomorphisms

(1.5) $$i_{m*}: H_k(Z_{2^m}; \Lambda) \longrightarrow H_k(H_m; \Lambda), \quad t_{i_m*}: H_k(H_m; \Lambda) \longrightarrow H_k(Z_{2^m}; \Lambda)$$

($\Lambda = Z$ or Z_2), induced by the inclusion

(1.6) $$i_m: Z_{2^m} \longrightarrow H_m, \qquad \operatorname{Im} i_m = Z_{2^m}[x],$$

have the following properties for the generators in Lemma 1.3 and (1.4).

LEMMA 1.7. (i) *For $k=4l$ and $\Lambda=Z_2$, i_{m*} is isomorphic and t_{i_m*} is trivial.*

(ii) $i_{m*}(z_{4l+1}) = e_{1,m}^{4l+1}$; $t_{i_m*}(e_{1,m}^{4l+1}) = 0$, $t_{i_m*}(e_{2,m}^{4l+1}) = 2^{m-1}z_{4l+1}$.

(iii) *For $k=4l+2$ and $\Lambda=Z_2$, i_{m*} is trivial and*

$$t_{i_m*}(e_{1,m}^{4l+2}) = z_{4l+2}, \quad t_{i_m*}(e_{2,m}^{4l+2}) = 0.$$

(iv) *For $k=4l+3$, $i_{m*}(z_{4l+3})=2e_m^{4l+3}$ and t_{i_m*} is epimorphic.*

Also, we consider the cyclic subgroup Z_4 of H_2 generated by the element y. Then, (1.1) for $m=2$ induces the CW-decomposition of S^∞/Z_4 by setting $y=1$, and we see easily that the generators of (1.4) for $m=2$ are given by

$$z_{4l} = e^{4l}, \quad z_{4l+1} = e_2^{4l+1}, \quad z_{4l+2} = (1+x)e_\varepsilon^{4l+2}, \quad z_{4l+3} = (1+x)e^{4l+3},$$

and that the extension and the transfer homomorphisms

(1.8) $\quad j_{2*}\colon H_k(Z_4; \Lambda) \longrightarrow H_k(H_2; \Lambda), \quad t_{j_2*}\colon H_k(H_2; \Lambda) \longrightarrow H_k(Z_4; \Lambda)$

($\Lambda = Z$ or Z_2), induced by the inclusion

(1.9) $$j_2\colon Z_4 \longrightarrow H_2, \qquad \mathrm{Im}\, j_2 = Z_4[y],$$

have the following properties.

LEMMA 1.10. (i) *For $k=4l$ and $\Lambda=Z_2$, j_{2*} is isomorphic and t_{j_2*} is trivial.*

(ii) $j_{2*}(z_{4l+1}) = e_{2,2}^{4l+1}$; $\quad t_{j_2*}(e_{1,2}^{4l+1}) = 2z_{4l+1}, \quad t_{j_2*}(e_{2,2}^{4l+1}) = 0.$

(iii) *For $k=4l+2$ and $\Lambda=Z_2$, j_{2*} is trivial and*

$$t_{j_2*}(e_{1,2}^{4l+2}) = z_{4l+2} = t_{j_2*}(e_{2,2}^{4l+2}).$$

(iv) *For $k=4l+3$, $j_{2*}(z_{4l+3})=2e_2^{4l+3}$ and t_{j_2*} is epimorphic.*

Now, consider the automorphism

(1.11) $$\gamma\colon H_2 \longrightarrow H_2, \qquad \gamma(x) = y, \quad \gamma(y) = x.$$

LEMMA 1.12. *For the isomorphism*

$$\gamma_*\colon H_{4l+2}(H_2; Z_2) \longrightarrow H_{4l+2}(H_2; Z_2)$$

induced by γ, we have

$$\gamma_*(e_{1,2}^{4l+2}) = e_{1,2}^{4l+2}, \qquad \gamma_*(e_{2,2}^{4l+2}) = e_{1,2}^{4l+2} + e_{2,2}^{4l+2}.$$

PROOF. Set $\gamma_*(e_{\varepsilon,2}^{4l+2}) = a_\varepsilon e_{1,2}^{4l+2} + b_\varepsilon e_{2,2}^{4l+2}$. Since $\gamma \circ i_2 = j_2$, we have the commutative diagram

$$\begin{tikzcd} H_{4l+2}(H_2; Z_2) \arrow[r, "\gamma_*"] \arrow[d, "t_{i_2*}"] & H_{4l+2}(H_2; Z_2) \arrow[d, "t_{j_2*}"] \\ H_{4l+2}(Z_4; Z_2) \arrow[r, "id"] & H_{4l+2}(Z_4; Z_2). \end{tikzcd}$$

Therefore we see that $a_2 + b_2 = 0$ by Lemmas 1.7 (iii) and 1.10 (iii), and so that $a_2 = b_2 = 1$ since γ_* is isomorphic. This result and the equality $\gamma_*\gamma_*(e_{2,2}^{4l+2}) = e_{2,2}^{4l+2}$ show that $a_1 = 1$, $b_1 = 0$, as desired. q. e. d.

Let

$$(1.13)\qquad k_{l,m}\colon H_l \longrightarrow H_m,\quad k_m = k_{2,m}\colon H_2 \longrightarrow H_m,\qquad (2 \leqq l \leqq m),$$

be the inclusions such that $k_{l,m}(x)=x^{2^{m-l}}$, $k_{l,m}(y)=y$.

LEMMA 1.14. *For the extension homomorphism*

$$k_{m*}\colon H_k(H_2; A) \longrightarrow H_k(H_m; A)\qquad (A = Z \text{ or } Z_2,\ m > 2),$$

induced by k_m and the generators of Lemma 1.3, *we have the following equalities*:

$$k_{m*}(e_2^{4l}) = e_m^{4l},\quad k_{m*}(e_{1,2}^{4l+1}) = 0,\quad k_{m*}(e_{2,2}^{4l+1}) = e_{2,m}^{4l+1},$$

$$k_{m*}(e_{1,2}^{4l+2}) = e_{1,m}^{4l+2},\quad k_{m*}(e_{2,2}^{4l+2}) = 0,\quad k_{m*}(e_2^{4l+3}) = 2^{m-2}e_m^{4l+3}.$$

PROOF. Consider the H_m-equivariant CW-decomposition (1.1) and the H_{m-1}-equivariant CW-decomposition

$$S^\infty = \{qe'^{4l+s},\ qe_\varepsilon'^{4l+t} | s = 0, 3;\ t = 1, 2;\ \varepsilon = 1, 2;\ q \in H_{m-1}\}$$

of [4, § 2]. Then, we see easily by the definition in [4, § 2] that

$$e'^{4l} = e^{4l},\quad e_1'^{4l+1} = (1+x)e_1^{4l+1},\quad e_2'^{4l+1} = e_2^{4l+1},$$

$$e_1'^{4l+2} = e_1^{4l+2},\quad e_2'^{4l+2} = (1+x)e_2^{4l+2},\quad e'^{4l+3} = (1+x)e^{4l+3}.$$

Therefore the extension homomorphisms $k_* = k_{m-1,m*}\colon H_k(H_{m-1}; A) \to H_k(H_m; A)$ are given by

$$k_*(e_{m-1}^{4l}) = e_m^{4l},\quad k_*(e_{1,m-1}^{4l+1}) = 0,\quad k_*(e_{2,m-1}^{4l+1}) = e_{2,m}^{4l+1},$$

$$k_*(e_{1,m-1}^{4l+2}) = e_{1,m}^{4l+2},\quad k_*(e_{2,m-1}^{4l+2}) = 0,\quad k_*(e_{m-1}^{4l+3}) = 2e_m^{4l+3}.$$

These show the desired results since $k_m = k_{m-1,m}\circ\cdots\circ k_{2,3}$. q.e.d.

Finally consider the inclusion

$$(1.15)\qquad j_m = k_m\circ j_2 = k_m\circ\gamma\circ i_2\colon Z_4 \longrightarrow H_m,\quad \operatorname{Im} j_m = Z_4[y].$$

Then, by Lemmas 1.10 and 1.14, we see that the extension homomorphism

$$j_{m*}\colon H_k(Z_4; A) \longrightarrow H_k(H_m; A)\qquad (A = Z \text{ or } Z_2)$$

satisfies the following

LEMMA 1.16. (i) *j_{m*} is isomorphic for $k=4l$ and $A=Z_2$, and is trivial for $k=4l+2$ and $A=Z_2$.*

(ii) $$j_{m*}(z_{4l+1}) = e_{2,m}^{4l+1},\quad j_{m*}(z_{4l+3}) = 2^{m-1}e_m^{4l+3}.$$

§2. The unoriented bordism modules $\mathfrak{N}_*(H_m)$ and $\mathfrak{N}_*(NS^1)$

For a given compact Lie group G, an n-dimensional principal G-manifold $(G, B^n)=(\alpha, B^n)$ is a pair of a compact (smooth) n-manifold B^n and a free (smooth) action $\alpha: G\times B^n \to B^n$, and two closed principal G-manifolds (G, M^n) and (G, N^n) are G-equivariantly bordant, if there is a principal G-manifold (G, B^{n+1}) with $(G, \dot{B}^{n+1})=(G, M^n \cup N^n)$. Denote by the G-bordism class of (G, M^n) by $[G, M^n]$, and the collection of all such classes by $\mathfrak{N}_n(G)$. $\mathfrak{N}_n(G)$ is a module with respect to the disjoint union, and the direct sum

(2.1) $$\mathfrak{N}_*(G) = \textstyle\sum_{n=l}^{\infty} \mathfrak{N}_n(G) \qquad (l = \dim G)$$

is the principal unoriented G-bordism module. For the unit group e,

$$\mathfrak{N}_* = \textstyle\sum_{n=0}^{\infty} \mathfrak{N}_n, \qquad \mathfrak{N}_n = \mathfrak{N}_n(e),$$

is the Thom bordism ring with respect to the multiplication induced by the cartesian product $M\times N$, and $\mathfrak{N}_*(G)$ of (2.1) can be given a structure of (left) $\mathfrak{N}_*$-module by

$$[N][G, M] = [G, N\times M],$$

where G acts on $N\times M$ by $g(n, m)=(n, gm)$ (cf. [3, §§2, 19]).

For an element $[G, M]\in \mathfrak{N}_n(G)$, let $f: M/G \to BG$ be the classifying map of the principal G-bundle $M\to M/G$. Put $l=\dim G$. Then we see easily the following result in a way similar to the case of a finite group G ([3, §19]):

(2.2) *There is an isomorphism*

$$\varphi: \mathfrak{N}_*(G) \longrightarrow \mathfrak{N}_*(BG)$$

of $\mathfrak{N}_*$-modules of degree $-l$, defined by $\varphi[G, M]=[M/G, f]$, where $\mathfrak{N}_*(BG)$ is the $\mathfrak{N}_*$-module in [3, §8].

The element $\mu[G, M]=f_*(M/G)\in H_{n-l}(BG; Z_2)$ is defined, where $M/G\in H_{n-l}(M/G; Z_2)$ means the fundamental class. Using [3, (8.1)] and (2.2), we have

(2.3) $\mu: \mathfrak{N}_n(G) \longrightarrow H_{n-l}(BG; Z_2)$ *is epimorphic.*

For a base $\{c_{n,i}\}$ of $H_n(BG; Z_2)$, we can take $C_{n+l,i}\in \mathfrak{N}_{n+l}(G)$ with $\mu C_{n+l,i} = c_{n,i}$ by (2.3). Then a homomorphism of $\mathfrak{N}_*$-modules

$$h: \mathfrak{N}_*\otimes H_*(BG; Z_2) \longrightarrow \mathfrak{N}_*(G)$$

is obtained by $h(1\otimes c_{n,i})=C_{n+l,i}$, and

(2.4) (cf. [3, (19.3)]) *h is an isomorphism of $\mathfrak{N}_*$-modules.*

Let H be a subgroup of G. For the inclusion $i: H \subset G$, consider the extension homomorphism

$$i: \mathfrak{N}_n(H) \longrightarrow \mathfrak{N}_n(G) \tag{2.5}$$

defined by $i[H, M]=[i(H, M)]$, where $i(H, M)$ is the principal G-manifold consisting of the quotient manifold $(G \times M)/H$ of $G \times M$ by $(g, m) \equiv (gh^{-1}, hm)$ and the G-action on $(G \times M)/H$ given by $g'[g, m]=[g'g, m]$. Consider also the transfer homomorphism

$$t_i: \mathfrak{N}_n(G) \longrightarrow \mathfrak{N}_n(H) \tag{2.6}$$

defined by restricting the G-action on H by the inclusion $i: H \subset G$.

Now, we study the $\mathfrak{N}_*$-module structure of $\mathfrak{N}_*(H_m)$ $(m \geqq 2)$.

We consider the principal Z_2-manifolds

$$(a, S^n) \qquad (a \text{ is the antipodal action}), \tag{2.7}$$

and the principal Z_{2^m}-manifolds

$$\begin{aligned} & l(a, S^n) \qquad (l: Z_2 \subset Z_{2^m}), \\ & (T_m, S^{2n+1}), \quad T_m(x, (z_0, z_1, \ldots, z_n)) = (xz_0, xz_1, \ldots, xz_n). \end{aligned} \tag{2.8}$$

For any principal H_2-manifold (H_2, M), denote by

$$\gamma(H_2, M) \tag{2.9}$$

the manifold M with the new action $g*m=\gamma(g)m$ $(g \in H_2,\ m \in M)$, where $\gamma: H_2 \to H_2$ is the automorphism (1.11).

Now we consider the principal H_m-manifolds

$$\begin{aligned} & (\alpha_m, S^{4n+3}) && ((0.1)), \\ & i_m(T_m, S^{2n+1}) && (i_m: Z_{2^m} \subset H_m \text{ of } (1.6)), \\ & j_m(T_2, S^{2n+1}) && (j_m: Z_4 \subset H_m \text{ of } (1.15)), \\ & i(a, S^n) && (i = i_m \circ l: Z_2 \subset Z_{2^m} \subset H_m), \\ & (\beta_m, S^{2n+1} \times S^{2n+1}), && \begin{cases} \beta_m(x, (z, z')) = (xz, x^{-1}z'), \\ \beta_m(y, (z, z')) = (-z', z), \end{cases} \\ & k_m\gamma(\beta_2, S^{2n+1} \times S^{2n+1}) && (k_m: H_2 \subset H_m \text{ of } (1.13)). \end{aligned} \tag{2.10}$$

LEMMA 2.11. *Consider the* Z_4*-manifold* $(Z_4, S^{2n+1}\times S^{2n+1})$ *with* $y(z, z')=(-z', z)$ *for the generator* $y\in Z_4$. *Then* $\mu[Z_4, S^{2n+1}\times S^{2n+1}]\in H_{4n+2}(Z_4; Z_2)$ *is non-zero, where* $\mu\colon \mathfrak{N}_*(Z_4)\to H_*(Z_4; Z_2)$ *is the homomorphism of* (2.3).

PROOF. By using the Z_4-equivariant CW-decomposition $S^{2n+1}\times S^{2n+1}=\{e^k_\pm\times e^l_\pm\}$, we see easily that $\mu[Z_4, S^{2n+1}\times S^{2n+1}]$ is non-zero in $H_{4n+2}(\cup_n(S^{2n+1}\times S^{2n+1})/Z_4; Z_2)=H_{4n+2}(Z_4; Z_2)$. q.e.d.

For the homomorphism

$$\mu\colon \mathfrak{N}_n(H_m)\longrightarrow H_n(H_m; Z_2) \qquad (m\geqq 2)$$

of (2.3) and the bordism classes of the manifolds in (2.10), we have the following lemma, where e's are the generators in Lemma 1.3.

LEMMA 2.12. (i) $\mu[\alpha_m, S^{4n+3}]=e^{4n+3}_m$,

(ii) $\mu i_m[T_m, S^{4n+1}] = e^{4n+1}_{1,m}$,

(iii) $\mu j_m[T_2, S^{4n+1}] = e^{4n+1}_{2,m}$,

(iv) $\mu i[a, S^{4n}] = e^{4n}_m$,

(v) $\mu[\beta_m, S^{2n+1}\times S^{2n+1}] = e^{4n+2}_{2,m}$,

(vi) $\mu k_m\gamma[\beta_2, S^{2n+1}\times S^{2n+1}] = \begin{cases} e^{4n+2}_{1,2}+e^{4n+2}_{2,2} & \text{for } m=2, \\ e^{4n+2}_{1,m} & \text{for } m>2. \end{cases}$

PROOF. We have (i) from (1.2). Since μ is natural for maps, (ii), (iii) and (iv) follow from Lemmas 1.7, 1.16 and [5, Prop. 1.7 (i)]. (vi) follows from (v), Lemmas 1.12 and 1.14. Therefore, it is sufficient to prove (v).

(v) Consider the commutative diagram (cf. [3, §20])

$$\begin{array}{ccccc} \mathfrak{N}_{4n+2}(Z_4) & \xleftarrow{t_{j_m}} & \mathfrak{N}_{4n+2}(H_m) & \xrightarrow{t_{i_m}} & \mathfrak{N}_{4n+2}(Z_{2^m}) \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mu} \\ H_{4n+2}(Z_4, Z_2) & \xleftarrow{t_{j_m*}} & H_{4n+2}(H_m; Z_2) & \xrightarrow{t_{i_m*}} & H_{4n+2}(Z_{2^m}; Z_2). \end{array}$$

Then we see that

$$\beta = \mu[\beta_m, S^{2n+1}\times S^{2n+1}] \neq 0 \quad \text{in} \quad H_{4n+2}(H_m; Z_2),$$

since $t_{j_m*}(\beta)=\mu[Z_4, S^{2n+1}\times S^{2n+1}]\neq 0$ by the above lemma. On the other hand, we see that $t_{i_m*}(\beta)=0$ since $t_{i_m}(\beta_m, S^{2n+1}\times S^{2n+1})$ is the boundary of $(Z_{2^m}, D^{2n+2}\times S^{2n+1})$, where D^{2n+2} is the disk bounded by S^{2n+1} and Z_{2^m} acts on

$D^{2n+2}\times S^{2n+1}$ by $x(z, z')=(xz, x^{-1}z')$. Therefore, we see $\beta=e_{2,m}^{4n+2}$ by Lemma 1.7 (iii). q.e.d.

By (2.4) and Lemmas 1.3 and 2.12, we have immediately

THEOREM 2.13. *$\mathfrak{N}_*(H_m)$ $(m\geqq 2)$ is a free $\mathfrak{N}_*$-module with basis* $\{[\alpha_m, S^{4n+3}],\ i_m[T_m, S^{4n+1}],\ j_m[T_2, S^{4n+1}],\ i[a, S^{4n}],\ [\beta_m, S^{2n+1}\times S^{2n+1}],\ k_m\gamma[\beta_2, S^{2n+1}\times S^{2n+1}] | n\geqq 0\}$.

LEMMA 2.14. (i) *By the extension homomorphism $k_{l,m}$: $\mathfrak{N}_*(H_l)\to\mathfrak{N}_*(H_m)$ $(2\leqq l\leqq m)$ induced by $k_{l,m}$: $H_l\to H_m$ of* (1.13), *the free $\mathfrak{N}_*$-submodule*

$$\mathfrak{N}_*\{\{k_l\gamma[\beta_2, S^{2n+1}\times S^{2n+1}], j_l[T_2, S^{4n+1}] | n \geqq 0\}\} \subset \mathfrak{N}_*(H_l)$$

is mapped isomorphically onto the free $\mathfrak{N}_*$-submodule

$$\mathfrak{N}_*\{\{k_m\gamma[\beta_2, S^{2n+1}\times S^{2n+1}], j_m[T_2, S^{4n+1}] | n \geqq 0\}\} \subset \mathfrak{N}_*(H_m).$$

(ii) *By the transfer homomorphism $t_{k_{l,m}}$: $\mathfrak{N}_*(H_m)\to\mathfrak{N}_*(H_l)$, the free $\mathfrak{N}_*$-submodule*

$$\mathfrak{N}_*\{\{[\beta_m, S^{2n+1}\times S^{2n+1}], i_m[T_m, S^{4n+1}] | n \geqq 0\}\} \subset \mathfrak{N}_*(H_m)$$

is mapped isomorphically onto the free $\mathfrak{N}_*$-submodule

$$\mathfrak{N}_*\{\{[\beta_l, S^{2n+1}\times S^{2n+1}], i_l[T_l, S^{4n+1}] | n \geqq 0\}\} \subset \mathfrak{N}_*(H_l).$$

PROOF. By Theorem 2.13, (i) is clear and (ii) follows immediately from the facts that $k_{l,m}\circ k_l=k_m$, $k_{l,m}\circ j_l=j_m$ and $t_{k_{l,m}}[\beta_m, S^{2n+1}\times S^{2n+1}]=[\beta_l, S^{2n+1}\times S^{2n+1}]$ and $t_{k_{l,m}}i_m[T_m, S^{4n+1}]=i_l[T_l, S^{4n+1}]$. q.e.d.

For the transfer homomorphism t_{j_m}: $\mathfrak{N}_*(H_m)\to\mathfrak{N}_*(Z_4)$ induced by the inclusion j_m: $Z_4\subset H_m$ of (1.15), we have

LEMMA 2.15. $t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}] = l[a, S^{4n+2}] + \sum_{q=0}^{2n} b_q l[a, S^{2q}]$, *for some $b_q\in\mathfrak{N}_*$, where l: $Z_2\subset Z_4$.*

PROOF. Since $\mu t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]\neq 0$ by Lemma 2.11, we can write

$$t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]$$
$$= l[a, S^{4n+2}]+\sum_{q=0}^{2n} b_q l[a, S^{2q}]+\sum_{q=0}^{2n} y_q[T_2, S^{2q+1}],$$

by [5, Prop. 1.7, Th. 1.22]. Consider the transfer homomorphism t_l: $\mathfrak{N}_*(Z_4)\to\mathfrak{N}_*(Z_2)$. Then, $t_l t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]=0$ and $t_l l[a, S^{2q}]=2[a, S^{2q}]=0$ $(q\geqq 0)$. Therefore, $\sum_{q=0}^{2n} y_q[a, S^{2q+1}]=0$ in $\mathfrak{N}_*(Z_2)$. By [3, Th. 23.2], $y_q=0$ $(q\geqq 0)$ in $\mathfrak{N}_*$. Hence, we have the desired result. q.e.d.

Let NS^1 be the normalizer of S^1 in S^3. For the rest of this section, we study the free $\mathfrak{N}_*$-module structure of $\mathfrak{N}_*(NS^1)$. Consider the fiber bundle

$$RP(2) \longrightarrow BNS^1 \longrightarrow BS^3,$$

where $RP(2)=S^3/NS^1$ is the real projective plane. Consider the homology spectral sequence $\{E^*_{p,q}\}$ for this bundle. Then

$$E^2_{q,q} = H_p(BS^3; \Lambda)\otimes H_q(RP(2); \Lambda) \qquad (\Lambda = Z \text{ or } Z_2),$$

and so this spectral sequence is trivial. Hence, we have immediately

PROPOSITION 2.16. *The homology groups of the classifying space BNS^1 of NS^1 are given by*

$$H_k(BNS^1; Z_2) = \begin{cases} Z_2 & \text{for} \quad k \not\equiv 3 \quad (4), \\ 0 & \text{for} \quad k \equiv 3 \quad (4). \end{cases}$$

Now, we consider the principal S^1-manifolds

$$(S^1, S^{2n+1}), \quad c(z_0,\dots, z_n)=(cz_0,\dots, cz_n) \qquad (c\in S^1),$$

and the principal NS^1-manifolds

$$(\alpha, S^{4n+3}), \quad \alpha(q, (q_0,\dots, q_n)) = (qq_0,\dots, qq_n) \quad (q\in NS^1),$$

$$(2.17) \qquad (\beta, S^{2n+1}\times S^{2n+1}), \begin{cases} \beta(c, (z, z')) = (cz, c^{-1}z') \quad (c\in S^1), \\ \beta(j, (z, z')) = (-z', z), \end{cases}$$

$$i(S^1, S^{2n+1}) \qquad (i: S^1 \subset NS^1).$$

Then we have easily by definition the following

LEMMA 2.18. *For the transfer homomorphism*

$$t: \mathfrak{N}_*(NS^1) \longrightarrow \mathfrak{N}_*(H_m)$$

induced by the inclusion $H_m\subset NS^1$, we have

(i) $t[\alpha, S^{4n+3}] = [\alpha_m, S^{4n+3}]$, $t[\beta, S^{2n+1}\times S^{2n+1}] = [\beta_m, S^{2n+1}\times S^{2n+1}]$,

(ii) $ti[S^1, S^{4n+1}]=i_m[T_m, S^{4n+1}]$.

PROPOSITION 2.19. *$\mathfrak{N}_*(NS^1)$ is a free $\mathfrak{N}_*$-module with basis $\{[\alpha, S^{4n+3}], i[S^1, S^{4n+1}], [\beta, S^{2n+1}\times S^{2n+1}]|n\geqq 0\}$.*

PROOF. We have the desired result from (2.4), Theorem 2.13, Proposition

2.16 and Lemma 2.18. q.e.d.

Also, by Lemma 2.18, Theorem 2.13 and Proposition 2.19, we have

LEMMA 2.20. *The transfer homomorphism t in Lemma* 2.18 *is monomorphic.*

§3. Preliminaries to the oriented bordism module $\Omega_*(G)$

The principal oriented G-bordism module and the oriented bordism ring

$$\Omega_*(G) = \sum_{n=0}^{\infty}\Omega_n(G) \quad\text{and}\quad \Omega_* = \Omega_*(e) = \sum_{n=0}^{\infty}\Omega_n$$

are defined in the same way as $\mathfrak{N}_*(G)$ and $\mathfrak{N}_*$ in §2, provided that manifolds are oriented and G-actions preserve the orientations (cf. [3, §§2, 19]). $\Omega_*(G)$ is a module over Ω_*, and there are homomorphisms

$$(3.1)\qquad r\colon \Omega_*(G) \longrightarrow \mathfrak{N}_*(G), \qquad r\colon \Omega_* \longrightarrow \mathfrak{N}_*,$$

obtained by ignoring the orientations. Also, the augmentation homomorphism

$$(3.2)\qquad \varepsilon_*\colon \Omega_*(G) \longrightarrow \Omega_*, \qquad \varepsilon_*[G, M] = [M/G],$$

defines the direct sum decomposition of Ω_*-modules:

$$\Omega_*(G) = \tilde{\Omega}_*(G)\oplus\Omega_*, \quad \tilde{\Omega}_*(G) = \operatorname{Ker}\varepsilon_*.$$

It is known that

(3.3) (Rohlin's Theorem, cf. [3, Th. 16.2]) *There are exact sequences*

$$\Omega_n(G) \xrightarrow{\times 2} \Omega_n(G) \xrightarrow{r} \mathfrak{N}_n(G), \quad \Omega_n \xrightarrow{\times 2} \Omega_n \xrightarrow{r} \mathfrak{N}_n.$$

Let H be a subgroup of G. For the inclusion $i\colon H\subset G$, the extension homomorphism

$$(3.4)\qquad i\colon \Omega_n(H) \longrightarrow \Omega_n(G)$$

and the transfer homomorphism

$$(3.5)\qquad t_i\colon \Omega_n(G) \longrightarrow \Omega_n(H)$$

are defined in the same way as (2.5) and (2.6) (cf. [3, §20]).

Wall's results on Ω_* can be stated as follows: Let π denote the set of partitions $\omega=(a_1,\dots, a_r)$ with unequal parts a_j, none of which is a power of 2, and set $|\omega|=r$. Let $\omega\cap\omega'$, $\omega\ominus\omega'$ and $\omega_j\in\pi$ for ω, $\omega'\in\pi$ be the intersection, the symmetric difference and the partition obtained from $\omega=(a_1,\dots, a_r)$ by omitting a_j, respectively. Then

THEOREM 3.6. (C. T. C. Wall [9]) *The oriented bordism ring Ω_* is the quotient ring of the integral polynomial ring*

$$Z[h_{4k}, g(\omega)|k \geqq 0, \omega \in \pi]$$

by the ideal generated by the elements

$$2g(\omega), \quad \textstyle\sum_j g(a_j)g(\omega_j) \qquad (|\omega| \geqq 3),$$

$$g(\omega)g(\omega')-\textstyle\sum_j h(\omega_j \cap \omega')g(a_j)g(\omega_j \ominus \omega'),$$

where $h(\omega)=h_{4a_1}\cdots h_{4a_r}$ for $\omega=(a_1,\dots, a_r)$.

We study in [5] the principal oriented Z_{2^k}-bordism module $\Omega_*(Z_{2^k})$ $(k>1)$. Consider the following elements in $\Omega_*(Z_{2^k})$:

$$(3.7) \qquad \begin{aligned} &[T_k, S^{2n+1}] && (\text{in } (2.8)), \\ &lE^{2n+1}W(\omega) && (l\colon Z_2 \subset Z_{2^k}), \end{aligned}$$

the second of which is the extension of $E^{2n+1}W(\omega) \in \tilde{\Omega}_*(Z_2)$ defined by K. Shibata [6, §§3, 4],

$$(3.8) \qquad \begin{aligned} A_{n,k}(\omega) &= \textstyle\sum_j g(a_j)lE^{2n+1}W(\omega_j) \qquad (|\omega| \geqq 2), \\ B_{n,k}(\omega, \omega') &= \textstyle\sum_j h(\omega_j \cap \omega')g(a_j)lE^{2n+1}W(\omega_j \ominus \omega') \\ &\quad -g(\omega)lE^{2n+1}W(\omega'), \end{aligned}$$

for $\omega, \omega' \in \pi$, where $g(\omega)lE^{2n+1}W(\phi)=0$.

THEOREM 3.9. ([5, Th. 2.18]) *The principal oriented Z_{2^k}-bordism module $\tilde{\Omega}_*(Z_{2^k})$ $(k>1)$ is the direct sum*

$$\tilde{\Omega}_*(Z_{2^k}) = \mathfrak{H}_k \oplus \mathfrak{G}_k,$$

where the submodule $\mathfrak{H}_k$ is the quotient module of the free Ω_*-module

$$\Omega_*\{\{[T_k, S^{2n+1}]|n \geqq 0\}\}$$

by the submodule generated by the elements $2^k[T_k, S^{2n+1}]$ $(n \geqq 0)$, and $\mathfrak{G}_k$ is the quotient module of the free Ω_*-module

$$\Omega_*\{\{lE^{2n+1}W(\omega)|n \geqq 0, \omega \in \pi\}\}$$

by the submodule generated by the elements $2lE^{2n+1}W(\omega)$ and $A_{n,k}(\omega)$ $(|\omega| \geqq 2)$,

$B_{n,k}(\omega, \omega')$, $(n \geqq 0, \omega, \omega' \in \pi)$, *of* (3.8).

By [5, Th. 2.22], Rohlin's Theorem (3.3) and the above theorem, we see the following proposition for the homomorphism $r: \Omega_*(Z_{2^k}) \to \mathfrak{N}_*(Z_{2^k})$ of (3.1).

PROPOSITION 3.10. (i) *$rlE^{2n+1}W(\omega) = rg(\omega)(\sum_{j=0}^{n+1} a_{2j} l[a, S^{2n-2j+2}])$, where a_{2j} is defined by $a_0 = 1$ and $\sum_{j=0}^{m} a_{2j}[RP(2m-2j)] = 0$ in $\mathfrak{N}_*$ for any $m \geqq 1$.*

(ii) *The submodule $\mathfrak{G}_k$ of $\tilde{\Omega}_*(Z_{2^k})$ in the above theorem is mapped by r monomorphically into $\mathfrak{N}_*(Z_{2^k})$.*

§4. Some Ω_*-submodules in $\Omega_*(H_m)$

Now, we begin to study the principal oriented bordism module $\Omega_*(H_m)$ of the generalized quaternion group H_m $(m \geqq 2)$.

For our purpose, we use the following theorem which follows immediately from [3, Th. 14.2] and Lemma 1.3.

THEOREM 4.1. *The canonical homomorphism*

$$\theta: \tilde{\Omega}_n(H_m) \longrightarrow \sum_{p+q=n} \tilde{H}_p(H_m; \Omega_q) \qquad (m \geqq 2)$$

of [3, §14] *is isomorphic.*

We see easily that the canonical homomorphisms θ are natural by the proof of [3, pp. 39–41], and so we have the commutative diagram

$$\begin{array}{ccccc} \mathfrak{N}_n(Z_{2^m}) & \xleftarrow{r} & \tilde{\Omega}_n(Z_{2^m}) & \xrightarrow[\cong]{\theta} & H'^1_{n,m} \oplus H'^3_{n,m} \oplus H'^0_{n,m} \oplus H'^2_{n,m} \\ \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_m} & & \downarrow{\scriptstyle i_{m*}} \\ \mathfrak{N}_n(H_m) & \xleftarrow{r} & \tilde{\Omega}_n(H_m) & \xrightarrow[\cong]{\theta} & H^1_{n,m} \oplus H^3_{n,m} \oplus H^0_{n,m} \oplus H^2_{n,m} \\ \uparrow{\scriptstyle j_m} & & \uparrow{\scriptstyle j_m} & & \uparrow{\scriptstyle j_{m*}} \\ \mathfrak{N}_n(Z_4) & \xleftarrow{r} & \tilde{\Omega}_n(Z_4) & \xrightarrow[\cong]{\theta} & H'^1_{n,2} \oplus H'^3_{n,2} \oplus H'^0_{n,2} \oplus H'^2_{n,2}, \end{array} \tag{4.2}$$

where r's are the orientation ignoring homomorphisms of (3.1) and $i_m: Z_{2^m} \subset H_m$, $j_m: Z_4 \subset H_m$ are the inclusions of (1.6) and (1.15), and

$$H'^{\varepsilon}_{n,k} = \sum_q \tilde{H}_{4q+\varepsilon}(Z_{2^k}; \Omega_{n-4q-\varepsilon}),$$

$$H^{\varepsilon}_{n,l} = \sum_q \tilde{H}_{4q+\varepsilon}(H_l; \Omega_{n-4q-\varepsilon}), \quad (k, l \geqq 2, \varepsilon = 0, 1, 2, 3).$$

LEMMA 4.3. *In $\tilde{\Omega}_*(H_m)$ $(m \geqq 2)$, we have the following relations.*

(i) *The elements $[\alpha_m, S^{4n+3}]$, $i_m[T_m, S^{4n+1}]$ and $j_m[T_2, S^{4n+1}]$ of* (2.10) *are of order 2^{m+1}, 2 and 2, respectively.*

(ii) $\qquad x[\alpha_m, S^{4n+3}] = 0 \quad$ *if and only if* $\quad x \in 2^{m+1}\Omega_*$,

$$xi_m[T_m, S^{4n+1}]=0 \quad \textit{if and only if} \quad x\in 2\Omega_*,$$

$$xj_m[T_2, S^{4n+1}]=0 \quad \textit{if and only if} \quad x\in 2\Omega_*,$$

for $x\in\Omega_*$.

PROOF. Consider the natural homomorphisms

$$\mu\colon \Omega_n(G) \longrightarrow H_n(G; Z) \tag{4.4}$$

defined for finite groups G in the same way as μ of (2.3). Then, we see easily that

$$(*)\qquad \begin{aligned} \mu[\alpha_m, S^{4n+3}] &= e_m^{4n+3}, \\ \mu i_m[T_m, S^{4n+1}] &= i_{m*}\mu[T_m, S^{4n+1}] = e_{1,m}^{4n+1}, \\ \mu j_m[T_2, S^{4n+1}] &= j_{m*}\mu[T_2, S^{4n+1}] = e_{2,m}^{4n+1}, \end{aligned}$$

by (1.2), the proof of [5, Lemma 2.13 (i)] and Lemmas 1.7 and 1.16.

(i) We see that the order of $[\alpha_m, S^{4n+3}]$ is 2^{m+1} by the first equality of $(*)$, Lemma 1.3 and Theorem 4.1.

Since $\theta([T_k, S^{4n+1}])\in H'^{1}_{4n+1,k}\oplus H'^{3}_{4n+1,k}$ by [5, Prop. 2.14 (ii)], we have

$$\theta(i_m[T_m, S^{4n+1}]),\ \ \theta(j_m[T_2, S^{4n+1}]) \in H^1_{4n+1,m}$$

by the commutativity of the diagram (4.2) and Lemmas 1.7 and 1.16. These and the last two equalities of $(*)$ and Lemma 1.3 show the rest of (i).

(ii) By (i), it is sufficient to prove the necessity. Since the bordism spectral sequence of BH_m is trivial (cf. [3, Th. 15.2]), there is a commutative diagram

$$\begin{array}{ccc} \Omega_l\otimes\Omega_{2n+1}(H_m) & \xrightarrow{\ \kappa\ } & J_{2n+1,l}\subset\Omega_{2n+l+1}(H_m) \\ \Big\downarrow{\scriptstyle 1\otimes\mu} & & \Big\downarrow \\ \Omega_l\otimes H_{2n+1}(H_m;Z) & \xrightarrow{\ \kappa\ } & H_{2n+1}(H_m;\Omega_l) \end{array}$$

by [3, §7], where κ's are the homomorphisms defined by the multiplication and μ is the one in (4.4), and the lower κ is monomorphic. Therefore, we have the desired results by using $(*)$, Lemma 1.3 and the structure of Ω_l. q.e.d.

LEMMA 4.5. *By the composition* $r\circ i_m$ *in* (4.2), *the submodule in* $\tilde{\Omega}_*(Z_{2^m})$ *generated by the elements* $lE^{4n+3}W(\omega)$ $(n\geqq 0,\ \omega\in\pi)$ *in* (3.7) *is mapped monomorphically into* $\mathfrak{N}_*(H_m)$, *and*

$$ri_m lE^{4n+3}W(\omega) = riE^{4n+3}W(\omega) = rg(\omega)\big(\textstyle\sum_{j=0}^{2n+2}a_{2j}i[a, S^{4(n+1)-2j}]\big),$$

where $i=i_m\circ l\colon Z_2\subset Z_{2^m}\subset H_m$ *and the coefficients* a_{2j} *are the ones in Proposition* 3.10.

PROOF. The equality follows from Proposition 3.10 (i). Then, we have the desired results from Proposition 3.10, [5, Prop. 1.7 (i)], Theorem 2.13 and the facts that $i_m \circ r = r \circ i_m$ and $a_0 = 1$. q. e. d.

Let

$$(4.6) \qquad \Delta: \Omega_n(H_m) \longrightarrow \Omega_{n-4}(H_m), \quad \Delta: \mathfrak{N}_n(H_m) \longrightarrow \mathfrak{N}_{n-4}(H_m)$$

be the Smith homomorphisms defined as follows (cf. [3, §26 and (34.7)]): For a principal (oriented) H_m-manifold (H_m, M^n), we can take a differentiable equivariant map $\varphi: (H_m, M^n) \to (\alpha_m, S^{4N+3})$ which is transverse regular on S^{4N-1}, since S^{4N+3}/α_m is the $(4N+3)$-skeleton of BH_m, where (α_m, S^{4N+3}) is the one in (2.10) and $4N+3>n$. Then,

$$\Delta[H_m, M^n] = [H_m, \varphi^{-1}(S^{4N-1})].$$

It is easy to see that Δ is a homomorphism of Ω_*- ($\mathfrak{N}_*$-) modules, and

LEMMA 4.7. (i) $\qquad \Delta[\alpha_m, S^{4n+3}] = [\alpha_m, S^{4n-1}]$,

(ii) $\qquad \Delta i_m[T_m, S^{4n+1}] = i_m[T_m, S^{4n-3}]$,

(iii) $\qquad \Delta j_m[T_2, S^{4n+1}] = j_m[T_2, S^{4n-3}]$.

PROOF. (i) is clear.

(ii) Consider the composition

$$f: H_m \times S^{4n+1} \xrightarrow{1\times f_1} H_m \times S^{4n+3} \xrightarrow{1\times f_2} H_m \times S^{4n+3} \xrightarrow{\alpha_m} S^{4n+3},$$

where $f_1(z_0, \dots, z_{2n}) = (0, z_0, \dots, z_{2n})$, $f_2(z_0, \dots, z_{2n+1}) = (z_0, z_1^{-1}, \dots, z_{2n}, z_{2n+1}^{-1})$. Then, it is easy to see that $f(q, z) = f(qx^{-1}, xz)$, and so f induces an H_m-equivariant differentiable map

$$f: i_m(T_m, S^{4n+1}) \longrightarrow (\alpha_m, S^{4n+3}),$$

which is transverse regular on S^{4n-1}. Therefore, we have (ii).

(iii) In the same way, we have the desired result by considering the composition

$$\bar{f}: H_m \times S^{4n+1} \xrightarrow{1\times f_1} H_m \times S^{4n+3} \xrightarrow{1\times f_3} H_m \times S^{4n+3} \xrightarrow{\alpha_m} S^{4n+3},$$

where $f_3(x_0, y_0, x_1, y_1, \dots, x_{2n}, y_{2n}, x_{2n+1}, y_{2n+1}) = (x_0, x_1, y_0, y_1, \dots, x_{2n}, x_{2n+1}, y_{2n}, y_{2n+1})$. q. e. d.

Now, we have the following

THEOREM 4.8. (i) *The Ω_*-submodule $\mathfrak{L}_m$ generated by* $\{[\alpha_m, S^{4n+3}] | n \geqq 0\}$

in (2.10) *is the quotient module of the free* Ω_**-module*

$$\Omega_*\{\{[\alpha_m, S^{4n+3}]|n \geqq 0\}\}$$

by the submodule generated by the elements $2^{m+1}[\alpha_m, S^{4n+3}]$ $(n\geqq 0)$.

(ii) *The* Ω_**-submodules* $\mathfrak{I}_m$ *and* $\mathfrak{J}_m$ *generated by* $\{i_m[T_m, S^{4n+1}]|n\geqq 0\}$ *and* $\{j_m[T_2, S^{4n+1}]|n\geqq 0\}$ *in* (2.10), *respectively, are the quotient modules of the free* Ω_**-modules*

$$\Omega_*\{\{i_m[T_m, S^{4n+1}]|n\geqq 0\}\} \quad and \quad \Omega_*\{\{j_m[T_2, S^{4n+1}]|n \geqq 0\}\}$$

by the submodules generated by the twices of the generators.

(iii) *Consider the extension*

$$iE^{4n+3}W(\omega) = i_m lE^{4n+3}W(\omega), \qquad for \quad n \geqq 0, \quad \omega \in \pi,$$

of the class of (3.7) *by* $i_m: Z_{2^m} \subset H_m$. *Then the* Ω_**-submodule* $\mathfrak{W}_m$ *generated by these elements is the quotient module of the free* Ω_**-module*

$$\Omega_*\{\{iE^{4n+3}W(\omega)|n \geqq 0, \omega \in \pi\}\}$$

by the submodule generated by the elements $2iE^{4n+3}W(\omega)$, $i_m A_{2n+1,m}(\omega)$ $(|\omega|\geqq 2)$ *and* $i_m B_{2n+1,m}(\omega, \omega')$, $(n\geqq 0,\ \omega,\ \omega' \in \pi)$, *which are the extensions of* (3.8).

PROOF. (i) Assume that

$$\textstyle\sum_{l=0}^{n} x_l[\alpha_m, S^{4l+3}] = 0 \qquad (x_l \in \Omega_*).$$

Then, the image of the left hand side of this equality by $\Delta^n = \Delta\circ\cdots\circ\Delta$ (n-times) of (4.6) is equal to $x_n[\alpha_m, S^3]$ by Lemma 4.7 (i). And so, we have $x_n \in 2^{m+1}\Omega_*$ by Lemma 4.3 (ii). Therefore, we have (i).

(ii) We have the desired results in the same way as (i) by Lemmas 4.7 and 4.3 (ii).

(iii) By Lemma 4.5 and Theorem 3.9, this result follows immediately. q.e.d.

Denote by ${}^{\#}A$ the order of a group A.

PROPOSITION 4.9. (i) *The submodule* $\mathfrak{L}_m+\mathfrak{I}_m+\mathfrak{J}_m+\mathfrak{W}_m$ *in* $\tilde{\Omega}_*(H_m)$ *is the direct sum*

$$\mathfrak{L}_m \oplus \mathfrak{I}_m \oplus \mathfrak{J}_m \oplus \mathfrak{W}_m.$$

(ii) ${}^{\#}(\mathfrak{L}_m \cap \tilde{\Omega}_n(H_m)) = {}^{\#}H^3_{n,m}$, ${}^{\#}(\mathfrak{I}_m \cap \tilde{\Omega}_n(H_m)) = {}^{\#}(\mathfrak{J}_m \cap \tilde{\Omega}_n(H_m)) = ({}^{\#}H^1_{n,m})^{1/2}$ *and* ${}^{\#}(\mathfrak{W}_m \cap \tilde{\Omega}_n(H_m)) = {}^{\#}H^0_{n,m}$.

(iii) *By the isomorphism* θ *in* (4.2), $2^k\mathfrak{L}_m \cap \tilde{\Omega}_n(H_m)$ *is mapped isomorphically* *onto* $2^k H^3_{n,m}$ $(k \geqq 1)$.

(iv) *For the transfer homomorphism* $t\colon \Omega_n(H_{m+1}) \to \Omega_n(H_m)$ *induced by* $k_{m,m+1}\colon H_m \subset H_{m+1}$ *of* (1.13), *we have the exact sequence*

$$0 \longrightarrow 2^{m+1}\,\mathfrak{L}_{m+1} \longrightarrow \mathfrak{L}_{m+1} \xrightarrow{\ t\ } \mathfrak{L}_m \longrightarrow 0.$$

PROOF. (i) Assume that

$$\Sigma x_n[\alpha_m, S^{4n+3}] + \Sigma y_n i_m[T_m, S^{4n+1}] + \Sigma z_n j_m[T_2, S^{4n+1}] + w = 0,$$

where $w = \Sigma w_n i E^{4n+3} W(\omega) \in \mathfrak{W}_m$. We consider the image of this equality by $r\colon \Omega_n(H_m) \to \mathfrak{N}_n(H_m)$. Then, by Theorem 2.13 and the equality of Lemma 4.5, we see that $rx_n = ry_n = rz_n = 0$ in $\mathfrak{N}_*$ for any n and $rw = 0$. The last equality and Lemma 4.5 show $w = 0$. Also, we have $y_n, z_n \in 2\Omega_*$ by (3.3) and so $y_n i_m[T_m, S^{4n+1}] = z_n j_m[T_2, S^{4n+1}] = 0$ by Lemma 4.3.

(ii) There is a group homomorphism

$$\varphi\colon H^3_{n,m} \oplus H^1_{n,m} \longrightarrow (\mathfrak{L}_m \oplus \mathfrak{J}_m \oplus \mathfrak{J}_m) \cap \tilde{\Omega}_n(H_m)\,,$$

defined by $\varphi(e^{4l+3}_m \otimes x) = x[\alpha_m, S^{4l+3}]$, $\varphi(e^{4l+1}_{1,m} \otimes x) = x i_m[T_m, S^{4l+1}]$ and $\varphi(e^{4l+1}_{2,m} \otimes x) = x j_m[T_2, S^{4l+1}]$ $(x \in \Omega_*)$. By Theorem 4.8 (i) and (ii), it is clear that φ is isomorphic. Therefore, we have the first two equalities.

Now, $\mathfrak{W}_m \cap \tilde{\Omega}_n(H_m) \approx r(\mathfrak{W}_m \cap \tilde{\Omega}_n(H_m))$ by Lemma 4.5. Furthermore, by using Theorem 2.13 and the equality $riE^{4l-1}W(\omega) = rg(\omega)(\sum_{j=0}^{2l} a_{2j} i[a, S^{4l-2j}])$ $(a_0 = 1)$ of Lemma 4.5, we see easily that $r(\mathfrak{W}_m \cap \tilde{\Omega}_n(H_m))$ corresponds bijectively to $\sum_l r(\mathrm{Tor}\,\Omega_{n-4l}) \approx \sum_l (\tilde{H}_{4l}(H_m, \Omega_{n-4l})) = H^0_{n,m}$ by sending $r(\sum_l \sum_\omega x_{l,\omega} i E^{4l-1} W(\omega)) \in r(\mathfrak{W}_m \cap \tilde{\Omega}_n(H_m))$ to $\sum_l \sum_\omega r(x_{l,\omega} g(\omega)) \in \sum_l r(\mathrm{Tor}\,\Omega_{n-4l})$.

(iii) By Theorems 4.8 (i), 4.1 and Lemma 1.3, we have immediately the desired result.

(iv) Since $t[\alpha_{m+1}, S^{4n+3}] = [\alpha_m, S^{4n+3}]$, we have (iv) by Theorem 4.8 (i).

q.e.d.

§5. Some Z_2-manifolds

Let $CP(n)$ be the complex projective n-space, and $P(2m+1, n)$ be the Dold manifold obtained from $S^{2m+1} \times CP(n)$ by the identification

$$(z_0, \dots, z_m, \eta_0, \dots, \eta_n) \equiv (-z_0, \dots, -z_m, \bar{\eta}_0, \dots, \bar{\eta}_n) \qquad (z_i, \eta_i \in \boldsymbol{C})\,.$$

Then, we have the Z_2-manifold $(\tau, P(2m+1, n))$ where τ is the involution given by $\tau[z_0, \dots, z_{m-1}, z_m, \eta_0, \dots, \eta_n] = [z_0, \dots, z_{m-1}, \bar{z}_m, \eta_0, \dots, \eta_n]$. Furthermore, we set

$$Q(2m+1, n) = (S^1 \times P(2m+1, n))/(a \times \tau)\,.$$

By C. T. C. Wall [9], the subalgebra $\mathfrak{M}_*$ of $\mathfrak{N}_*$, consisting of all classes of manifolds whose first Stiefel-Whitney classes w_1 are integral, is given as follows:

(5.1) $$\mathfrak{M}_* = Z_2[X_{2l}, X_{2l-1}, (X_{2^j})^2 | l \neq 2^*],$$

where $X_{2^r(2s+1)-1} = [P(2^r-1, 2^r s)]$, $X_{2^r(2s+1)} = [Q(2^r-1, 2^r s)]$ and $X_{2^j} = [RP(2^j)]$, the real projective 2^j-space. Also, there is a homomorphism

(5.2) $$\partial: \mathfrak{M}_* \longrightarrow \Omega_*$$

obtained by sending the class of M to the class of the submanifold N of M dual to $w_1(M)$, and the homomorphism

(5.3) $$\partial' = r \circ \partial: \mathfrak{M}_* \longrightarrow \mathfrak{M}_*$$

is a derivation such that $\partial' X_{2l} = X_{2l-1}$, $\partial' X_{2l-1} = 0$ and $\partial'(X_{2^j})^2 = 0$.

Put $X(\omega) = X_{2a_1} \cdots X_{2a_r}$ for $\omega = (a_1, \dots, a_r) \in \pi$. Then

(5.4) $$g(\omega) = \partial X(\omega) \qquad \textit{for any} \quad \omega \in \pi,$$

where $g(\omega)$ is the element in Theorem 3.6.

By the consideration of K. Kawakubo [6], we can define the orientable manifold $N(\omega)$, which represents $g(\omega)$ and admits an orientation reversing involution, as follows: Define an orientation reversing involution T on $P(2m+1, 2n)$ by

$$T[z_0, z_1, \dots, z_m, \eta_0, \dots, \eta_{2n}] = [\bar{z}_0, z_1, \dots, z_m, \eta_0, \dots, \eta_{2n}].$$

Set $N_{2^r(2s+1)} = Q(2^{r+1}-1, 2^{r+1}s)$ which admits an involution T induced by $1 \times T$. Furthermore, for any $\omega = (a_1, \dots, a_r) \in \pi$, $a_1 < a_2 < \cdots < a_r$, we can consider the involution $T = T \times 1 \times \cdots \times 1$ on $N_{a_1} \times \cdots \times N_{a_r}$. Also, the map $p: N_{a_1} \times \cdots \times N_{a_r} \to S^1$ is defined by $p([t_1, u_1], \dots, [t_r, u_r]) = t_1^2 \cdots t_r^2$, and it is easy to see that p is transverse regular on $1 \in S^1$ and realizes $w_1(N_{a_1} \times \cdots \times N_{a_r})$. Therefore, the submanifold $N(\omega) = p^{-1}(1)$ of $N_{a_1} \times \cdots \times N_{a_r}$ represents $g(\omega)$ of Theorem 3.6 by (5.4). (This construction is due to Anderson [1] and Stong [8].) Now, we obtain a Z_2-manifold

(5.5) $$(T, N(\omega)), \qquad T = T | N(\omega),$$

where the involution T reverses the orientation on $N(\omega)$. It is easy to see that

(5.6) $$(T, N(\omega)) = (T, P(2^{r+1}-1, 2^{r+1}s))$$

if $|\omega| = 1$ and $\omega = (2^r(2s+1))$.

LEMMA 5.7. *There is an oriented differentiable manifold W with an orientation reversing involution T' such that*

$$(T',\ \dot{W}) \cong 2(T,\ N(\omega)).$$

PROOF. The desired result follows in the same way as the proof that $2[N(\omega)] = 2g(\omega) = 0$ in Ω_* (cf. [9, Lemma 1]). q.e.d.

§6. Some new H_m-manifolds

Let G be a group, and (G, M_0) be a given closed (not necessarily principal) G-manifold. Then for any principal G-manifold (G, M), the product $M \times M_0$ is a principal G-manifold by the action $g(m, n) = (gm, gn)$ for $g \in G$, $m \in M$, $n \in M_0$. This G-manifold $(G, M \times M_0)$ is denoted by

(6.1) $$(G,\ M) \times (G,\ M_0).$$

Then the following lemma is clear.

LEMMA 6.2. *For a given closed G-manifold* (G, M_0),

$$\mathfrak{N}_*(G) \longrightarrow \mathfrak{N}_*(G), \quad [G,\ M] \longrightarrow [(G,\ M) \times (G,\ M_0)],$$

is a homomorphism of $\mathfrak{N}_*$*-modules.*

Now, for the Z_2-manifold $(T, N(\omega))$ of (5.5), we define the H_m-manifold $N(\omega)$ by the action

$$x \cdot u = u, \ y \cdot u = Tu \ \text{ for the generators } x,\ y \in H_m,$$

which is again denoted by $(T, N(\omega))$. By taking the products (6.1) of this H_m-manifold and $(\beta_m, S^{2n+1} \times S^{2n+1})$ in (2.10), we define the principal oriented H_m-manifolds

(6.3)
$$\begin{aligned} \bar{\beta}_m(n,\ \omega) &= (\bar{\beta}_m,\ S^{2n+1} \times S^{2n+1} \times N(\omega)) \\ &= (\beta_m,\ S^{2n+1} \times S^{2n+1}) \times (T,\ N(\omega)), \end{aligned}$$

due to K. Shibata. By using γ of (2.9), we consider also

(6.4) $$k_m \gamma \bar{\beta}_2(n,\ \omega) \qquad (k_m \colon H_2 \subset H_m \text{ of } (1.13)).$$

LEMMA 6.5. *The bordism classes* $[\bar{\beta}_m(n, \omega)]$ *and* $k_m \gamma [\bar{\beta}_2(n, \omega)]$ *of the above manifolds are contained in* $\tilde{\Omega}_*(H_m)$.

PROOF. Since H_m is the normal subgroup of H_{m+1}, there is a principal Z_2-bundle

$$Z_2 \longrightarrow S^{2n+1} \times S^{2n+1} \times N(\omega)/\bar{\beta}_m \longrightarrow S^{2n+1} \times S^{2n+1} \times N(\omega)/\bar{\beta}_{m+1}.$$

So, $[S^{2n+1} \times S^{2n+1} \times N(\omega)/\bar{\beta}_m] = 2[S^{2n+1} \times S^{2n+1} \times N(\omega)/\bar{\beta}_{m+1}]$ in Ω_* by [3, (19.4)]. Furthermore, since $S^{2n+1}\times S^{2n+1}\times N(\omega)/\bar{\beta}_{m+1}$ is odd-dimensional, its class belongs to $\operatorname{Tor}\Omega_*$ by Theorem 3.6. Therefore, we have $\varepsilon_*[\bar{\beta}_m(n,\omega)]=0$ by $2\operatorname{Tor}\Omega_*=0$. Also,

$$\varepsilon_* k_m \gamma[\bar{\beta}_2(n,\omega)] = \varepsilon_*[\bar{\beta}_2(n,\omega)] = 0$$

by (i). q.e.d.

In the commutative diagram

$$(6.6)\qquad \begin{array}{ccccc} \Omega_*(H_m) & \xrightarrow{t_{j_m}} & \Omega_*(Z_4) & \xleftarrow{t_{i_2}} & \Omega_*(H_2) \\ \downarrow r & & \downarrow r & & \downarrow r \\ \mathfrak{N}_*(H_m) & \xrightarrow{t_{j_m}} & \mathfrak{N}_*(Z_4) & \xleftarrow{t_{i_2}} & \mathfrak{N}_*(H_2) \end{array}$$

of the transfer homomorphisms and the orientation ignoring homomorphisms, we have

LEMMA 6.7. (i) $rt_{j_m} i_m[T_m, S^{4n+1}]=0.$

(ii) $rt_{j_m}[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = rt_{i_2}\gamma[\bar{\beta}_2, S^{2n+1}\times S^{2n+1}\times N(\omega)]$

$$= rg(\omega)t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}].$$

(iii) $r[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = rg(\omega)[\beta_m, S^{2n+1}\times S^{2n+1}]$

$$+\textstyle\sum_{l\geq 0} y_l i_m[T_m, S^{4l+1}],\ \textit{for some } y_l\in\mathfrak{N}_*.$$

PROOF. (i) It is clear that $rt_{j_m} i_m[T_m, S^{4n+1}]=l[a, S^{4n+1}]$, which is zero by [5, Prop. 1.7 (ii)].

(ii) The first equality is clear by the definitions (6.3) and (6.4). By applying the homomorphism of Lemma 6.2 for $(G, M_0)=(T, N(\omega))$ to the equality of Lemma 2.15, we have

$$rt_{j_m}[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = \textstyle\sum_{q=0}^{2n+1} b_q[l(a, S^{2q})\times(T, N(\omega))],\ (b_{2n+1}=1).$$

On the other hand, there is a Z_4-equivariant diffeomorphism

$$\varphi\colon l(a, S^{2q})\times(T, N(\omega)) \longrightarrow l(a, S^{2q})\times N(\omega)$$

defined by $\varphi([1, s], u)=([1, s], u)$, $\varphi([y, s], u)=([y, s], Tu)$. Thus the last sum is equal to

$$r[N(\omega)](\textstyle\sum_{q=0}^{2n+1} b_q l[a, S^{2q}]) = rg(\omega)t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]$$

by Lemma 2.15 and the definition of $N(\omega)$.

(iii) First, we consider the principal NS^1-manifolds

(6.8) $(\bar{\beta}, S^{2n+1}\times S^{2n+1}\times N(\omega)) = (\beta, S^{2n+1}\times S^{2n+1})\times(T, N(\omega))$,

where $(T, N(\omega))$ is the NS^1-manifold $N(\omega)$ with the action $c\cdot u=u$, $j\cdot u=Tu$. Then it is clear that $r[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)]=t[\bar{\beta}, S^{2n+1}\times S^{2n+1}\times N(\omega)]$, where $t\colon \mathfrak{N}_*(NS^1)\to\mathfrak{N}_*(H_m)$ is the transfer homomorphism. Hence,

$$r[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = \Sigma x_l[\beta_m, S^{2l+1}\times S^{2l+1}]+\Sigma y_l i_m[T_m, S^{4l+1}] + \Sigma z_l[\alpha_m, S^{4l+3}],$$

by Proposition 2.19 and Lemma 2.18. The image of this equality by t_{j_m} is equal to $rg(\omega)t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]=\Sigma x_l t_{j_m}[\beta_m, S^{2l+1}\times S^{2l+1}]+\Sigma z_l t_{j_m}[\alpha_m, S^{4l+3}]$ by (i) and (ii). Since $\mu t_{j_m}[\alpha_m, S^{4l+3}]=\mu t_{j_2}[\alpha_2, S^{4l+3}]=t_{j_2*}\mu[\alpha_2, S^{4l+3}]\neq 0$ in $H_{4l+3}(Z_4; Z_2)$ by Lemmas 2.12 (i) and 1.10 (iv) and $\mu t_{j_m}[\beta_m, S^{2l+1}\times S^{2l+1}]\neq 0$ by Lemma 2.11, we have $x_n=rg(\omega)$, $x_l=0$ $(l\neq n)$ and $z_l=0$ by (2.4). q.e.d.

LEMMA 6.9. (i) $[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)]\in\tilde{\Omega}_*(H_m)$ *is of order* 2.
(ii) $x[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)]=0$ *if and only if* $x\in 2\Omega_*$, *for* $x\in\Omega_*$.

PROOF. (i) follows from Lemmas 5.7 and 6.7 (iii), Theorem 2.13 and the fact that $rg(\omega)\neq 0$ in $\mathfrak{N}_*$.

(ii) By Lemma 6.7 (ii), $rt_{j_m}x[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = rxrg(\omega)t_{j_m}[\beta_m, S^{2n+1}\times S^{2n+1}]\in\mathfrak{N}_*(Z_4)$. Therefore, if $x[\bar{\beta}_m(n, \omega)]=0$, we have $rxrg(\omega)=0$ by Lemma 2.15 and [5, Prop. 1.7 (i)], which implies $rx=0$ since $\mathfrak{N}_*$ is a polynomial ring over Z_2 and $rg(\omega)\neq 0$. Therefore, we have $x\in 2\Omega_*$ by Rohlin's Theorem (3.3). q.e.d.

§7. Some new Ω_*-submodules of $\Omega_*(H_m)$

We consider the Ω_*-submodules

(7.1) $\mathfrak{Q}_m$ *and* $\mathfrak{Q}'_m$ *of* $\tilde{\Omega}_*(H_m)$ $(m\geqq 2)$

generated by $\{[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)], i_m[T_m, S^{4n+1}]\mid n\geqq 0, \omega\in\pi\}$ and $\{k_m\gamma[\bar{\beta}_2, S^{2n+1}\times S^{2n+1}\times N(\omega)], j_m[T_2, S^{4n+1}]\mid n\geqq 0, \omega\in\pi\}$, respectively, where the manifolds are those in (2.10), (6.3) and (6.4).

LEMMA 7.2. *For the induced isomorphism*

$$\gamma\colon \Omega_n(H_2)\longrightarrow\Omega_n(H_2)$$

of γ *of* (1.11), *we have* $\gamma\mathfrak{Q}_2=\mathfrak{Q}'_2$ *and* $\gamma\mathfrak{Q}'_2=\mathfrak{Q}_2$.

LEMMA 7.3. (i) $2\mathfrak{Q}_m=2\mathfrak{Q}'_m=0$.
(ii) *The orientation ignoring homomorphism* $r\colon\Omega_*(H_m)\to\mathfrak{N}_*(H_m)$ *maps* $\mathfrak{Q}_m$ *and* $\mathfrak{Q}'_m$ *into the free* $\mathfrak{N}_*$-*submodules*

$$\mathfrak{N}_*\{\{[\beta_m, S^{2n+1}\times S^{2n+1}], i_m[T_m, S^{4n+1}]|n\geqq 0\}\}$$

and

$$\mathfrak{N}_*\{\{k_m\gamma[\beta_2, S^{2n+1}\times S^{2n+1}], j_m[T_2, S^{4n+1}]|n\geqq 0\}\},$$

respectively.

(iii) *Furthermore, r is monomorphic on* $\mathfrak{Q}_m$ *and* $\mathfrak{Q}'_m$.

(iv) *By the extension homomorphism* $k_m\colon \Omega_*(H_2)\to\Omega_*(H_m)$, $\mathfrak{Q}'_2$ *is mapped isomorphically onto* $\mathfrak{Q}'_m$.

PROOF. (i) follows from Lemmas 4.3 (i) and 6.9 and the fact that $\mathfrak{Q}'_m=k_m\gamma\mathfrak{Q}_2$.

(ii) follows from Lemma 6.7 (iii) and the naturality.

(iii) Consider the commutative diagram

$$\begin{array}{ccc}\tilde{\Omega}_*(H_{m+1}) & \xrightarrow{r} & \mathfrak{N}_*(H_{m+1})\\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t}\\ \tilde{\Omega}_*(H_m) & \xrightarrow{r} & \mathfrak{N}_*(H_m)\end{array},$$

where t's are the transfer homomorphisms of $k_{m,m+1}\colon H_m\subset H_{m+1}$. Assume

$$rq=0 \qquad \text{for} \quad q\in\mathfrak{Q}_m.$$

Since $ti_{m+1}[T_{m+1}, S^{4n+1}]=i_m[T_m, S^{4n+1}]$ and $t[\bar{\beta}_{m+1}, S^{2n+1}\times S^{2n+1}\times N(\omega)]=[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)]$, there is an element $q'\in\mathfrak{Q}_{m+1}$ such that $tq'=q$. Then, we see that

$$rq'=0$$

by (ii) and Lemma 2.14. Hence, $q'\in 2\mathfrak{L}_{m+1}$ by (3.3), Lemma 1.3, (4.2) and Proposition 4.9 (iii). Therefore $q'\in 2^{m+1}\mathfrak{L}_{m+1}$ from (i) and Theorem 4.8 (i), and so $q=tq'=0$ by Proposition 4.9 (iv). Thus we have the result for $\mathfrak{Q}_m$.

We have the result for $\mathfrak{Q}'_2$ from the result for $\mathfrak{Q}_2$ by using the isomorphism γ, Lemma 7.2 and $r\circ\gamma=\gamma\circ r$. Finally, consider the commutative diagram

$$\begin{array}{ccc}\tilde{\Omega}_*(H_2) & \xrightarrow{k_m} & \tilde{\Omega}_*(H_m)\\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r}\\ \mathfrak{N}_*(H_2) & \xrightarrow{k_m} & \mathfrak{N}_*(H_m)\end{array}.$$

Then it is clear that $\mathfrak{Q}'_m=k_m\mathfrak{Q}'_2$ by definition. Since $r|\mathfrak{Q}'_2$ is monomorphic by the above proof and so is $k_m|r(\mathfrak{Q}'_2)$ by (ii) and Lemma 2.14 (i), we see that $r\circ k_m=k_m\circ r$ is monomorphic on $\mathfrak{Q}'_2$. Therefore, so is r on $\mathfrak{Q}'_m$.

(iv) The result is shown in the above. q.e.d.

LEMMA 7.4. *The Ω_*-submodule $\mathfrak{L}_m+\mathfrak{Q}_m+\mathfrak{Q}'_m+\mathfrak{W}_m$ of $\tilde{\Omega}_*(H_m)$ is the direct sum*

$$\mathfrak{L}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \oplus \mathfrak{W}_m,$$

where $\mathfrak{L}_m$ and $\mathfrak{W}_m$ are the ones in Theorem 4.8.

PROOF. Assume that

$$l+q+q'+w = 0 \qquad (l \in \mathfrak{L}_m,\ q \in \mathfrak{Q}_m,\ q' \in \mathfrak{Q}'_m,\ w \in \mathfrak{W}_m).$$

Then by Lemmas 4.5, 7.3 (ii) and Theorem 2.13, we have

$$rl = rq = rq' = rw = 0 \qquad \text{in} \quad \mathfrak{N}_*(H_m).$$

Therefore, we get

$$q = 0 = q' \qquad \text{by Lemma 7.3 (iii)},$$

$$w = 0 \qquad \text{by Lemma 4.5.}$$

q.e.d.

Now, we have the following

THEOREM 7.5. *The principal oriented H_m-bordism module $\tilde{\Omega}_*(H_m)$ $(m \geqq 2)$ is the direct sum*

$$\mathfrak{L}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \oplus \mathfrak{W}_m$$

of the Ω_*-submodules $\mathfrak{L}_m$, $\mathfrak{W}_m$ in Theorem 4.8 (i), (iii) *and $\mathfrak{Q}_m$, $\mathfrak{Q}'_m$ in* (7.1).

PROOF. By Lemma 7.4, it is sufficient to prove that

$$^{\#}((\mathfrak{L}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \oplus \mathfrak{W}_m) \cap \tilde{\Omega}_n(H_m)) = {}^{\#}\tilde{\Omega}_n(H_m).$$

Consider the homomorphisms

$$\tilde{\Omega}_n(H_m) \xrightarrow{r} \mathfrak{N}_n(H_m) \xrightarrow{p_1 \oplus p_2} P_1 \oplus P_2, \tag{7.6}$$

where $P_1 = \mathfrak{N}_*\{\{[\beta_m, S^{2l+1} \times S^{2l+1}] | l \geqq 0\}\} \cap \mathfrak{N}_n(H_m)$, $P_2 = \mathfrak{N}_*\{\{k_m\gamma[\beta_2, S^{2l+1} \times S^{2l+1}] | l \geqq 0\}\} \cap \mathfrak{N}_n(H_m)$ and p_i are the projections. Then,

$$\begin{aligned} p_1 r[\bar{\beta}_m(l, \omega)] &= rg(\omega)[\beta_m, S^{2l+1} \times S^{2l+1}], \\ p_2 r[\bar{\beta}_m(l, \omega)] &= 0 = p_1 r k_m \gamma[\bar{\beta}_2(l, \omega)], \\ p_2 r k_m \gamma[\bar{\beta}_2(l, \omega)] &= rg(\omega) k_m \gamma[\beta_2, S^{2l+1} \times S^{2n+1}], \end{aligned} \tag{7.7}$$

by Lemma 6.7 (iii). Also, $(p_1 \oplus p_2) r((\mathfrak{L}_m \oplus \mathfrak{I}_m \oplus \mathfrak{J}_m \oplus \mathfrak{W}_m) \cap \tilde{\Omega}_n(H_m)) = 0$ by Theorem 2.13 and Lemma 4.5, where $\mathfrak{I}_m$ and $\mathfrak{J}_m$ are the ones in Theorem 4.8, and so $(p_1 \oplus p_2)r((\mathfrak{L}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m \oplus \mathfrak{W}_m) \cap \tilde{\Omega}_n(H_m)) = p_1 r(\mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)) \oplus p_2 r(\mathfrak{Q}'_m \cap \tilde{\Omega}_n(H_m))$. Hence, using (7.7), we have

$$(7.8) \qquad {}^{\#}p_1 r(\mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)) = {}^{\#}p_2 r(\mathfrak{Q}'_m \cap \tilde{\Omega}_n(H_m)) = ({}^{\#}H^2_{n,m})^{1/2}$$

in a way similar to the proof of Proposition 4.9 (ii). Therefore,

$$\begin{aligned} {}^{\#}\Omega_n(H_m) &\geqq {}^{\#}((\mathfrak{L}_m \oplus \mathfrak{J}_m \oplus \mathfrak{J}_m \oplus \mathfrak{W}_m) \cap \tilde{\Omega}_n(H_m)) \cdot {}^{\#}((p_1 r\mathfrak{Q}_m \oplus p_2 r\mathfrak{Q}'_m) \cap \tilde{\Omega}_n(H_m)) \\ &= {}^{\#}(H^3_{n,m} \oplus H^1_{n,m} \oplus H^0_{n,m}) \cdot {}^{\#}H^2_{n,m} \end{aligned}$$

by Proposition 4.9 (ii). Hence, we have the desired result by Theorem 4.1. q.e.d.

LEMMA 7.9. (i) ${}^{\#}(\mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)) = {}^{\#}(\mathfrak{Q}'_m \cap \tilde{\Omega}_n(H_m)) = ({}^{\#}(H^1_{n,m} \oplus H^2_{n,m}))^{1/2}$.

(ii) *For the homomorphism* $p_1 \circ r \colon \tilde{\Omega}_n(H_m) \to P_1$ *in* (7.6), *if* $p_1 r(q) = 0$ *for* $q \in \mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)$, *then* $q \in \mathfrak{J}_m \cap \tilde{\Omega}_n(H_m)$.

PROOF. (i) In the same way as the proof of Theorem 7.5, we see ${}^{\#}(\mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)) \geqq ({}^{\#}H^1_{n,m})^{1/2} \cdot ({}^{\#}H^2_{n,m})^{1/2}$, ${}^{\#}(Q'_m \cap \tilde{\Omega}_n(H_m)) \geqq ({}^{\#}H^1_{n,m})^{1/2} \cdot ({}^{\#}H^2_{n,m})^{1/2}$ by using Proposition 4.9 (ii) and (7.8). Hence, we have the desired results from Theorems 4.1, 7.5 and Proposition 4.9 (ii).

(ii) We see that $\mathrm{Ker}(p_1 \circ r | \mathfrak{Q}_m \cap \tilde{\Omega}_n(H_m)) = \mathfrak{J}_m \cap \tilde{\Omega}_n(H_m)$ from (i), (7.8) and Proposition 4.9 (ii). q.e.d.

§8. The main theorem

Now, we shall determine completely the Ω_*-submodules $\mathfrak{Q}_m$ and $\mathfrak{Q}'_m$ of (7.1).

For the principal NS^1-manifold $(\bar{\beta}, S^1 \times S^1 \times N(\omega))$ given by (6.8), the bordism class in $\mathfrak{N}_*$ of the orbit manifold

$$S^1 \times S^1 \times N(\omega)/\bar{\beta} = S^1 \times N(\omega)/(a \times T)$$

is contained in $\mathfrak{M}_*$ of (5.1) since the orientation bundle $S^1 \times N(\omega) \to S^1 \times N(\omega)/(a \times T)$ is classified by the map $S^1 \times N(\omega)/(a \times T) \to S^1/a$, $[t, z] \to [t]$. Also for the derivation ∂' of (5.3), we see that

$$\partial'[S^1 \times N(\omega)/(a \times T)] = rg(\omega).$$

Since $\mathrm{Ker}\,\partial' = \mathrm{Im}\,r$ by [9] and $\partial' X(\omega) = rg(\omega)$ by (5.3) and (5.4), we have

$$[S^1 \times N(\omega)/(a \times T)] - X(\omega) \quad \in \mathrm{Im}\,r.$$

Hence, we can take $Y(\omega) \in \Omega_* \pmod{2\Omega_*}$ such that

$$(8.1) \qquad rY(\omega) = [S^1 \times N(\omega)/(a \times T)] - X(\omega) \qquad in \quad \mathfrak{N}_*.$$

LEMMA 8.2. *For the elements of* (6.3), *we have*

$$\Delta[\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)] = \begin{cases} [\bar{\beta}_m, S^{2n-1}\times S^{2n-1}\times N(\omega)] & (n>0), \\ 0 & (n=0), \end{cases}$$

where $\Delta\colon \Omega_*(H_m)\to\Omega_{*-4}(H_m)$ *and* $\Delta\colon \mathfrak{N}_*(H_m)\to\mathfrak{N}_{*-4}(H_m)$ *are the Smith homomorphisms of* (4.6).

PROOF. There is an H_m-equivariant differentiable map

$$\varphi\colon (\bar{\beta}_m, S^{2n+1}\times S^{2n+1}\times N(\omega)) \longrightarrow (\alpha_m, S^{4n+3})$$

defined by $\varphi(z_0,\dots, z_n, z'_0,\dots, z'_n, u) = (z_0/\sqrt{2},\ z'_0/\sqrt{2},\dots, z_n/\sqrt{2},\ z'_n/\sqrt{2})$ which is transverse regular on S^{4n-1}. Hence, we have the desired result. q.e.d.

LEMMA 8.3. $r[\bar{\beta}_m, S^1\times S^1\times N(\omega)] = rg(\omega)[\beta_m, S^1\times S^1]$

$$+(X(\omega)+rY(\omega))i_m[T_m, S^1].$$

PROOF. By Lemma 6.7 (iii), we can write

$$r[\bar{\beta}_m, S^1\times S^1\times N(\omega)] = rg(\omega)[\beta_m, S^1\times S^1]+\textstyle\sum_l y_l i_m[T_m, S^{4l+1}].$$

Since $\Delta r[\bar{\beta}_m, S^1\times S^1\times N(\omega)]=0=\Delta[\beta_m, S^1\times S^1]$ by Lemma 8.2, we have $y_l=0$ $(l\neq 0)$ by Lemma 4.7 (ii) and Theorem 2.13, and so

$$r[\bar{\beta}_m, S^1\times S^1\times N(\omega)] = rg(\omega)[\beta_m, S^1\times S^1]+y_0 i_m[T_m, S^1].$$

Consider the transfer homomorphism $t\colon \mathfrak{N}_*(NS^1)\to\mathfrak{N}_*(H_m)$ induced by the inclusion $H_m\subset NS^1$. Then,

$$r[\bar{\beta}_m, S^1\times S^1\times N(\omega)] = t[\bar{\beta}, S^1\times S^1\times N(\omega)]$$

by the definition of $\bar{\beta}$ of (6.8). Therefore, we have

$$[\bar{\beta}, S^1\times S^1\times N(\omega)] = rg(\omega)[\beta, S^1\times S^1]+y_0 i[S^1, S^1],$$

by Lemmas 2.18 and 2.20. By applying the augmentation homomorphism $\varepsilon_*\colon \mathfrak{N}_*(NS^1)\to\mathfrak{N}_*$ to this equality, we see

$$[S^1\times S^1\times N(\omega)/\bar{\beta}] = rg(\omega)[S^1]+y_0 = y_0.$$

Hence, we have $y_0=X(\omega)+rY(\omega)$ by (8.1). q.e.d.

Put

$$N(n,\omega) = \varepsilon_*[\bar{\beta}, S^{2n+1}\times S^{2n+1}\times N(\omega)] \in \mathfrak{N}_*, \tag{8.4}$$

where ε_* is the augmentation homomorphism of (3.2).

LEMMA 8.5. *For each* $\omega=(a_1,\dots,a_{|\omega|})\in\pi$, $|\omega|\geqq 2$, *there are elements* $K_{2n+1,m}(\omega)\in\Omega_*$ *such that*

$$K_{1,m}(\omega) = \sum_j g(a_j)Y(\omega_j)+g(\omega),$$

$$(8.6)\qquad rK_{2n+1,m}(\omega) = \sum_j rg(\omega_j)N(n,\omega_j) + \sum_{l=0}^{n-1}[CP(2n-2l)]rK_{2l+1,m}(\omega) \quad (n\geqq 1),$$

and that the following elements $C_{n,m}(\omega)$ *vanish in* $\tilde{\Omega}_*(H_m)$:

$$(8.7)\quad C_{n,m}(\omega) = \sum_j g(a_j)[\bar{\beta}_m(n,\omega_j)]+\sum_{l=0}^{n} K_{2l+1,m}(\omega)i_m[T_m, S^{4n-4l+1}].$$

PROOF. We notice that $r\colon \tilde{\Omega}_n(H_m)\to\mathfrak{N}_n(H_m)$ is monomorphic on $\mathfrak{Q}_m$ by Lemma 7.3 (iii). Define $K_{1,m}(\omega)$ by (8.6). Then we see $C_{0,m}(\omega)=0$ as follows:

$$\begin{aligned} rC_{0,m}(\omega) &= r(\textstyle\sum_j g(a_j)[\bar{\beta}_m(0,\omega_j)]+(\sum_j g(a_j)Y(\omega_j)+g(\omega))i_m[T_m, S^1]) \\ &= r(\textstyle\sum_j g(a_j)g(\omega_j))[\beta_m, S^1\times S^1]+(\sum_j rg(a_j)X(\omega_j) \\ &\quad + rg(\omega))i_m[T_m, S^1] \qquad \text{by Lemma 8.3} \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{by Theorem 3.6, (5.3) and (5.4).} \end{aligned}$$

Now we assume that there exist $K_{2l+1,m}(\omega)$ for $l<n$ in this lemma, and put

$$A = \sum_j g(a_j)[\bar{\beta}_m(n,\omega_j)]+\sum_{l=0}^{n-1}K_{2l+1,m}(\omega)i_m[T_m, S^{4n-4l+1}].$$

Then $\Delta A=0$ by Lemmas 4.7, 8.2 and the assumption $C_{n-1,m}(\omega)=0$, where Δ is the Smith homomorphism of (4.6). Also, by the first equality of (7.7), Theorems 3.6 and 2.13, we see that $p_1rA=0$ and so $A\in\mathfrak{I}_m$ by Lemma 7.9 (ii). Therefore, $A\in\mathfrak{I}_m\cap\operatorname{Ker}\Delta$ and we see that

$$A+x_0i_m[T_m, S^1] = 0 \qquad \text{for some } x_0\in\Omega_*$$

by Lemma 4.7 (ii) and Theorem 4.8 (ii). Take $K_{2n+1,m}(\omega)=x_0$. Then this equality shows $C_{n,m}(\omega)=0$.

Consider the transfer homomorphism $t\colon\mathfrak{N}_*(NS^1)\to\mathfrak{N}_*(H_m)$. The element

$$A' = \sum_j rg(a_j)[\bar{\beta}, S^{2n+1}\times S^{2n+1}\times N(\omega)]+\sum_{l=0}^{n-1}rK_{2l+1,m}(\omega)i[S^1, S^{4n-4l+1}]$$

of $\mathfrak{N}_*(NS^1)$ satisfies $tA'=A$ by Lemma 2.18 and the definition of (6.8). Therefore, we see that

$$A'+rx_0i[S^1, S^1] = 0 \qquad \text{in} \quad \mathfrak{N}_*(NS^1)$$

because $C_{n,m}(\omega)=0$ and t is monomorphic (Lemma 2.20). Thus $rK_{2n+1,m}(\omega)$

$=rx_0=\varepsilon_* A'$ which is the equality (8.6), and the proof is complete by the induction on n. q.e.d.

LEMMA 8.8. *For each $\omega, \omega' \in \pi$, there are the elements $P_{2n+1,m}(\omega, \omega') \in \Omega_*$ such that*

$$P_{1,m}(\omega, \omega') = g(\omega)Y(\omega') - \sum_j h(\omega_j \cap \omega')g(a_j)Y(\omega_j \ominus \omega'),$$

$$\text{(8.9)} \quad rP_{2n+1,m}(\omega, \omega') = rg(\omega)N(n, \omega') - \sum_j rh(\omega_j \cap \omega')rg(a_j)N(n, \omega_j \ominus \omega') + \sum_{l=0}^{n-1}[CP(2n-2l)]rP_{2l+1,m}(\omega, \omega') \qquad (n \geqq 1),$$

and that the following elements $D_{n,m}(\omega, \omega')$ vanish in $\tilde{\Omega}_*(H_m)$:

$$\text{(8.10)} \quad D_{n,m}(\omega, \omega') = g(\omega)[\bar{\beta}_m(n, \omega')] - \sum_j h(\omega_j \cap \omega')g(a_j)[\bar{\beta}_m(n, \omega_j \ominus \omega')] + \sum_{l=0}^{n} P_{2l+1,m}(\omega, \omega')i_m[T_m, S^{4n-4l+1}].$$

PROOF. Define $P_{1,m}(\omega, \omega')$ by (8.9). Then, we see $D_{0,m}(\omega, \omega')=0$ as follows:

$$\begin{aligned}
rD_{0,m}(\omega, \omega') &= r(g(\omega)[\bar{\beta}_m(0, \omega')] - \sum_j h(\omega_j \cap \omega')g(a_j)[\bar{\beta}_m(0, \omega_j \ominus \omega')] \\
&\quad + (g(\omega)Y(\omega') - \sum_j h(\omega_j \cap \omega')g(a_j)Y(\omega_j \ominus \omega'))i_m[T_m, S^1]) \\
&= r(g(\omega)g(\omega') - \sum_j h(\omega_j \cap \omega')g(a_j)g(\omega_j \ominus \omega'))[\beta_m, S^1 \times S^1] \\
&\quad + (rg(\omega)X(\omega') - \sum_j rh(\omega_j \cap \omega')rg(a_j)X(\omega_j \ominus \omega'))i_m[T_m, S^1] \\
&= 0
\end{aligned}$$

by Lemma 8.3, Theorem 3.6, (5.3), (5.4) and the fact that $rh(\omega)=X(\omega)^2$ ([9, Lemma 14]).

The rest of the lemma can be proved in the same way as the proof of the above lemma. q.e.d.

By the definitions of (8.7) and (8.10), we have easily the following equality.

LEMMA 8.11.

$$\begin{aligned}
&D_{n,m}(\omega, \omega') + D_{n,m}(\omega', \omega) + h(\omega \cap \omega')C_{n,m}(\omega \ominus \omega') \\
&= g(\omega)[\bar{\beta}_m(n, \omega')] + g(\omega')[\bar{\beta}_m(n, \omega)] + \sum_{l=0}^{n}(h(\omega \cap \omega')K_{2l+1,m}(\omega \ominus \omega') \\
&\quad + P_{2l+1,m}(\omega, \omega') + P_{2l+1,m}(\omega', \omega))i_m[T_m, S^{4n-4l+1}].
\end{aligned}$$

Now, we are ready to prove our main theorem.

THEOREM 8.12. *The principal oriented H_m-bordism module $\tilde{\Omega}_*(H_m)$ $(m \geqq 2)$*

is the direct sum

$$\tilde{\Omega}_*(H_m) = \mathfrak{L}_m \oplus \mathfrak{W}_m \oplus \mathfrak{Q}_m \oplus \mathfrak{Q}'_m,$$

where the Ω_**-submodules* $\mathfrak{L}_m$ *and* $\mathfrak{W}_m$ *are given by Theorem* 4.8 (i) *and* (iii), *and* $\mathfrak{Q}_m$ *and* $\mathfrak{Q}'_m$ *are given as follows:*

The Ω_**-submodule* $\mathfrak{Q}_m$ *of* (7.1) *is the quotient module of the free* Ω_**-module*

$$\Omega_*\{\{[\bar{\beta}_m(n, \omega)],\ i_m[T_m, S^{4n+1}] \mid n \geqq 0,\ \omega \in \pi\}\}$$

by the Ω_**-submodule generated by the elements*

$$2[\bar{\beta}_m(n, \omega)], \quad 2i_m[T_m, S^{4n+1}], \quad C_{n,m}(\omega) \quad (|\omega| \geqq 2), \quad D_{n,m}(\omega, \omega'),$$

$(n \geqq 0,\ \omega,\ \omega' \in \pi)$, *where C and D are the ones in* (8.7) *and* (8.10).

The Ω_**-submodule* $\mathfrak{Q}'_m$ *of* (7.1) *is isomorphic to* $\mathfrak{Q}_2$ *by the composition*

$$\tilde{\Omega}_*(H_2) \xrightarrow[\approx]{\gamma} \tilde{\Omega}_*(H_2) \xrightarrow{k_m} \tilde{\Omega}_*(H_m)$$

of the isomorphism γ *induced by the automorphism* $\gamma\colon H_2 \to H_2$ *of* (1.11) *and the extension homomorphism* k_m *induced by the inclusion* $k_m\colon H_2 \subset H_m$ *of* (1.13).

Proof. The first result is Theorem 7.5.

We shall determine the submodule $\mathfrak{Q}_m$. Consider the element

$$X = \sum_n X_n + Y, \quad X_n = \sum_\omega x_{n,\omega}[\bar{\beta}_m(n, \omega)], \qquad Y \in \mathfrak{J}_m,$$

of $\mathfrak{Q}_m$ such that $X=0$ in $\tilde{\Omega}_*(H_m)$. Applying the composition $p_1 \circ r\colon \tilde{\Omega}_*(H_m) \xrightarrow{r} \mathfrak{N}_*(H_m) \xrightarrow{p_1} P_1$ of (7.6), we have

$$\sum_n (\sum_\omega r(x_{n,\omega} g(\omega)) [\beta_m, S^{2n+1} \times S^{2n+1}]) = 0$$

by (7.7). Since $\{[\beta_m, S^{2n+1} \times S^{2n+1}] \mid n \geqq 0\}$ is a free $\mathfrak{N}_*$-base of $\mathfrak{N}_*(H_m)$ by Theorem 2.13, we have $r(\sum_\omega x_{n,\omega} g(\omega)) = 0$ in $\mathfrak{N}_*$ for each $n \geqq 0$. On the other hand, $\sum_\omega x_{n,\omega} g(\omega) \in \mathrm{Tor}\, \Omega_*$ by Theorem 3.6. Therefore we see that $\sum_\omega x_{n,\omega} g(\omega) = 0$ in Ω_* by (3.3) and the fact $2\,\mathrm{Tor}\, \Omega_* = 0$.

Thus, according to Theorem 3.6, we can write

$$\begin{aligned} \sum_\omega x_{n,\omega} g(\omega) = {} & \sum_\omega 2A_{n,\omega} g(\omega) + \sum_\omega B_{n,\omega}(\sum_j g(a_j) g(\omega_j)) \\ & + \sum_{\omega,\omega'} C_{n,\omega,\omega'}(g(\omega) g(\omega') - \sum_j h(\omega_j \cap \omega') g(a_j) g(\omega_j \ominus \omega')) \end{aligned}$$

for some $A_{n,\omega}$, $B_{n,\omega}$, $C_{n,\omega,\omega'} \in \Omega_*$, and we consider the linear combination

$$X'_n = \sum_\omega 2A_{n,\omega}[\bar{\beta}_m(n, \omega)] + \sum_\omega B_{n,\omega} C_{n,m}(\omega) + \sum_{\omega,\omega'} C_{n,\omega,\omega'} D_{n,m}(\omega, \omega')$$

of $R_n = \{2[\bar{\beta}_m(n, \omega)],\ C_{n,m}(\omega),\ D_{n,m}(\omega, \omega')\}$. Then we see that

$$p_1 r X'_n = \sum_\omega r x_{n,\omega} r g(\omega)[\beta_m, S^{2n+1}\times S^{2n+1}] = p_1 r X_n$$

by (7.7). Furthermore, the only two elements $g(\omega)[\bar{\beta}_m(n, \omega')]$ and $g(\omega')[\bar{\beta}_m(n, \omega)]$ are mapped by $p_1 \circ r$ to $rg(\omega)rg(\omega')[\beta_m, S^{2n+1}\times S^{2n+1}]$ by the proof of Theorem 7.5, and $g(\omega)[\bar{\beta}_m(n, \omega')] - g(\omega')[\bar{\beta}_m(n, \omega)]$ is the sum of a linear combination of R_n and an element of $\mathfrak{I}_m$ by Lemma 8.11. Therefore, we see that

$$X_n - X'_n = X''_n + Y_n,$$

where X''_n is a linear combination of R_n and $Y_n \in \mathfrak{I}_m$. These show that

$$X = \sum_n X_n + Y = \sum_n (X'_n + X''_n) + (\sum_n Y_n + Y).$$

Since the elements of R_n are zero in $\tilde{\Omega}_*(H_m)$ by Lemmas 6.9, 8.5, and 8.8, the assumption $X=0$ in $\tilde{\Omega}_*(H_m)$ implies that $\sum_n Y_n + Y = 0$ in $\mathfrak{I}_m$, and hence that $\sum_n Y_n + Y$ is a linear combination of $\{2i_m[T_m, S^{4l+1}]\}$ by Theorem 4.8 (ii). Therefore, X is a linear combination of $\cup_n R_n$ and $\{2i_m[T_m, S^{4l+1}]\}$, as desired.

The result for the submodule $\mathfrak{Q}'_m$ is Lemmas 7.2 and 7.3 (iv). q.e.d.

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