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Stability of Difference Schemes for Nonsymmetric Linear Hyperbolic Systems

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1. Introduction

Let us consider the Cauchy problem for a hyperbolic system

$$(1.1)\quad \frac{\partial u}{\partial t}(x, t) = \textstyle\sum_{j=1}^{n} A_j(x, t)\frac{\partial u}{\partial x_j}(x, t) \qquad (0 \leqq t \leqq T,\ -\infty < x_j < \infty),$$

$$(1.2)\qquad u(x, 0) = u_0(x), \quad u_0(x) \in L_2,$$

where $u(x, t)$ and $u_0(x)$ are N-vectors and $A_j(x, t)$ $(j=1, 2,\dots, n)$ are $N \times N$ matrices, and assume that this problem is well posed. For the numerical solution of this problem we consider the following difference scheme:

$$(1.3)\qquad v(x, t+k) = S_h(t, h)v(x, t) \qquad (0 \leqq t \leqq T,\ -\infty < x_j < \infty),$$

$$(1.4)\qquad v(x, 0) = u_0(x), \quad k = \lambda h \qquad (\lambda > 0),$$

where $S_h(t, \mu)$ is a sum of products of operators of the form $\sum_\alpha c_\alpha(x, t, \mu)T_h^\alpha$ $(\mu \geqq 0)$, α is a multi-index, $c_\alpha(x, t, \mu)$ is an $N \times N$ matrix, T_h is the translation operator and h is a space mesh width.

In our previous paper [5] we treated the case where $A_j(x, t)$ $(j=1, 2,\dots, n)$ are independent of t, and obtained sufficient conditions for L_2-stability of the scheme (1.3). In this paper we extend the results to the system (1.1) that satisfies the following conditions: Eigenvalues of $A(x, t, \xi) = \sum_{j=1}^{n} A_j(x, t)\xi_j/|\xi|$ $(\xi \neq 0)$ are all real and their multiplicities are independent of x, t and ξ; elementary divisors of $A(x, t, \xi)$ are all linear; there exists a positive constant δ such that

$$|\lambda_i(x, t, \xi) - \lambda_j(x, t, \xi)| \geqq \delta \qquad (i \neq j;\ i, j = 1, 2,\dots, s),$$

where $\lambda_i(x, t, \xi)$ $(i=1, 2,\dots, s)$ are all the distinct eigenvalues of $A(x, t, \xi)$.

Our proof of stability is based on the following result: The scheme (1.3) is stable if $S_h(t, h)$ and $S_h(t, 0)$ are the families of bounded linear operators in L_2 and if there exist positive constants c_j $(j=0, 1, 2)$ and a norm $|||\cdot|||_t$ which depends on t and is equivalent to the L_2-norm such that

$$(1.5)\qquad |||u|||_{t+k} \leqq (1 + c_0 k)\,|||u|||_t \qquad (t + k \leqq T),$$

(1.6) $$|||S_h(t, 0)u|||_t \leqq (1 + c_1 h)\, |||u|||_t,$$

(1.7) $$\|(S_h(t, h) - S_h(t, 0))u\| \leqq c_2 h \|u\| \quad \text{for all} \quad u \in L_2,\ t \in [0, T],\ h > 0.$$

The lemmas and theorems stated without proofs can be shown by the arguments similar to those of the corresponding ones in [5].

2. Notations and preliminaries

2.1. Notations

Let $\boldsymbol{C}$ be the field of complex numbers and let a^* stand for the conjugate transpose of a matrix a. We denote by $|a|$, $|z|$ and $\rho(a)$ the spectral norm of an $N \times N$ matrix a, the Euclidean norm of an N-vector z and the spectral radius of a respectively. For any hermitian matrices a and b we use the notation $a \geqq b$ if $a-b$ is positive semidefinite.

We denote by R^n the real n-space and write it as R^n_x, R^n_ω, R^n_χ, etc. to specify its space variables. Unless otherwise stated, we denote by $u(x)$, $\varphi(x)$, etc. the N-vector functions defined on R^n. We put $J=[0, T]$ and $I_\infty=[0, \infty)$.

The space L_p $(p \geqq 1)$ consists of all measurable functions $u(x)$ in R^n such that $|u(x)|^p$ is integrable, i.e. $\int |u(x)|^p dx < \infty$. The scalar product and the norm in L_2 are denoted by $(\ ,\ )$ and $\|\cdot\|$ respectively.

We denote by $\hat{p}(\chi, t, \omega)$ $(\chi \in R^n)$ the Fourier transform of $p(x, t, \omega)$ with respect to x.

Let $\mathscr{S}$ be the space of all C^∞ functions on R^n_x which, together with all their derivatives, decrease faster than any negative power of $|x|$ as $|x| \to \infty$. Then, for each $\varphi(x)$ in $\mathscr{S}$, $\hat{\varphi}(\chi)$ can be written as follows:

(2.1) $$\hat{\varphi}(\chi) = \kappa \int e^{-ix\cdot\chi} \varphi(x) dx \qquad \text{for all} \quad \varphi \in \mathscr{S},$$

where

(2.2) $$\kappa = (2\pi)^{-n/2}, \quad x \cdot \chi = \textstyle\sum_{j=1}^n x_j \chi_j.$$

For simplicity we make use of the notations

$$\partial_t = \frac{\partial}{\partial t}, \quad D_j = \frac{\partial}{\partial x_j}, \quad \partial_j = \frac{\partial}{\partial \omega_j} \qquad (j = 1, 2, \dots, n).$$

We denote by $\sup\limits_{\omega \neq 0} u(x, t, \omega)$ and $\sup\limits_{\omega \notin z} u(x, t, \omega)$ the supremum of $u(x, t, \omega)$ on $R^n_\omega - \{0\}$ for each fixed (x, t) and that on $R^n_\omega - Z$ respectively, where Z is a subset of R^n_ω.

We say that $l(\chi, t, \omega)$ is absolutely continuous with respect to ω_k if it is so on any finite closed interval for each fixed χ, t and ω_j $(j=1, 2,\dots, n; j\neq k)$, and that $l(\chi, t, \omega)$ is absolutely continuous with respect to t if it is so on J for each fixed χ and ω. We say that a scalar function $c(x, t, \omega)$ satisfies the condition imposed on matrix functions, if $c(x, t, \omega)I$ does.

2.2. The difference approximations

We consider a mesh imposed on (x, t)-space with a spacing of h in each x_j-direction $(j=1, 2,\dots, n)$ and a spacing of k in the t-direction. The ratio $\lambda = k/h$ is to be kept constant as h varies. We approximate (1.1) and (1.2) by the difference scheme of the form:

$$v(x, t+k) = S_h(t, h)v(x, t) \qquad (t, t+k\in J) \tag{2.3}$$

$$v(x, 0) = u_0(x), \tag{2.4}$$

where

$$S_h(t, \mu) = \textstyle\sum_m \prod_{j=1}^{\nu} C_{m_j}(x, t, \mu, T_h), \quad m = (m_1, m_2,\dots, m_\nu), \tag{2.5}$$

$$C_{m_j}(x, t, \mu, T_h) = \textstyle\sum_\alpha c_{\alpha m_j}(x, t, \mu)T_h^\alpha, \quad \alpha = (\alpha_1, \alpha_2,\dots, \alpha_n), \tag{2.6}$$

$$T_h^\alpha = T_{1h}^{\alpha_1}T_{2h}^{\alpha_2}\cdots T_{nh}^{\alpha_n}, \quad T_{jh}u(x) = u(x_1,\dots, x_{j-1}, x_j+h, x_{j+1},\dots, x_n), \tag{2.7}$$

m_j $(m_j\geqq 0; j=0, 1,\dots, \nu)$ and α_j $(j=1, 2,\dots, n)$ are integers, $\mu\in I_\infty$ and $c_{\alpha m_j}(x, t, \mu)$'s are $N\times N$ matrices.

We approximate the partial differential operator hD_j $(1\leqq j\leqq n)$ by the difference operator Δ_{jh} of the form

$$\Delta_{jh} = \textstyle\sum_l b_l(T_{jh}^l - T_{jh}^{-l})/2, \tag{2.8}$$

where the summation is over a finite set of l $(l\geqq 0)$ and b_l's are real constants. We put

$$s_j(\omega) = \textstyle\sum_l b_l \sin l\omega_j \qquad (j = 1, 2,\dots, n), \tag{2.9}$$

$$s(\omega) = (s_1(\omega), s_2(\omega),\dots, s_n(\omega)), \tag{2.10}$$

and assume that, for some positive integer r, $s_j(\omega)$ can be written as follows:

$$s_j(\omega) = \omega_j + O(|\omega_j|^{r+1}) \qquad (|\omega_j|\leqq \pi). \tag{2.11}$$

For example the following difference operators are well known:

$$F_h(t) = C_h + \lambda P_h(t), \tag{2.12}$$

(2.13) $$M_h(t) = I + \lambda P_h(t)C_h + \lambda^2\{(P_h(t))^2 + hQ_h(t)\}/2,$$

where

(2.14) $$P_h(t) = \sum_{j=1}^n A_j(x, t)\Delta_{jh}, \quad C_h = (1/n)\sum_{j=1}^n (T_{jh} + T_{jh}^{-1})/2,$$

$$Q_h(t) = \sum_{j=1}^n (\partial_t A_j(x, t))\Delta_{jh}, \quad \Delta_{jh} = (T_{jh} - T_{jh}^{-1})/2 \quad (j = 1, 2,\dots, n).$$

The schemes (2.3) with operators (2.12) and (2.13) are called Friedrichs' scheme and the modified Lax-Wendroff scheme respectively.

We say that the difference scheme (2.3) approximates (1.1) with accuracy of order r [4, 6] if all smooth solutions u of (1.1) satisfy

(2.15) $$|u(x, t + k) - S_h(t, h)u(x, t)| = O(h^{r+1}) \qquad (h \to 0)$$

for each (x, t).

The difference scheme is said to be stable in L_2 if there exists a constant M such that

(2.16) $$\|S_h(\nu k, h)S_h((\nu - 1)k, h)\cdots S_h(0, h)u\| \leqq M\|u\|$$

for all $u \in L_2$ and for all $h>0$ and integers $\nu \geqq 0$ such that $(\nu+1)k \leqq T$. Since $S_h(t, h)$ is a family of bounded linear operators in L_2 depending on h and t, we have to study the boundedness of products of the form $L_h(\nu k)L_h((\nu-1)k)\cdots L_h(0)$ of such families of operators $L_h(t)$.

Let $\mathscr{H}_h$ be the set of all families of bounded linear operators $H_h(t)$ in L_2 such that

(2.17) $$\|H_h(t)u\| \leqq c(h)\|u\| \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h > 0,$$

where $c(\mu)$ is a continuous function on I_∞.

For $A_h(t), B_h(t) \in \mathscr{H}_h$ and $\alpha \in \boldsymbol{C}$ let $A_h(t)+B_h(t)$, $A_h(t)B_h(t)$ and $\alpha A_h(t)$ be defined by

$$(A_h(t) + B_h(t))u = A_h(t)u + B_h(t)u,$$

$$(A_h(t)B_h(t))u = A_h(t)(B_h(t)u), \quad (\alpha A_h(t))u = \alpha(A_h(t)u).$$

Then $\mathscr{H}_h$ forms an algebra over $\boldsymbol{C}$ with unit element I_h. Since the adjoint $A_h^*(t)$ of a family $A_h(t)$ also belongs to $\mathscr{H}_h$, the operation $*$ is an involution in $\mathscr{H}_h$ and $\mathscr{H}_h$ is an algebra with involution [2].

For $A_h(t), B_h(t) \in \mathscr{H}_h$ we use the notation $A_h(t) \equiv B_h(t)$ if there exists a constant c such that

(2.18) $$\|(A_h(t) - B_h(t))u\| \leqq ch\|u\| \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h > 0.$$

Then we have the following

THEOREM 2.1. *Let $L_h(t) \in \mathscr{H}_h$ and suppose there exist a norm $|||\cdot|||_t$ $(t \in J)$ and positive constants d_j $(j=1, 2, 3)$ and c_0 such that*

$$d_1\|u\| \leqq |||u|||_t \leqq d_2\|u\|, \tag{2.19}$$

$$|||u|||_{t+k} \leqq (1+d_3k)\,|||u|||_t \qquad (t+k \in J), \tag{2.20}$$

$$|||L_h(t)u|||_t \leqq (1+c_0h)\,|||u|||_t \qquad \textit{for all} \quad u \in L_2,\ t \in J \quad \textit{and} \quad h>0. \tag{2.21}$$

Then there exists a constant M such that

$$\|L_h(\nu k)L_h((\nu-1)k)\cdots L_h(0)u\| \leqq M\|u\| \tag{2.22}$$

for all $u \in L_2$ and for all $h>0$ and integers $\nu \geqq 0$ such that $(\nu+1)k \leqq T$.

PROOF. Making use of (2.20) and (2.21), we have

$$\begin{aligned} &|||L_h(\nu k)L_h((\nu-1)k)\cdots L_h(0)u|||_{\nu k} \\ &\qquad \leqq (1+c_0h)\,|||L_h((\nu-1)k)\cdots L_h(0)u|||_{\nu k} \\ &\qquad \leqq (1+c_0h)(1+d_3k)\,|||L_h((\nu-1)k)\cdots L_h(0)u|||_{(\nu-1)k} \\ &\qquad \leqq \cdots \leqq (1+c_0h)^{\nu+1}(1+d_3k)^{\nu}\,|||u|||_0 \qquad \text{for all} \quad u \in L_2,\ h>0, \end{aligned}$$

and by (2.19)

$$d_1\|L_h(\nu k)L_h((\nu-1)k)\cdots L_h(0)u\| \leqq c_1d_2\|u\| \qquad \text{for all} \quad u \in L_2,\ h>0,$$

where $c_1 = \exp(c_0T/\lambda)\exp(d_3T)$. Hence (2.22) holds with $M = c_1d_2/d_1$.

COROLLARY 2.1. *For any $S_h(t) \in \mathscr{H}_h$ let $L_h(t)$ be a family such that $L_h(t) \equiv S_h(t)$ and which satisfies the assumption of the theorem. Then there exists a constant M such that*

$$\|S_h(\nu k)S_h((\nu-1)k)\cdots S_h(0)u\| \leqq M\|u\| \tag{2.23}$$

for all $u \in L_2$ and for all $h>0$ and integers $\nu \geqq 0$ such that $(\nu+1)k \leqq T$.

PROOF. Since there is a constant c_2 such that

$$\|(L_h(t)-S_h(t))u\| \leqq c_2h\|u\| \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h>0,$$

by (2.19) and (2.21) we have

$$\begin{aligned} |||S_h(t)u|||_t &\leqq |||L_h(t)u|||_t + |||(S_h(t)-L_h(t))u|||_t \\ &\leqq |||L_h(t)u|||_t + c_2d_2h\|u\| \end{aligned}$$

$$\leqq (1 + c_3 h) |||u|||_t,$$

where $c_3 = c_0 + c_2 d_2/d_1$. Hence (2.21) is satisfied and (2.23) follows from the theorem.

By Theorem 2.1 and its corollary, in proving the stability of the scheme (2.3), the problem is to find a norm $|||\cdot|||_t$ $(t \in J)$ and a family $L_h(t) \in \mathscr{H}_h$ such that $L_h(t) \equiv S_h(t, h)$ in order to establish (2.21).

3. The subalgebra $\mathscr{K}_h$ of $\mathscr{H}_h$

3.1. Definitions

Let $\mathscr{K}$ be the set of all $N \times N$ matrix functions $p(x, t, \omega)$ defined on $R_x^n \times J \times R_\omega^n$ with the properties:

1) $p(x, t, \omega)$ can be written as

$$p(x, t, \omega) = p_0(x, t, \omega) + p_\infty(t, \omega),$$

where $p_0(x, t, \omega)$ and $p_\infty(t, \omega)$ are bounded and measurable on $R_x^n \times J \times R_\omega^n$ and measurable on $R_x^n \times R_\omega^n$ for each $t \in J$,

$$\lim_{|x| \to \infty} p_0(x, t, \omega) = 0 \qquad \text{for each} \quad (t, \omega);$$

2) $p_0(x, t, \omega)$ is integrable as a function of x for each (t, ω);

3) $\hat{p}(\chi, t, \omega)$ is integrable as a function of χ for each (t, ω) and $\int \operatorname{ess\cdot sup}_\omega |\hat{p}_0(\chi, t, \omega)| d\chi$ is bounded on J.

The Fourier transform $\hat{p}(\chi, t, \omega)$ of the element $p(x, t, \omega)$ of $\mathscr{K}$ can be written as follows:

$$\hat{p}(\chi, t, \omega) = \hat{p}_0(\chi, t, \omega) + \delta(\chi) p_\infty(t, \omega), \tag{3.1}$$

where $\delta(\chi)$ is the delta function. We define $\|\hat{p}(t)\|_F$ by

$$\|\hat{p}(t)\|_F = \int \operatorname{ess\cdot sup}_\omega |\hat{p}_0(\chi, t, \omega)| d\chi + \operatorname{ess\cdot sup}_\omega |p_\infty(t, \omega)|. \tag{3.2}$$

Then we have the following two lemmas.

LEMMA 3.1. *If $p, q \in \mathscr{K}$ and $\alpha \in \boldsymbol{C}$, then $p+q, pq, \alpha p, p^* \in \mathscr{K}$ and*

$$\|\widehat{p+q}(t)\|_F \leqq \|\hat{p}(t)\|_F + \|\hat{q}(t)\|_F, \quad \|\widehat{pq}(t)\|_F \leqq \|\hat{p}(t)\|_F \|\hat{q}(t)\|_F, \tag{3.3}$$

$$\|\widehat{\alpha p}(t)\|_F = |\alpha| \, \|\hat{p}(t)\|_F, \quad \|\widehat{p^*}(t)\|_F = \|\hat{p}(t)\|_F. \tag{3.4}$$

LEMMA 3.2. *Let $p \in \mathscr{K}$ and $u \in \mathscr{S}$. Then*

$$\left\| \int \hat{p}(\xi - \xi', t, h\xi')\hat{u}(\xi')d\xi' \right\| \leqq \|\hat{p}(t)\|_F \|\hat{u}\| \quad \textit{for all} \quad t \in J, \ h > 0, \tag{3.5}$$

and for each $t \in J$ and $h > 0$

$$\begin{aligned} \text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} \int \hat{p}(\xi - \xi', t, h\xi')\hat{u}(\xi')d\xi' d\xi \\ = \kappa^{-1} \int e^{ix\cdot\xi} p(x, t, h\xi)\hat{u}(\xi)d\xi \end{aligned} \tag{3.6}$$

for almost all x.

With each $p \in \mathscr{K}$ we associate a family of operators $P_h(t)$ by the formula:

$$P_h(t)u(x) = \text{l.i.m.}\, \kappa^{-1} \int e^{ix\cdot\xi} \int \hat{p}(\xi - \xi', t, h\xi')\hat{u}(\xi')d\xi' d\xi \tag{3.7}$$

$$\text{for all} \quad u \in \mathscr{S},\ t \in J,\ h > 0.$$

Then by (3.5) $P_h(t)$ can be extended to the closure $\bar{\mathscr{S}} = L_2$ with preservation of norm and the extension is unique. Denoting this extension of $P_h(t)$ again by $P_h(t)$, we call $P_h(t)$ the family (of operators) associated with p and denote this mapping by ϕ i.e. $P_h(t) = \phi(p)$. Unless otherwise stated, we denote by $Q_h(t)$, $\tilde{L}_h(t)$, etc. the families associated with q, $\tilde{l}$, etc. respectively.

We note that by (3.6) $P_h(t)u$ ($u \in \mathscr{S}$) can be written as follows:

$$P_h(t)u(x) = \kappa^{-1} \int e^{ix\cdot\xi} p(x, t, h\xi)\hat{u}(\xi)d\xi \tag{3.8}$$

$$\text{for all} \quad u \in \mathscr{S},\ t \in J,\ h > 0.$$

Let $\mathscr{K}_h = \phi(\mathscr{K})$. Then we have

LEMMA 3.3. *The mapping ϕ is one-to-one.*

By Lemma 3.1 $\mathscr{K}$ forms an algebra with involution over $\boldsymbol{C}$. For $p, q \in \mathscr{K}$ and $\alpha \in \boldsymbol{C}$ we have

$$\phi(p) + \phi(q) = \phi(p + q), \quad \alpha\phi(p) = \phi(\alpha p),$$

because $\mathscr{K}_h \subset \mathscr{H}_h$. Let

$$\phi(p) \circ \phi(q) = \phi(pq), \quad \phi(p)^{\#} = \phi(p^*).$$

Then $\mathscr{K}_h$ forms an algebra with involution over $\boldsymbol{C}$ and the mappings ϕ and ϕ^{-1} are morphisms [1].

3.2. Products and adjoints

We introduce the following three conditions.

CONDITION I. 1) $p \in \mathscr{K}$;

2) $\hat{p}_0(\chi, t, \omega)$ and $p_\infty(t, \omega)$ are absolutely continuous with respect to ω_j $(j=1, 2,\dots, n)$ and $\partial_j \hat{p}_0(\chi, t, \omega)$ and $\partial_j p_\infty(t, \omega)$ $(j=1, 2,\dots, n)$ are measurable in $R^n_\chi \times R^n_\omega$ for each t;

3) $\int \underset{\omega}{\mathrm{ess\cdot sup}}\, |\partial_j \hat{p}_0(\chi, t, \omega)| d\chi$ and $\underset{\omega}{\mathrm{ess\cdot sup}}\, |\partial_j p_\infty(t, \omega)|$ $(j=1, 2,\dots, n)$ are bounded on J.

CONDITION II. $q \in \mathscr{K}$ and $\int \underset{\omega}{\mathrm{ess\cdot sup}}\, (|\chi|\, |\hat{q}_0(\chi, t, \omega)|) d\chi$ is bounded on J.

CONDITION III. 1) $r \in \mathscr{K}$;

2) $\hat{r}_0(\chi, t, \omega)$ is absolutely continuous with respect to ω_j $(j=1, 2,\dots, n)$ and $\partial_j \hat{r}_0(\chi, t, \omega)$ $(j=1, 2,\dots, n)$ are measurable in $R^n_\chi \times R^n_\omega$ for each t;

3) $\int \underset{\omega}{\mathrm{ess\cdot sup}}\, (|\chi_j|\, |\partial_j \hat{r}_0(\chi, t, \omega)|) d\chi$ $(j=1, 2,\dots, n)$ are bounded on J.

We have

THEOREM 3.1. *If p, q and r satisfy Conditions* I, II *and* III *respectively, then*

$$P_h(t)Q_h(t) \equiv P_h(t)\circ Q_h(t), \quad R^*_h(t) \equiv R^\sharp_h(t). \tag{3.9}$$

COROLLARY 3.1. *If $a(x, t)$, $b(\omega, t)$, $p(x, t, \omega) \in \mathscr{K}$, then*

$$A_h(t)P_h(t) = A_h(t)\circ P_h(t), \quad P_h(t)B_h(t) = P_h(t)\circ B_h(t), \tag{3.10}$$

$$B^*_h(t) = B^\sharp_h(t). \tag{3.11}$$

3.3. Construction of a new norm

We construct a norm $|||\cdot|||_t$ $(t \in J)$ stated in Theorem 2.1.

Let ε and R $(R \geqq \varepsilon)$ be positive numbers and let $S(R, \varepsilon) = \{x \mid |x| < R+\varepsilon\}$. Let $\{x^{(i)}\}$ $(i=1, 2,\dots, s)$ be all the lattice-points $(\varepsilon\eta_1, \varepsilon\eta_2,\dots, \varepsilon\eta_n)$ contained in $S(R, \varepsilon)$ $(\eta_j = m_j/\sqrt{n};\ m_j = 0, \pm 1, \pm 2,\dots;\ j=1, 2,\dots, n)$ and let

$$V_0 = \{x \mid |x| > R\}, \quad V_i = \{x \mid |x - x^{(i)}| < \varepsilon\} \qquad (i = 1, 2,\dots, s).$$

Then we can construct a partition of unity $\{\alpha_i^2(x)\}_{i=0,1,\dots,s}$ with the properties:

1) $\alpha_i(x) \geqq 0, \quad \alpha_i(x) \in C^\infty, \quad \mathrm{supp}\, \alpha_i(x) \subset V_i \qquad (i = 0, 1,\dots, s)$;

2) $\sum_{i=0}^s \alpha_i^2(x) = 1$;

3) $\alpha_0(x)$ and all its derivatives are bounded uniformly with respect to R for each ε.

We introduce the following

CONDITION N. 1) $g \in \mathscr{K}$ and $D_j g(x, t, \omega)$ $(j=1, 2,\dots, n)$ are bounded on $R_x^n \times J \times R_\omega^n$ and continuous on R_x^n for each (t, ω); $D_j g(x, t, \omega)$ $(j=1, 2,\dots, n)$ are integrable as functions of x for each (t, ω); $\widehat{D_j g}(\chi, t, \omega)$ $(j=1, 2,\dots, n)$ are integrable as functions of χ for each (t, ω) and $\int \operatorname{ess\cdot sup}_\omega |\widehat{D_j g}(\chi, t, \omega)| d\chi$ $(j=1, 2, \dots, n)$ are bounded on J;

2) $\|\widehat{\alpha_0 g_0}(t)\|_F$ converges to zero uniformly on J as $R \to \infty$.

Then we have the following lemma and theorem.

LEMMA 3.4. *If p and q satisfy Condition* N, *so also do $p+q$, pq and p^*.*

THEOREM 3.2. *Suppose*
1) *$g(x, t, \omega)$ satisfies Condition* N;
2) *$g(x, t, \omega) \geqq eI$ for some constant $e>0$.*
Then for sufficiently small ε and large R there exist positive constants d_j $(j=1, 2)$ independent of u, t and h such that

$$(3.12) \qquad d_1^2 \|u\|^2 \leqq \textstyle\sum_{i=0}^s \operatorname{Re}(G_h(t)\alpha_i u, \alpha_i u) \leqq d_2^2 \|u\|^2$$

$$\textit{for all} \quad u \in L_2,\ t \in J,\ h > 0.$$

This theorem enables us to introduce the norm

$$(3.13) \qquad |||u|||_t = \{\textstyle\sum_{i=0}^s \operatorname{Re}(G_h(t)\alpha_i u, \alpha_i u)\}^{1/2} \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h > 0,$$

which has the property (2.19) by (3.12). (For simplicity the dependence of $|||\cdot|||_t$ on h is not expressed explicitly.)

To obtain sufficient conditions for (2.20), we introduce the following

CONDITION L. 1) $g \in \mathscr{K}$;

2) $\hat{g}_0(\chi, t, \omega)$ and $g_\infty(t, \omega)$ are absolutely continuous with respect to t; there exist measurable functions $\varphi_0(\chi, t, \omega)$ and $\varphi_\infty(t, \omega)$ in $R_\chi^n \times J \times R_\omega^n$ such that for each (χ, ω) and for almost all $t \in J$

$$\partial_t \hat{g}_0(\chi, t, \omega) = \varphi_0(\chi, t, \omega), \quad \partial_t g_\infty(t, \omega) = \varphi_\infty(t, \omega);$$

3) There exists a constant $M>0$ such that for almost all $t \in J$

$$\int \operatorname{ess\cdot sup}_\omega |\varphi_0(\chi, t, \omega)| d\chi \leqq M, \quad \operatorname{ess\cdot sup}_\omega |\varphi_\infty(t, \omega)| \leqq M.$$

We have

LEMMA 3.5. *If g satisfies Condition* L, *then there exists a positive constant c independent of u, t, t' and h such that*

(3.14) $\|(G_h(t') - G_h(t))u\| \leqq c|t' - t| \, \|u\|$ *for all* $u \in L_2$, $t, t' \in J$, $h > 0$.

PROOF. By Lemma 3.2 it suffices to show that for some constant $c>0$

(3.15) $\|\hat{g}(t') - \hat{g}(t)\|_F \leqq c(t' - t)$ for all $t, t' \in J$ $(t' \geqq t)$.

From Condition L–2) it follows that for each (χ, ω)

$$|\hat{g}_0(\chi, t', \omega) - \hat{g}_0(\chi, t, \omega)| = \left|\int_t^{t'} \partial_t \hat{g}_0(\chi, \theta, \omega) d\theta\right|$$
$$\leqq \int_t^{t'} |\varphi_0(\chi, \theta, \omega)| d\theta.$$

Taking the essential suprema of both sides over R^n_ω and integrating them with respect to χ, we have by Condition L–3)

$$\|\hat{g}_0(t') - \hat{g}_0(t)\|_F \leqq \iint_t^{t'} \operatorname{ess\cdot sup}_{\omega} |\varphi_0(\chi, \theta, \omega)| d\theta d\chi \tag{3.16}$$
$$\leqq \int_t^{t'} M d\theta = M(t' - t).$$

Similarly we have

$$\|g_\infty(t') - g_\infty(t)\|_F \leqq M(t' - t). \tag{3.17}$$

Hence (3.15) holds with $c = 2M$ by (3.16) and (3.17).

Combining Theorem 3.2 with Lemma 3.5, we have

THEOREM 3.3. *Let g satisfy Conditions* N *and* L *and suppose $g(x, t, \omega) \geqq eI$ for some constant $e>0$. Then the norm $|||\cdot|||_t$ given by* (3.13) *satisfies* (2.19) *and* (2.20).

PROOF. It suffices to show (2.20). By Lemma 3.5 for some constant c independent of u, t, t' and h we have

$$|\,|||u|||_{t'}^2 - |||u|||_t^2| = |\textstyle\sum_{i=0}^s \operatorname{Re}((G_h(t') - G_h(t))\alpha_i u, \alpha_i u)|$$
$$\leqq \textstyle\sum_{i=0}^s \|(G_h(t') - G_h(t))\alpha_i u\| \, \|\alpha_i u\|$$
$$\leqq \textstyle\sum_{i=0}^s c|t' - t| \, \|\alpha_i u\|^2 = c|t' - t| \, \|u\|^2$$

for all $u \in L_2$, $t, t' \in J$, $h > 0$.

The choice $t'=t+k$ yields (2.20) with $d_3=c/d_1^2$ by (2.19).

3.4. Lax-Nirenberg Theorem

We have the following analogue of Lax-Nirenberg Theorem [3] which plays an important role in establishing (2.21).

THEOREM 3.4. *Suppose $p\in\mathscr{K}$ satisfies the conditions:*

1) $\partial_j\hat{p}_0(\chi, t, \omega)$ *and* $\partial_j p_\infty(t, \omega)$ $(j=1, 2,\dots, n)$ *are continuous on* R^n_ω *for each* (χ, t) *and absolutely continuous with respect to* ω_k $(k=1, 2,\dots, n)$;

2) $\partial_k\partial_j\hat{p}_0(\chi, t, \omega)$ *and* $\partial_k\partial_j p_\infty(t, \omega)$ $(j, k=1, 2,\dots, n)$ *are measurable in* $R^n_\chi\times R^n_\omega$ *for each* t; $\int \operatorname{ess\cdot sup}_\omega |\partial_k\partial_j\hat{p}_0(\chi, t, \omega)|d\chi$ *and* $\operatorname{ess\cdot sup}_\omega |\partial_k\partial_j p_\infty(t, \omega)|$ $(j, k=1, 2,\dots, n)$ *are bounded on* J;

3) $\int \operatorname{ess\cdot sup}_\omega (|\chi|^2|\hat{p}_0(\chi, t, \omega)|)d\chi$ *is bounded on* J;

4) $p(x, t, \omega)\geqq 0$.

Then there exists a positive constant c independent of u, t and h such that

$$\operatorname{Re}(P_h(t)u, u)\geqq -ch\|u\|^2 \qquad \textit{for all} \quad u\in L_2,\ t\in J,\ h>0. \tag{3.18}$$

4. Products of families of operators

4.1. The family of operators Λ_h

In this section $s(\omega)$ denotes a real-valued vector function with the properties:

1) $s_l(\omega)$, $\partial_j s_l(\omega)$ and $\partial_k\partial_j s_l(\omega)$ $(j, k, l=1, 2,\dots, n)$ are bounded and continuous on R^n_ω;

2) Zeros of $|s(\omega)|$ are isolated points.

It is readily seen that $|s(\omega)|I$ satisfies Condition I. Let $Z=\{\omega \mid |s(\omega)|=0\}$ and Λ_h be the family associated with $|s(\omega)|I$. Then by Corollary 3.1 we have $\Lambda_h=\Lambda_h^\sharp=\Lambda_h^*$.

Let $p(x, t, \omega)$ be an element of $\mathscr{K}$ such that $p(x, t, \omega)/|s(\omega)|$ is bounded on $R^n_x\times J\times(R^n_\omega-Z)$. For any constant α let

$$q_\alpha(x, t, \omega)=\begin{cases} p(x, t, \omega)/|s(\omega)| & \text{for} \quad \omega\in R^n_\omega-Z,\\ \alpha I & \text{for} \quad \omega\in Z,\end{cases} \tag{4.1}$$

and suppose $q_\alpha(x, t, \omega)\in\mathscr{K}$. Then, since Z is a set of measure zero, we have for each t

$$\widehat{Q_{\alpha h}(t)u}(\xi)=\widehat{Q_{\beta h}(t)u}(\xi) \quad \text{a.e.} \tag{4.2}$$

for all $u\in\mathscr{S}$, where $Q_{\alpha h}(t)$ and $Q_{\beta h}(t)$ are the families associated with q_α and q_β

$(\beta \neq \alpha)$ respectively. In the following we identify $q_\alpha(x, t, \omega)$ with $q_\beta(x, t, \omega)$ and denote them by $p(x, t, \omega)/|s(\omega)|$. Then we have $P_h(t) = P_{1h}(t) \circ \Lambda_h$, where $P_{1h}(t)$ is the family associated with $p/|s|$.

When $e(\omega)$ is a scalar function with isolated zeros such that $e(\omega)I \in \mathscr{K}$, $p(x, t, \omega)/e(\omega)$ can be defined similarly by replacing $|s(\omega)|$ by $e(\omega)$.

Now we introduce the following conditions.

CONDITION I′. 1) $p \in \mathscr{K}$;

2) $\hat{p}_0(\chi, t, \omega)$ is bounded on $R^n_\chi \times J \times (R^n_\omega - Z)$;

3) $\partial_j l_0(\chi, t, \omega)$ and $\partial_j l_\infty(t, \omega)$ $(j = 1, 2, \ldots, n)$ are bounded on $R^n_\chi \times J \times (R^n_\omega - Z)$ and continuous on $R^n_\omega - Z$ for each (χ, t), where $l_0(\chi, t, \omega) = \hat{p}_0 |s|$, $l_\infty(t, \omega) = p_\infty |s|$;

4) $\int \operatorname{ess\cdot sup}_\omega |\partial_j l_0(\chi, t, \omega)| d\chi$ $(j = 1, 2, \ldots, n)$ are bounded on J.

CONDITION III′. 1), 2) the same as I′–1), I′–2) respectively;

3) $\partial_j l_0(\chi, t, \omega)$ $(j = 1, 2, \ldots, n)$ are bounded on $R^n_\chi \times J \times (R^n_\omega - Z)$ and continuous on $R^n_\omega - Z$ for each (χ, t);

4) $\int \operatorname{ess\cdot sup}_\omega (|\chi_j|\,|\partial_j l_0(\chi, t, \omega)|) d\chi$ $(j = 1, 2, \ldots, n)$ are bounded on J.

CONDITION IV. $p \in \mathscr{K}$ and $\int \operatorname{ess\cdot sup}_\omega (|\chi|^2 |\hat{p}_0(\chi, t, \omega)|) d\chi$ is bounded on J.

CONDITION V. 1) p satisfies Condition I′;

2) $\partial_k m_{j0}(\chi, t, \omega)$ and $\partial_k m_{j\infty}(t, \omega)$ $(j, k = 1, 2, \ldots, n)$ are bounded on $R^n_\chi \times J \times (R^n_\omega - Z)$ and continuous on $R^n_\omega - Z$ for each (χ, t), where $m_{j0}(\chi, t, \omega) = (\partial_j l_0)\,|s|$, $m_{j\infty}(t, \omega) = (\partial_j l_\infty)\,|s|$, $l_0 = \hat{p}_0 |s|$, $l_\infty = p_\infty |s|$;

3) $\int \operatorname{ess\cdot sup}_\omega |\partial_k m_{j0}(\chi, t, \omega)| d\chi$ $(j, k = 1, 2, \ldots, n)$ are bounded on J.

We have the following lemmas.

LEMMA 4.1. (i) *If p satisfies Condition* I′, *then $p|s|$ satisfies Condition* I.
(ii) *If p satisfies Condition* III′, *then $p|s|$ satisfies Condition* III.

LEMMA 4.2. (i) *If p satisfies Condition* I′ *and q satisfies Condition* II, *then*

$$P_h(t) Q_h(t) \Lambda_h \equiv P_h(t) \circ Q_h(t) \circ \Lambda_h. \tag{4.3}$$

(ii) *If p satisfies Condition* III′, *then*

$$(P_h(t) \Lambda_h)^* \equiv P^\#_h(t) \circ \Lambda_h. \tag{4.4}$$

LEMMA 4.3. *If p satisfies Conditions* IV *and* V, *then $p(x, t, \omega)|s(\omega)|^2$ satisfies conditions* 1), 2) *and* 3) *of Theorem* 3.4.

4.2. Subalgebras $\mathscr{M}$ and $\mathscr{L}$ of $\mathscr{K}$

Let $\mathscr{M}$ be the set of all elements of $\mathscr{K}$ that satisfy Conditions I′, II and III′ and let the set $\mathscr{L}$ consist of all elements of $\mathscr{M}$ that satisfy Conditions IV and V. For instance $|s(\omega)|I$ and $(s_j(\omega)/|s(\omega)|)I$ $(j=1, 2,\dots, n)$ belong to $\mathscr{M}$ and $\mathscr{L}$.

LEMMA 4.4. (i) *If p and q satisfy Condition* II, *so also do $p+q$, pq and p^*.*

(ii) *If $p, q \in \mathscr{M}$, then $p+q$, pq, $p^* \in \mathscr{M}$.*

(iii) *If $p, q \in \mathscr{L}$, then $p+q$, pq, $p^* \in \mathscr{L}$.*

LEMMA 4.5. *Let $g(x, t, \omega)$ satisfy Conditions* I′ *and* II, *and let*

$$l(x, t, \omega) = c(\omega)I + q(x, t, \omega)\,|s(\omega)|, \tag{4.5}$$

where $q(x, t, \omega) \in \mathscr{M}$ and $c(\omega)$ is a scalar function satisfying Condition I. *Then*

$$L_h^*(t)G_h(t)L_h(t) \equiv L_h^\sharp(t)\circ G_h(t)\circ L_h(t). \tag{4.6}$$

COROLLARY 4.1. *Under the assumption of Lemma* 4.5 *let*

$$g(x, t, \omega) = w^*(x, t, \omega)w(x, t, \omega), \tag{4.7}$$

where $w, w^{-1} \in \mathscr{K}$. Then

$$\begin{aligned} G_h(t) - L_h^*(t)G_h(t)L_h(t) &\equiv G_h(t) - L_h^\sharp(t)\circ G_h(t)\circ L_h(t) \\ &= W_h^\sharp(t)\circ(I_h - \tilde{L}_h^\sharp(t)\circ\tilde{L}_h(t))\circ W_h(t), \end{aligned} \tag{4.8}$$

$$g - l^*gl = w^*(I - \tilde{l}^*\tilde{l})w, \qquad \tilde{l} = wlw^{-1}. \tag{4.9}$$

4.3. Integrability of Fourier transforms

We introduce

CONDITION VI. 1) $p(x, t, \omega)$ can be written as

$$p(x, t, \omega) = p_0(x, t, \omega) + p_\infty(t, \omega),$$

where $p_0(x, t, \omega)$ and $p_\infty(t, \omega)$ are bounded and measurable on $R_x^n \times J \times R_\omega^n$ and measurable on $R_x^n \times R_\omega^n$ for each t,

$$\lim_{|x|\to\infty} p_0(x, t, \omega) = 0 \qquad \text{for each} \quad (t, \omega);$$

2) $D_l^m p_0(x, t, \omega)$ $(l=1, 2,\dots, n;\ m=0, 1,\dots, n+3)$ are continuous on $R_x^n \times J \times (R_\omega^n - Z)$ and continuous on $R_x^n \times J$ for each $\omega \in Z$; $\sup_\omega |D_l^m p_0(x, t, \omega)|$ and $\int \sup_\omega |D_l^m p_0(x, t, \omega)|dx$ $(l=1, 2,\dots, n;\ m=0, 1,\dots, n+3)$ are bounded on $R_x^n \times J$

and on J respectively;

3) $\{(D_l^q\partial_j p_0(x, t, \omega))|s(\omega)|\}$ and $\{(\partial_j p_\infty(t, \omega))|s(\omega)|\}$ $(j, l=1, 2,\dots, n;\ q=0, 1,\dots, n+2)$ are bounded and continuous on $R_x^n\times J\times(R_\omega^n-Z)$;

4) $\int\sup_{\omega\notin z}(|D_l^q\partial_j p_0(x, t, \omega)|\,|s(\omega)|)dx$ $(j, l=1, 2,\dots, n;\ q=0, 1,\dots, n+2)$ are bounded on J;

5) $\{(D_l^r\partial_k\partial_j p_0(x, t, \omega))|s(\omega)|^2\}$ and $\{(\partial_k\partial_j p_\infty(t, \omega))|s(\omega)|^2\}$ $(j, k, l=1, 2,\dots, n;\ r=0, 1,\dots, n+1)$ are bounded and continuous on $R_x^n\times J\times(R_\omega^n-Z)$;

6) $\int\sup_{\omega\notin z}(|D_l^r\partial_k\partial_j p_0(x, t, \omega)||s(\omega)|^2)dx$ $(j, k, l=1, 2,\dots, n;\ r=0, 1,\dots, n+1)$ are bounded on J;

7) $\int_{|x|\geqq R}\sup_\omega|D_l^r p_0(x, t, \omega)|dx$ $(l=1, 2,\dots, n;\ r=0, 1,\dots, n+1)$ converge to zero uniformly on J as $R\to\infty$;

8) $\partial_t p_0(x, t, \omega)$ and $\partial_t p_\infty(t, \omega)$ are bounded on $R_x^n\times J\times R_\omega^n$; $D_l^r\partial_t p_0(x, t, \omega)$ $(l=1, 2,\dots, n;\ r=0, 1,\dots, n+1)$ are continuous on $R_x^n\times J\times(R_\omega^n-Z)$ and continuous on $R_x^n\times J$ for each $\omega\in Z$; $\sup_\omega|D_l^r\partial_t p_0(x, t, \omega)|$ and $\int\sup_\omega|D_l^r\partial_t p_0(x, t, \omega)|dx$ $(l=1, 2,\dots, n;\ r=0, 1,\dots, n+1)$ are bounded on $R_x^n\times J$ and on J respectively.

We have

LEMMA 4.6. (i) *If p satisfies Conditions* VI–1) *and* VI–2), *then p satisfies Conditions* II *and* IV.

(ii) *If p satisfies Conditions* VI–1)–VI–4), *then* $p\in\mathscr{M}$.

(iii) *If p satisfies Conditions* VI–1)–VI–6), *then* $p\in\mathscr{L}$.

COROLLARY 4.2. *Let $a(x, t)$ be an $N\times N$ matrix such that*

$$a(x, t) = a_0(x, t) + a_\infty(t), \tag{4.10}$$

where $a_0(x, t)$ and $a_\infty(t)$ are bounded on $R_x^n\times J$ and $\lim_{|x|\to\infty} a_0(x, t)=0$ for each t. Suppose $D_l^m a_0(x, t)$ $(l=1, 2,\dots, n;\ m=0, 1,\dots, n+1+p;\ p=0, 1, 2)$ are bounded and continuous on $R_x^n\times J$ and $\int|D_l^m a_0(x, t)|dx$ are bounded on J. Then $\int|\chi|^p|\hat{a}_0(\chi, t)|d\chi$ $(p=0, 1, 2)$ are bounded on J.

LEMMA 4.7. (i) *If g satisfies Conditions* VI–1), VI–2) *and* VI–7), *then it satisfies Condition* N.

(ii) *If g satisfies Conditions* VI–1), VI–2) *and* VI–8), *then it satisfies Condition* L.

PROOF. We have only to prove (ii). By Lemma 4.6 $g\in\mathscr{K}$, and g satisfies Condition L–1).

By Condition VI–8) we have for any fixed (χ, ω)

$$(4.11)\qquad e^{-ix\cdot\chi}g_0(x,t',\omega)-e^{-ix\cdot\chi}g_0(x,t,\omega)$$
$$=\int_t^{t'}e^{-ix\cdot\chi}\partial_t g_0(x,\theta,\omega)d\theta\qquad\text{for all}\quad t,t'\in J.$$

Integrating both sides of (4.11) with respect to x, we have

$$\hat{g}_0(\chi,t',\omega)-\hat{g}_0(\chi,t,\omega)=\int_t^{t'}\widehat{\partial_t g_0}(\chi,\theta,\omega)d\theta\qquad\text{for all}\quad t,t'\in J.$$

Hence $\hat{g}_0(\chi,t,\omega)$ is absolutely continuous with respect to t, so that for each (χ,ω) and for almost all $t\in J$

$$(4.12)\qquad \partial_t\hat{g}_0(\chi,t,\omega)=\widehat{\partial_t g_0}(\chi,t,\omega).$$

Since $\widehat{\partial_t g_0}(\chi,t,\omega)$ is measurable on $R^n_\chi\times J\times R^n_\omega$, $g_0(x,t,\omega)$ satisfies Condition L–2) with $\varphi_0(\chi,t,\omega)=\widehat{\partial_t g_0}(\chi,t,\omega)$ by (4.12). Similarly g_∞ satisfies Condition L–2) with $\varphi_\infty(t,\omega)=\partial_t g_\infty(t,\omega)$.

By the argument similar to that of Lemma 4.6 in [5] it can be shown that Condition L–3) is satisfied.

4.4. Products of families of operators

To prove the boundedness of $L_h(\nu k)L_h((\nu-1)k)\cdots L_h(0)$, in view of Theorem 2.1, it suffices to show that $L_h(t)$ satisfies (2.21). We have

THEOREM 4.1. *Let $g(x,t,\omega)\in\mathscr{M}$ satisfy conditions of Theorem 3.3 and let*

$$(4.13)\qquad l(x,t,\omega)=c(\omega)I+q(x,t,\omega)\,|s(\omega)|,$$

$$(4.14)\qquad g(x,t,\omega)-l^*(x,t,\omega)g(x,t,\omega)l(x,t,\omega)$$
$$=a(x,t,\omega)\,|s(\omega)|^2+b(x,t,\omega)\,|e(\omega)|^2,$$

where $q\in\mathscr{M}$ and $c(\omega)$ and $e(\omega)$ are scalar functions satisfying Condition I. *Suppose*

1) $a\in\mathscr{L}$ *and* $a(x,t,\omega)\geqq 0$;
2) $b(x,t,\omega)$ *satisfies Conditions* II *and* N;
3) $b(x,t,\omega)\geqq\beta I$ *for some* $\beta>0$.

Then for some $c_0\geqq 0$

$$(4.15)\qquad |||L_h(t)u|||_t^2\leqq(1+c_0h)\,|||u|||_t^2\qquad\textit{for all}\quad u\in L_2,\ t\in J,\ h>0,$$

where $|||\cdot|||_t$ is the norm given by (3.13).

PROOF. Let $\{\alpha_i^2(x)\}_{i=0,1,\dots,s}$ be the partition of unity given in 3.3 and let $\alpha_i=\phi(\alpha_i I)$ $(i=0,1,\dots,s)$. Then $\alpha_i(x)u(x)=(\alpha_i u)(x)$ $(i=0,1,\dots,s)$ and by Theorem 3.2 there exist positive constants d_j, ε_j $(j=1, 2)$, ε and R such that

$$d_1^2\|u\|^2 \leqq \textstyle\sum_{i=0}^{s} \mathrm{Re}\,(G_h(t)\alpha_i u,\ \alpha_i u) \leqq d_2^2\|u\|^2, \tag{4.16}$$

$$\varepsilon_1^2\|u\|^2 \leqq \textstyle\sum_{i=0}^{s} \mathrm{Re}\,(B_h(t)\alpha_i u,\ \alpha_i u) \leqq \varepsilon_2^2\|u\|^2. \tag{4.17}$$

By Lemma 4.5

$$L_h^*(t)G_h(t)L_h(t) \equiv L_h^\#(t)\circ G_h(t)\circ L_h(t),$$

and for some $c_1 \geqq 0$

$$|((L_h^*(t)G_h(t)L_h(t) - L_h^\#(t)\circ G_h(t)\circ L_h(t))u,\ u)| \leqq c_1 h\|u\|^2 \tag{4.18}$$

$$\text{for all} \quad u \in L_2,\ t \in J,\ h > 0.$$

Since $\alpha_i(x)$ $(i=0, 1,\dots, s)$ satisfy Condition II, by Theorem 3.1 we have $L_h(t)\alpha_i \equiv \alpha_i L_h(t)$ $(i=0, 1,\dots, s)$. Hence for some $c_2 \geqq 0$

$$\begin{aligned} &|(G_h(t)\alpha_i L_h(t)u,\ \alpha_i L_h(t)u) - (G_h(t)L_h(t)\alpha_i u,\ L_h(t)\alpha_i u)| \\ &\qquad \leqq c_2 h\|u\|^2\ (i = 0, 1,\dots, s) \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h > 0. \end{aligned} \tag{4.19}$$

Since by definition

$$|||L_h(t)u|||_t^2 = \textstyle\sum_{i=0}^{s} \mathrm{Re}\,(G_h(t)\alpha_i L_h(t)u,\ \alpha_i L_h(t)u),$$

by (4.18) and (4.19) we have

$$\begin{aligned} |||L_h(t)u|||_t^2 &\leqq \textstyle\sum_{i=0}^{s} \mathrm{Re}\,(G_h(t)L_h(t)\alpha_i u,\ L_h(t)\alpha_i u) + c_3 h\|u\|^2 \\ &\leqq \textstyle\sum_{i=0}^{s} \mathrm{Re}\,((L_h^\#(t)\circ G_h(t)\circ L_h(t))\alpha_i u,\ \alpha_i u) + c_4 h\|u\|^2, \end{aligned} \tag{4.20}$$

where $c_3=(s+1)c_2$, $c_4=c_1+c_3$. Hence

$$\begin{aligned} &|||u|||_t^2 - |||L_h(t)u|||_t^2 \\ &\qquad \geqq \textstyle\sum_{i=0}^{s} \mathrm{Re}\,((G_h(t) - L_h^\#(t)\circ G_h(t)\circ L_h(t))\alpha_i u,\ \alpha_i u) - c_4 h\|u\|^2. \end{aligned} \tag{4.21}$$

The condition (4.14) yields

$$G_h(t) - L_h^\#(t)\circ G_h(t)\circ L_h(t) = A_h(t)\circ \Lambda_h^2 + B_h(t)\circ E_h^\#\circ E_h, \tag{4.22}$$

where $E_h=\phi(eI)$. By Lemma 4.3 and Theorem 3.4 from condition 1) it follows that for some $c_5 \geqq 0$

$$\mathrm{Re}\,((A_h(t)\circ \Lambda_h^2)u,\ u) \geqq -c_5 h\|u\|^2 \qquad \text{for all} \quad u \in L_2,\ t \in J,\ h > 0. \tag{4.23}$$

By Theorem 3.1 and its corollary we have $E_h\alpha_i \equiv \alpha_i E_h$ $(i=0, 1,\dots, s)$ and

$$B_h(t)\circ E_h^\sharp\circ E_h = (E_h^\sharp\circ B_h(t))\circ E_h = (E_h^\sharp\circ B_h(t))E_h$$
$$\equiv E_h^\sharp B_h(t)E_h = E_h^* B_h(t)E_h,$$

so that

$$\alpha_i^*(B_h(t)\circ E_h^\sharp\circ E_h)\alpha_i \equiv (E_h\alpha_i)^* B_h(t)(E_h\alpha_i) \equiv (\alpha_i E_h)^* B_h(t)(\alpha_i E_h). \tag{4.24}$$

By (4.17) and (4.24) we have for some $c_6 \geqq 0$

$$\begin{aligned}\textstyle\sum_{i=0}^s \operatorname{Re}((B_h(t)\circ E_h^\sharp\circ E_h)\alpha_i u, \alpha_i u) &\geqq \textstyle\sum_{i=0}^s\{\operatorname{Re}(B_h(t)\alpha_i E_h u, \alpha_i E_h u) - c_6 h\|u\|^2\}\\ &\geqq \varepsilon_1^2\|E_h u\|^2 - c_7 h\|u\|^2,\end{aligned} \tag{4.25}$$

where $c_7=(s+1)c_6$. Hence by (4.21)–(4.23) and (4.25)

$$|||u|||_t^2 - |||L_h(t)u|||_t^2 \geqq \varepsilon_1^2\|E_h u\|^2 - c_8 h\|u\|^2 \geqq -c_8 h\|u\|^2,$$

where $c_8=c_4+c_5+c_7$. Thus (4.15) holds by (4.16) with $c_0=c_8/d_1^2$.

5. Two algebras of difference operators

5.1. Algebra $\mathscr{F}_h$

Let $\mathscr{A}_0$ be the set of all $N\times N$ matrix functions $a(x, t)$ defined on $R_x^n\times J$ with the properties:

1) $a(x, t)$ can be written as

$$a(x, t) = a_0(x, t) + a_\infty(t),$$

where $a_0(x, t)$ and $a_\infty(t)$ are bounded and measurable on $R_x^n\times J$ and $\lim_{|x|\to\infty} a_0(x, t) = 0$ for each t;

2) $a_0(x, t)$ is integrable as a function of x for each t;

3) $\int|\chi|^p|\hat{a}_0(\chi, t)|d\chi$ $(p=0, 1, 2)$ are bounded on J.

We denote by α an n-tuple $(\alpha_1, \alpha_2,\dots, \alpha_n)$ of integers, i.e. $\alpha=(\alpha_1, \alpha_2,\dots, \alpha_n)$. Let $\mathscr{A}$ be the set of all matrices $a(x, t, \omega)$ such that $a(x, t, \omega)=\sum_\alpha a_\alpha(x, t)e^{i\alpha\cdot\omega}$, where $a_\alpha\in\mathscr{A}_0$ and the summation is over a finite set of α. It is clear that $a(x, t, \omega)$ satisfies Conditions I, II and III. Let

$$a(x, t, \omega) = \textstyle\sum_\alpha a_\alpha(x, t)e^{i\alpha\cdot\omega}, \quad b(x, t, \omega) = \sum_\beta b_\beta(x, t)e^{i\beta\cdot\omega}. \tag{5.1}$$

Then

$$a(x, t, \omega) + b(x, t, \omega) = \sum_\gamma (a_\gamma(x, t) + b_\gamma(x, t))e^{i\gamma\cdot\omega}, \tag{5.2}$$

$$a(x, t, \omega)b(x, t, \omega) = \sum_\gamma (\sum_{\alpha+\beta=\gamma} a_\alpha(x, t)b_\beta(x, t))e^{i\gamma\cdot\omega}, \tag{5.3}$$

$$a^*(x, t, \omega) = \sum_\alpha a_\alpha^*(x, t)e^{-i\alpha\cdot\omega}. \tag{5.4}$$

Hence $\mathscr{A}$ is a subalgebra of $\mathscr{K}$ with involution.

Since for $a(x, t) \in \mathscr{A}_0$

$$\|a(x, t)T_h^\alpha u(x)\| \leqq (\sup_{x,t} |a(x, t)|) \|u\| \qquad \text{for all} \quad u \in L_2, \ t \in J, \ h > 0,$$

the family $a(x, t)T_h^\alpha$ belongs to $\mathscr{H}_h$. We define a mapping ψ from $\mathscr{A}$ into $\mathscr{H}_h$ by

$$\psi(\sum_\alpha a_\alpha(x, t)e^{i\alpha\cdot\omega}) = \sum_\alpha a_\alpha(x, t)T_h^\alpha, \tag{5.5}$$

and let $\mathscr{A}_h = \psi(\mathscr{A})$.

For $\sum_\alpha a_\alpha(x, t)e^{i\alpha\cdot\omega} \in \mathscr{A}$ let $A_h = \phi(\sum_\alpha a_\alpha(x, t)e^{i\alpha\cdot\omega})$. Then for each $u \in \mathscr{S}$ and $t \in J$

$$\begin{aligned}
&\kappa \int e^{ix\cdot\xi} \sum_\alpha a_\alpha(x, t)T_h^\alpha u(x)dx \\
&= \int \sum_\alpha \widehat{a_{\alpha 0}}(\xi - \xi', t)e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi' + \sum_\alpha a_{\alpha\infty}(t)e^{i\alpha\cdot h\xi}\hat{u}(\xi) \\
&= \int \sum_\alpha \hat{a}_\alpha(\xi - \xi', t)e^{i\alpha\cdot h\xi'}\hat{u}(\xi')d\xi' = \widehat{A_h(t)u}(\xi) \quad \text{a.e.,}
\end{aligned}$$

so that for $u \in \mathscr{S}$ we have in L_2

$$\sum_\alpha a_\alpha(x, t)T_h^\alpha u(x) = A_h(t)u(x). \tag{5.6}$$

It is clear that (5.6) holds for all $u \in L_2$, so that $\sum_\alpha a_\alpha(x, t)T_h^\alpha$ and $A_h(t)$ can be identified. Hence ψ is the restriction of ϕ to $\mathscr{A}$ and is a one-to-one mapping from $\mathscr{A}$ onto $\mathscr{A}_h$. We call $\sum_\alpha a_\alpha(x, t)\, e^{i\alpha\cdot\omega}$ the symbol of $\sum_\alpha a_\alpha(x, t)T_h^\alpha$.

Let $A_h(t), B_h(t) \in \mathscr{A}_h$ and let

$$A_h(t) = \sum_\alpha a_\alpha(x, t)T_h^\alpha, \quad B_h(t) = \sum_\beta b_\beta(x, t)T_h^\beta. \tag{5.7}$$

Then their symbols $a(x, t, \omega)$ and $b(x, t, \omega)$ are given by (5.1). Since $\mathscr{A}_h \subset \mathscr{K}_h$, the families $A_h(t) + B_h(t)$, $A_h(t) \circ B_h(t)$ and $A_h^\#(t)$ can be defined in $\mathscr{K}_h$. By (5.2)–(5.4) we have

$$A_h(t) + B_h(t) = \sum_\gamma (a_\gamma(x, t) + b_\gamma(x, t))T_h^\gamma, \tag{5.8}$$

$$A_h(t)\circ B_h(t) = \sum_\gamma (\sum_{\alpha+\beta=\gamma} a_\alpha(x, t) b_\beta(x, t)) T_h^\gamma, \tag{5.9}$$

$$A_h^\sharp(t) = \sum_\alpha a_\alpha^*(x, t) T_h^{-\alpha}. \tag{5.10}$$

Hence $\mathscr{A}_h$ is a subalgebra of $\mathscr{K}_h$ with involution and it follows that ψ and ψ^{-1} are morphisms.

LEMMA 5.1. *Let $F_{jh}(t) \in \mathscr{A}_h$ $(j=1, 2, \ldots, k)$ and let*

$$F_h(t) = F_{1h}(t) F_{2h}(t) \cdots F_{kh}(t), \quad L_h(t) = F_{1h}(t) \circ F_{2h}(t) \circ \cdots \circ F_{kh}(t). \tag{5.11}$$

Then $F_h(t) \equiv L_h(t)$ and $F_h^*(t) \equiv L_h^\sharp(t)$.

Let $\mathscr{F}_h$ be the subalgebra of $\mathscr{H}_h$ generated by $\mathscr{A}_h$. Then $F_h(t) \in \mathscr{F}_h$ can be expressed as

$$F_h(t) = \sum_r F_{1h}^{(r)}(t) F_{2h}^{(r)}(t) \cdots F_{kh}^{(r)}(t) \qquad (F_{jh}^{(r)}(t) \in \mathscr{A}_h). \tag{5.12}$$

Corresponding to this we put

$$L_h(t) = \sum_r F_{1h}^{(r)}(t) \circ F_{2h}^{(r)}(t) \circ \cdots \circ F_{kh}^{(r)}(t), \tag{5.13}$$

$$l(x, t, \omega) = \sum_r f_1^{(r)} f_2^{(r)} \cdots f_k^{(r)}, \tag{5.14}$$

where $f_j^{(r)}(x, t, \omega)$ is the symbol of $F_{jh}^{(r)}(t)$. Then $L_h(t) \in \mathscr{A}_h$, $F_h(t) \equiv L_h(t)$ and $l(x, t, \omega)$ is the symbol of $L_h(t)$. In the following we call $l(x, t, \omega)$ a symbol belonging to $F_h(t)$.

5.2. Algebra $\mathscr{G}_h$

Let $\mathscr{B}_0$ be the set of all $N \times N$ matrix functions $b(x, t, \mu)$ defined on $R_x^n \times J \times I_\infty$ with the properties:

1) $b(x, t, 0) \in \mathscr{A}_0$;
2) $b(x, t, \mu)$ can be written as

$$b(x, t, \mu) = b_0(x, t, \mu) + b_\infty(t, \mu),$$

where $b_0(x, t, \mu)$ and $b_\infty(t, \mu)$ are bounded and measurable on $R_x^n \times J$ for each μ and

$$\lim_{|x| \to \infty} b_0(x, t, \mu) = 0 \qquad \text{for each} \quad (t, \mu);$$

3) For each (t, μ) $b_0(x, t, \mu)$ is integrable as a function of x;
4) $\hat{b}_0(\chi, t, \mu)$ is integrable as a function of χ for each (t, μ);
5) There exists a constant $c \geqq 0$ such that

$$\int |\hat{b}_0(\chi, t, \mu) - \hat{b}_0(\chi, t, 0)| d\chi \leqq c\mu,$$

$$|b_\infty(t, \mu) - b_\infty(t, 0)| \leqq c\mu \qquad \text{for all} \quad t \in J, \ \mu \geqq 0.$$

For instance $\Delta_{j\mu}a(x, t)\,(j=1, 2,\dots, n)$ belong to $\mathscr{B}_0$ for $a(x, t) \in \mathscr{A}_0$.
We have

LEMMA 5.2. *Let $b(x, t, \mu) \in \mathscr{B}_0$ and let $B_h(t)$ be the family associated with $b(x, t, 0)e^{i\alpha\cdot\omega}$. Then $b(x, t, h)T_h^\alpha \in \mathscr{H}_h$ and*

$$b(x, t, h)T_h^\alpha \equiv B_h(t). \tag{5.15}$$

Let $\mathscr{B}_h$ be the set of all finite sums of families of the form $\sum_\alpha b_\alpha(x, t, h)T_h^\alpha$ $(b_\alpha(x, t, \mu) \in \mathscr{B}_0)$ and let $\mathscr{G}_h$ be the subalgebra of $\mathscr{H}_h$ generated by $\mathscr{B}_h$. It is clear that $\mathscr{A}_0 \subset \mathscr{B}_0$ and $\mathscr{F}_h \subset \mathscr{G}_h$.

Let $E_h(t, h) \in \mathscr{G}_h$. Then it can be expressed as

$$E_h(t, h) = \sum_r E_{1h}^{(r)}(t, h)E_{2h}^{(r)}(t, h)\cdots E_{kh}^{(r)}(t, h) \qquad (E_{jh}^{(r)}(t, h) \in \mathscr{B}_h), \tag{5.16}$$

where

$$E_{jh}^{(r)}(t, \mu) = \sum_\alpha e_{j\alpha}^{(r)}(x, t, \mu)T_h^\alpha \qquad (e_{j\alpha}^{(r)}(x, t, \mu) \in \mathscr{B}_0). \tag{5.17}$$

By the definition of $\mathscr{F}_h$ and by Lemma 5.2

$$E_h(t, 0) \in \mathscr{F}_h, \quad E_h(t, h) \equiv E_h(t, 0).$$

Thus we have

THEOREM 5.1. *Let $S_h(t, h)$ be the difference operator* (2.5) *with*

$$c_{\alpha m_j}(x, t, \mu) \in \mathscr{B}_0 \qquad (j = 1, 2,\dots, \nu). \tag{5.18}$$

Then

$$S_h(t, h) \in \mathscr{G}_h, \quad S_h(t, 0) \in \mathscr{F}_h.$$

Let $L_h(t)$ be the family associated with a symbol belonging to $S_h(t, 0)$. Then

$$L_h(t) \in \mathscr{A}_h, \quad S_h(t, h) \equiv S_h(t, 0) \equiv L_h(t).$$

By this theorem and Corollary 2.1, in proving the stability of the scheme (2.3) under the condition (5.18) the problem is to establish (2.21) for $L_h(t)$.

Let

$$s(x, t, \omega) = \sum_m \prod_{j=1}^{\nu} c_{m_j}(x, t, \omega), \tag{5.19}$$

where

$$c_{m_j}(x, t, \omega) = \sum_\alpha c_{\alpha m_j}(x, t, 0)e^{i\alpha\cdot\omega}, \quad c_{\alpha m_j}(x, t, \mu) \in \mathscr{B}_0. \tag{5.20}$$

Then $s(x, t, \omega)$ is a symbol belonging to $S_h(t, 0)$.

REMARK. The results obtained in Sections 2–5 are also valid when, for any $h_0>0$, the parameters h and μ are restricted to $(0, h_0]$ and $[0, h_0]$ respectively.

6. Stability of difference schemes

6.1. Assumptions and lemmas

Let

$$A(x, t, \omega) = \sum_{j=1}^{n} A_j(x, t)\omega_j \tag{6.1}$$

and let Δ_{jh} $(j=1, 2,\dots, n)$ be the difference operators such that $s_j(\omega)$ $(j=1, 2,\dots, n)$ satisfy (2.11).

We denote by ω' a point on the unit spherical surface in R^n_ω. Suppose the following conditions are satisfied:

CONDITION A. $A_j(x, t)$ $(j=1, 2,\dots, n)$ are bounded and continuous on $R^n_x \times J$ and can be written as

$$A_j(x, t) = A_{j0}(x, t) + A_{j\infty}(t) \qquad (j = 1, 2,\dots, n),$$

where $A_{j0}(x, t)$ converges to 0 uniformly on J as $|x|\to\infty$.

CONDITION B. 1) $D^m_l A_{j0}(x, t)$, $D^r_l \partial_t A_{j0}(x, t)$ and $\partial_t A_{j\infty}(t)$ $(j, l=1, 2,\dots, n;$ $m=0, 1,\dots, n+3;$ $r=0, 1,\dots, n+1)$ are bounded and continuous on $R^n_x\times J$;

2) $\int|D^m_l A_{j0}(x, t)|dx$ and $\int|D^r_l\partial_t A_{j0}(x, t)|dx$ $(j, l=1, 2,\dots, n;$ $m=0, 1,\dots, n+3;$ $r=0, 1,\dots, n+1)$ are bounded on J;

3) $\int_{|x|\geqq R}|D^r_l A_{j0}(x, t)|dx$ $(j, l=1, 2,\dots, n;$ $r=0, 1,\dots, n+1)$ converge to zero uniformly on J as $R\to\infty$.

CONDITION C. 1) Eigenvalues of $A(x, t, \omega')$ are all real and their multiplicities are independent of x, t and ω';

2) There exists a constant $\delta>0$ independent of x, t and ω' such that

$$|\lambda_i(x, t, \omega') - \lambda_j(x, t, \omega')| \geqq \delta \qquad (i \neq j;\ i, j = 1, 2,\dots, s),$$

where $\lambda_i(x, t, \omega')$ $(i=1, 2,\dots, s)$ are all the distinct eigenvalues of $A(x, t, \omega')$;

3) Elementary divisors of $A(x, t, \omega')$ are all linear.

By Corollary 4.2 $A_j(x, t)$ $(j=1, 2,\dots, n)$ belong to $\mathscr{A}_0$. Let

$$P_h(t) = \sum_{j=1}^{n} A_j(x, t)\Delta_{jh}, \tag{6.2}$$

$$p(x, t, \omega) = \sum_{j=1}^{n} A_j(x, t)s_j(\omega), \tag{6.3}$$

$$p_z(x, t, \omega) = \sum_{j=1}^{n} A_j(x, t)s_j(\omega)/|s(\omega)|, \tag{6.4}$$

$$e_r(x, t, \omega; \lambda) = \sum_{j=0}^{r} (i\lambda p)^j/j!. \tag{6.5}$$

Then $P_h(t) \in \mathscr{A}_h$ and $ip(x, t, \omega)$ is the symbol of $P_h(t)$. By Lemmas 4.6 and 4.7 $p_z(x, t, \omega)$ belongs to $\mathscr{L}$ and satisfies Condition N.

We have the following lemmas.

LEMMA 6.1. *There exists an element* $g(x, t, \omega)$ *of* $\mathscr{L}$ *satisfying the conditions of Theorem* 3.3 *such that*

$$\{g(x, t, \omega)p_z(x, t, \omega)\}^* = g(x, t, \omega)p_z(x, t, \omega) \qquad \textit{for} \quad \omega \in R^n_\omega - Z. \tag{6.6}$$

LEMMA 6.2. *There exist elements* $w(x, t, \omega)$ *and* $w^{-1}(x, t, \omega)$ *of* $\mathscr{L}$ *satisfying Condition* N *such that*

$$g(x, t, \omega) = w^*(x, t, \omega)w(x, t, \omega). \tag{6.7}$$

For $a \in \mathscr{K}$ we denote waw^{-1} by $\tilde{a}$. By these lemmas $\tilde{p}_z$ and $\tilde{p}$ are hermitian matrices on $R^n_x \times J \times (R^n_\omega - Z)$ and on $R^n_x \times J \times R^n_\omega$ respectively. By Lemma 3.4 $\tilde{p}_z$ satisfies Condition N and by Lemma 4.4 it belongs to $\mathscr{L}$.

In the following we assume that $S_h(t, h) \in \mathscr{G}_h$ and denote by $l(x, t, \omega; \lambda)$ a symbol belonging to $S_h(t, 0)$. Let the difference scheme (2.3) approximate (1.1) with accuracy of order r ($r \geqq 1$) and put

$$d = r + k, \quad k = \begin{cases} 1 & \text{if } r \text{ is odd,} \\ 2 & \text{if } r \text{ is even.} \end{cases} \tag{6.8}$$

We denote by λ_0, c_1 and c_2 positive constants and by $e(\omega)$ a scalar function such that $e(\omega)I \in \mathscr{K}$.

Let $P[\lambda; \mathscr{L}]$ be the set of all polynomials in λ of the form

$$a(x, t, \omega; \lambda) = \sum_{j=0}^{m} \lambda^j a_j(x, t, \omega), \quad a_j(x, t, \omega) \in \mathscr{L} \qquad (j = 0, 1, \dots, m),$$

and denote by $P[\lambda; p]$ the set of all polynomials in λ and $p(x, t, \omega)$. The set $P[\lambda; \mathscr{M}]$ is defined similarly. We use the notation

$$a(x, t, \omega)/e(\omega) = \sum_{j=0}^{m} \lambda^j a_j/e \in \mathscr{K} \quad (\text{or } \mathscr{L}, \mathscr{M}),$$

if $a_j(x, t, \omega)/e(\omega) \in \mathscr{K}$ (or $\mathscr{L}$, $\mathscr{M}$) ($j=0, 1, \dots, m$).

6.2. Stability theorems

We have the following theorems.

THEOREM 6.1. *Friedrichs' scheme is stable, if* $\lambda\rho(p_z(x, t, \omega))\leqq 1/\sqrt{n}$. *The modified Lax-Wendroff scheme is stable, if* $\lambda\rho(p_z(x, t, \omega))\leqq 2/\sqrt{n}$.

THEOREM 6.2. *Let* $l(x, t, \omega; \lambda)=e_r$, *where* $r=4m-1$ *or* $4m$ $(m\geqq 1)$. *Then the scheme* (2.3) *is stable for sufficiently small* λ.

THEOREM 6.3. *Let* $l(x, t, \omega; \lambda)=e_r-(\lambda p)^m v(\lambda p)^m$, *where* $r\geqq 2m$ $(m\geqq 1)$ *and* $v(x, t, \omega; \lambda)\in P[\lambda; \mathscr{L}]$. *Suppose*

1) $|s(\omega)|^\sigma \leqq c_1 e(\omega)$;

2) $v_1(x, t, \omega; \lambda) = v/e \in \mathscr{K}$;

3) $u(x, t, \omega; \lambda) \geqq c_2 e(\omega)I \quad$ *for* $\lambda \leqq \lambda_0$,

where $\sigma=d-2m$ *and* $u=\tilde{v}^*+\tilde{v}-\tilde{v}^*(\lambda\tilde{p})^{2m}\tilde{v}$. *Then the scheme* (2.3) *is stable for sufficiently small* λ.

THEOREM 6.4. *Let*

$$l(x, t, \omega; \lambda) = e_r - (i\lambda p)^{2m+1}a - (\lambda p)^{m+1}v(\lambda p)^{m+1}, \tag{6.9}$$

where $r\geqq 2m+2$ $(m\geqq 0)$, $v(x, t, \omega; \lambda)\in P[\lambda; \mathscr{L}]$ *and* $a(\omega)$ *is a real-valued scalar function such that* $a(\omega)I\in\mathscr{L}$ *and* $(a(\omega)/e(\omega))I\in\mathscr{K}$. *Suppose conditions* 1), 2) *and* 3) *of Theorem* 6.3 *are satisfied, where* $\sigma=d-2m-2$,

$$u = \tilde{v}^* + \tilde{v} + (-1)^m 2aI - \tilde{b}^*(\lambda\tilde{p})^{2m}\tilde{b}, \quad b = (-1)^m(ia) + \lambda p v.$$

Then the scheme (2.3) *is stable for sufficiently small* λ.

COROLLARY 6.1. *Let* $l(x, t, \omega; \lambda)=e_r-(i\lambda p)^{r-1}e$, *where* $r=4m+1$ *or* $4m+2$ $(m\geqq 1)$. *Suppose* $e(\omega)$, $\partial_j e(\omega)$ *and* $\partial_k\partial_j e(\omega)$ $(j, k=1, 2,\dots, n)$ *are bounded and continuous on* R^n_ω *and* $|s(\omega)|^2\leqq c_1 e(\omega)$. *Then the scheme* (2.3) *is stable for sufficiently small* λ.

THEOREM 6.5. *Let* $l(x, t, \omega; \lambda)=e_r-\lambda^{2m}v$, *where* $r\geqq 2m$ $(m\geqq 0, r\geqq 1)$,

$$v(x, t, \omega; \lambda) = a + \lambda^\alpha b \qquad (\alpha \geqq 0),$$

$$a(x, t, \omega; \lambda)\in P[\lambda; \mathscr{L}], \quad b(x, t, \omega; \lambda)\in P[\lambda; \mathscr{L}],$$

$$a_1(x, t, \omega; \lambda) = a/|s|^2 \in \mathscr{L}, \quad b_1(x, t, \omega; \lambda) = b/|s| \in \mathscr{L}.$$

Suppose

1) $\tilde{b}^* + \tilde{b} = 0$;

2) $|s(\omega)|^{d-2} \leqq c_1 e(\omega)$;

3) $a_2(x, t, \omega; \lambda) = a_1/e \in \mathscr{K}, \quad b_2(x, t, \omega; \lambda) = b_1/e \in \mathscr{K}$;

4) $u(x, t, \omega; \lambda) \geqq c_2 e|s|^2 I \qquad for \quad \lambda \leqq \lambda_0,$

where $u = \tilde{a}^* + \tilde{a} - \lambda^{2m}\tilde{v}^*\tilde{v}$. *Then the scheme* (2.3) *is stable for sufficiently small* λ.

THEOREM 6.6. *Let* $l(x, t, \omega; \lambda) = e_r - \lambda^\alpha v$, *where*

$$v(x, t, \omega; \lambda) = mI + \lambda^\beta a + \lambda^\gamma b \qquad (\beta, \gamma \geqq 0),$$

$$m(\omega; \lambda) = \textstyle\sum_{j=0}^{\mu} \lambda^j m_j(\omega) I, \qquad \gamma \geqq \alpha \geqq 0,$$

$$a(x, t, \omega; \lambda) \in P[\lambda; \mathscr{M}], \quad b(x, t, \omega; \lambda) \in P[\lambda; \mathscr{M}],$$

$$a_1(x, t, \omega; \lambda) = a/|s| \in \mathscr{M}, \quad b_1(x, t, \omega; \lambda) = b/|s| \in \mathscr{M},$$

$m_j(\omega)$ $(j = 0, 1,\dots, \mu)$ *are scalar functions satisfying Condition* I. *Suppose*

1) $\tilde{b}^* + \tilde{b} = 0$;

2) $e(\omega)$ *satisfies Condition* I;

3) $|s(\omega)|^d \leqq c_1 e^2(\omega), \quad |m_j(\omega)| \leqq c_1 e^2(\omega) \qquad (j = 0, 1,\dots, \mu)$;

4) $a_2(x, t, \omega; \lambda) = a/e^2 \in \mathscr{K}$, $b_2(x, t, \omega; \lambda) = b|s|/e^2 \in \mathscr{K}$ *and* a_2, b_1 *and* b_2 *satisfy Conditions* N *and* II;

5) $u(x, t, \omega; \lambda) \geqq c_2 e^2 I \qquad for \quad \lambda \leqq \lambda_0,$

where $u = (m^* + m)I + \lambda^\beta(\tilde{a}^* + \tilde{a}) - \lambda^\alpha \tilde{v}^*\tilde{v}$. *Then the scheme* (2.3) *is stable for sufficiently small* λ.

THEOREM 6.7. *For a regularly hyperbolic system with real coefficients let*

$$l(x, t, \omega; \lambda) = I + i\lambda p(x, t, \omega) + \lambda^2 q(x, t, \omega; \lambda)|s(\omega)|^2, \tag{6.10}$$

where q *is a polynomial in* λ *with coefficients satisfying Condition* VI. *Suppose*

$$\rho(l(x, t, \omega; \lambda)) \leqq 1 \qquad for \quad \lambda \leqq \lambda_0. \tag{6.11}$$

Then the scheme (2.3) *is stable for sufficiently small* λ.

7. Examples of schemes

In this section Conditions A, B and C are assumed. To construct difference schemes with accuracy of order r $(r = 3, 4)$, we assume that $\partial_t^q A_{j0}(x, t)$ and

$\partial_t^q A_{j\infty}(t)$ $(q=0, 1,\ldots, r-1; j=1, 2,\ldots, n)$ are bounded and continuous on $R_x^n \times J$ together with their partial derivatives up to the $(n+3)rd$ order with respect to x and that $\int |D_l^m \partial_t^q A_{j0}(x, t)|dx$ $(j, l=1, 2,\ldots, n;\ m=0, 1,\ldots, n+3;\ q=0, 1,\ldots, r-1)$ are bounded on J.

We introduce the following difference operators:

$$\Delta_{1jh} = (T_{jh} - T_{jh}^{-1})/2, \quad \Delta_{2jh} = [8(T_{jh} - T_{jh}^{-1}) - (T_{jh}^2 - T_{jh}^{-2})]/12,$$

$$\delta_{jh} = (T_{jh} + T_{jh}^{-1} - 2I)/4 \qquad (j = 1, 2,\ldots, n),$$

$$P_{mh}(t) = \textstyle\sum_{j=1}^n A_j(x, t)\Delta_{mjh} \qquad (m = 1, 2),$$

$$K_{1h}(t, \mu) = F_{1h}(t, \mu) + 4\textstyle\sum_{j=1}^n A_j^2\delta_{jh},$$

$$K_{2h}(t, \mu) = F_{2h}(t, \mu) + 4\textstyle\sum_{j=1}^n A_j^2\delta_{jh}(1 - \delta_{jh}/3),$$

$$L_h(t, \mu) = F_{2h}(t, \mu) + \textstyle\sum_{j=1}^n A_j^2\Delta_{1jh}^2(1 - 4\delta_{jh}/3),$$

$$E_{1h} = \textstyle\sum_{j=1}^n \Delta_{1jh}^2 \sum_{k=1}^n \delta_{kh}/n^2, \quad E_{2h} = \sum_{j=1}^n \delta_{jh}^2/n,$$

$$E_{3h} = \textstyle\sum_{j=1}^n \Delta_{1jh}^2 \sum_{k=1}^n \delta_{kh}^2/n^2, \quad E_{4h} = \sum_{j=1}^n \delta_{jh}^3/n,$$

$$W_{1h}(t, h) = M_{1h}(t, h), \quad W_{2h}(t, h) = M_{2h}(t, h) + \lambda^2 G_h(t, h)/24,$$

where

$$M_{mh}(t, h) = Q_{mh}(t)/2 + \lambda\{2Q_{1h}(t)P_{mh}(t) + P_{mh}(t)Q_{1h}(t) + hR_h(t)\}/6,$$

$$F_{mh}(t, \mu) = \textstyle\sum_{j\neq k} A_j\Delta_{mjh}(A_k\Delta_{mkh}) + \sum_{j=1}^n A_j(\Delta_{mj\mu}A_j)\Delta_{mjh} \qquad (m = 1, 2),$$

$$G_h(t, h) = (P_{1h}(t))^2 Q_{1h}(t) + 2P_{1h}(t)Q_{1h}(t)P_{1h}(t) + 3Q_{1h}(t)(P_{1h}(t))^2$$
$$+ h\{P_{1h}(t)R_h(t) + 3(Q_{1h}(t))^2 + 3R_h(t)P_{1h}(t)\} + h^2V_h(t),$$

$$Q_{mh}(t) = \textstyle\sum_{j=1}^n (\partial_t A_j(x, t))\Delta_{mjh} \qquad (m = 1, 2),$$

$$R_h(t) = \textstyle\sum_{j=1}^n (\partial_t^2 A_j(x, t))\Delta_{1jh}, \quad V_h(t) = \sum_{j=1}^n (\partial_t^3 A_j(x, t))\Delta_{1jh}.$$

Since by Corollary 4.2 $\partial_t^q A_j(x, t) \in \mathscr{A}_0$ and $\Delta_{mj\mu}A_j(x, t) \in \mathscr{B}_0$ $(j=1, 2,\ldots, n;$ $q=0, 1,\ldots, r-1;\ m=1, 2)$, $P_{mh}(t)$ $(m=1, 2)$ belong to $\mathscr{A}_h$ and $F_{mh}(t, h)$, $K_{mh}(t, h)$, $hW_{mh}(t, h)$ $(m=1, 2)$ and $L_h(t, h)$ belong to $\mathscr{G}_h$.

We consider the following difference operators:

$$(7.1) \qquad S_h(t) = I - E_{1h} + \lambda P_{2h}(t) + \lambda^2 P_{2h}(t)P_{1h}(t)/2 + (\lambda P_{1h}(t))^3/6$$
$$+ \lambda^2 h W_{1h}(t, h),$$

$$(7.2) \qquad S_h(t, h) = I - E_{2h} + \lambda P_{2h}(t) + (\lambda P_{1h}(t))^2/2 + \lambda^3 K_{1h}(t, h)P_{1h}(t)/6$$

$$+ \lambda^2 h W_{1h}(t, h),$$

$$(7.3) \qquad S_h(t, h) = I + E_{3h} + \lambda\{I + \lambda P_{2h}(t)/2 + \lambda^2 L_h(t, h)/6 + (\lambda P_{1h}(t))^3/24\} P_{2h}(t) + \lambda^2 h W_{2h}(t, h),$$

$$(7.4) \qquad S_h(t, h) = I + E_{4h} + \lambda\{I + \lambda P_{2h}(t)/2 + \lambda^2 K_{2h}(t, h)/6 + \lambda^3 K_{1h}(t, h) P_{1h}(t)/24\} P_{2h}(t) + \lambda^2 h W_{2h}(t, h).$$

Then by Theorems 6.5 and 6.6 the schemes (2.3) with the operators (7.1)–(7.4) are stable for sufficiently small λ.

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