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Classification Theory for Nonlinear Functional-Harmonic Spaces

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Introduction

In the classical classification theory of Riemann surfaces, the basic relations involving classes of harmonic functions are given by

$$O_G \subsetneqq O_{HP} \subsetneqq O_{HB} \subsetneqq O_{HD} = O_{HDB} \tag{1}$$

(see, e.g., [11] for notation and detailed account of the classical classification theory). The same relations have been shown to hold for the class H of solutions of the equation of the form

$$\Delta u = Pu \qquad (P \geqq 0) \tag{2}$$

on, in general, Riemannian manifolds Ω; furthermore, for the solutions of (2), additional relations

$$O_{HD} \subsetneqq O_{HE} = O_{HBE} \tag{3}$$

hold, where E indicates the finiteness of the energy integral

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega Pu^2 dx \qquad (dx\text{: the volume element}) \tag{4}$$

(see, e.g., [9], [5]).

Here, we note that (2) is the Euler equation of the variational integral (4). Thus we may generalize the above situation as follows. For simplicity, consider the case where Ω is a domain in the euclidean space $\mathbf{R}^d$. Suppose the "Dirichlet integral" of a function f is given in the form

$$D[f] = \int_\Omega \psi(x, \nabla f(x))dx \tag{5}$$

with a function $\psi(x, \tau)$: $\Omega \times \mathbf{R}^d \to \mathbf{R}$ which is non-negative and convex in τ, and the "energy" of f is given by

$$E[f] = D[f] + \int_\Omega \Gamma(x, f(x))dx \tag{6}$$

with another non-negative function $\Gamma(x, t)\colon \Omega\times\mathbf{R}\to\mathbf{R}$. The Euler equation for the variational integral (6) is formally written as

$$-\operatorname{div} \nabla_\tau\psi(x, \nabla u(x)) + \Gamma'_t(x, u(x)) = 0, \tag{7}$$

which is an elliptic quasi-linear equation.

Let H be the class of all "weak solutions" of (7) on Ω. Then, we may consider classes HP, HB, HD, HE, etc. as in the classical case, where P means the positivity, B the boundedness, D (resp. E) the finiteness of $D[u]$ (resp. $E[u]$) which is given by (5) (resp. (6)). Also, O_G may be replaced by O_{SHP}, where SH means the class of "supersolutions" of (7). In this way, we can pose a problem to find relations among null classes appearing in (1) and (3) in our general situation.

The same type of problem may be considered also for infinite networks; cf. [13] in which the class O_G is discussed for a non-linear case. Thus, we shall try to construct a theory on general locally compact spaces Ω. Given Ω, we fix a positive measure ξ on Ω and instead of ψ as described above we abstractly consider a convex mapping Ψ of a subspace $\mathbf{X}$ of $L^\infty_{loc}(\Omega; \xi)$ into $L^1_{loc}(\Omega; \xi)$ such that $\Psi(f)\geqq 0$ for all $f\in\mathbf{X}$, $\Psi(c)=0$ for constants c and Ψ has local property. Given $\Gamma\colon \Omega\times\mathbf{R}\to\mathbf{R}$ as above, we obtain a configuration $\mathfrak{H}=\{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\}$. Such a configuration may be regarded as a non-linear functional space (cf. [7]), which is of local type.

In order to obtain a satisfactory theory, we shall place several conditions under which (weak) solutions of the Euler equation corresponding to the variational integral

$$\int_\Omega \Psi(f)d\xi + \int_\Omega \Gamma(\cdot, f)d\xi$$

behave like classical harmonic functions, or, at least satisfy some of the properties which are assumed in the theory of (non-linear) harmonic spaces (cf. [1]). Thus we shall call $\mathfrak{H}$ a functional-harmonic space, or simply an FH-space.

We shall see that the relation $O_{HB}\subset O_{HD}$ cannot be expected for a general class of FH-spaces; in fact we shall see (in §6 and §7) that there are no inclusion relations between O_{HP} and O_{HD}. In §4 and §5, we give restricted classes of FH-spaces for which (1) and (3) are valid. Essential condition for an FH-space to belong to this class is the so called Orlicz' (Δ_2)-condition: $\Psi(2f)\leqq C\Psi(f)$ (C: const.).

As special cases, we treat infinite networks in §6 and the case where Ω is a differentiable manifold in §7.

§ 1. Functional spaces

Let Ω be a locally compact Hausdorff space which is connected, σ-compact and non-compact. We consider a positive Radon measure ξ on Ω whose support is the whole space Ω.

All functions considered in this paper are real-valued ξ-measurable functions on Ω and two functions which are equal ξ-a.e. are identified. Thus, for a ξ-measurable set A in Ω, "$f \geqq g$ on A" (resp. "$f=g$ on A") means that $f(x) \geqq g(x)$ (resp. $f(x)=g(x)$) for almost all $x \in A$ with respect to ξ. For a function f on Ω, let $Supp f$ denote the support of the measure $f d\xi$. We denote by $L^p_{loc}(\Omega)$ $(1 \leqq p \leqq \infty)$ the ordinary Lebesgue classes with respect to ξ.

We consider a space $\mathbf{X}$ of functions on Ω satisfying:

(X.1) $\mathbf{X}$ is a linear subspace of $L^\infty_{loc}(\Omega)$ containing all constant functions;
(X.2) $\mathbf{X}$ is closed under max. and min. operations.

Next, we introduce a mapping Ψ: $\mathbf{X} \to L^1_{loc}(\Omega)$ satisfying the following conditions:

(Ψ.1) $\Psi(c)=0$ for all constant functions c;
(Ψ.2) $\Psi(-f)=\Psi(f)$ for all $f \in \mathbf{X}$;
(Ψ.3) (Local property) $\Psi(f)=\Psi(g)$ on the set $\{x \in \Omega | f(x)=g(x)\}$;
(Ψ.4) Ψ is convex on $\mathbf{X}$, i.e.,

$$\Psi(tf+(1-t)g) \leqq t\Psi(f)+(1-t)\Psi(g)$$

for $t \in [0, 1]$, f, $g \in \mathbf{X}$; the equality holds for some (and hence for all) $0<t<1$ only when $f=g+\text{const.}$;
(Ψ.5) For any f, $g \in \mathbf{X}$, there is $\nabla\Psi(f; g) \in L^1_{loc}(\Omega)$ such that

$$\lim_{t \to 0} \frac{\Psi(f+tg)-\Psi(f)}{t} = \nabla\Psi(f; g) \qquad \text{a.e. on } \Omega. \tag{1.1}$$

REMARK. By convexity of Ψ, $\nabla\Psi(f; g)$ is uniquely determined by f and g, and Lebesgue's convergence theorem implies that the limit (1.1) can be taken in the topology of $L^1_{loc}(\Omega)$.

Finally, we consider a mapping Γ: $\Omega \times \mathbf{R} \to \mathbf{R}$ ($\mathbf{R}$: the real numbers) satisfying:

(Γ.1) $\Gamma(x, t) \geqq 0$, $\Gamma(x, 0)=0$ and $\Gamma(x, -t)=\Gamma(x, t)$ for all $x \in \Omega$, $t \in \mathbf{R}$;
(Γ.2) For each $x \in \Omega$, $\Gamma(x, t)$ is convex and continuously differentiable in $t \in \mathbf{R}$; $\frac{\partial \Gamma}{\partial t}(x, t)$ will be denoted by $\Gamma'(x, t)$;
(Γ.3) For each $t \in \mathbf{R}$, $\Gamma'(\cdot, t) \in L^1_{loc}(\Omega)$.

We call $\mathfrak{H}=\{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\}$ a *functional space* if $\mathbf{X}$, Ψ, Γ satisfy the above conditions. By (Γ.3), we see that $\Gamma'(\cdot, f)\in L^1_{loc}(\Omega)$ for any $f\in\mathbf{X}$, where

$$\Gamma'(\cdot, f)(x) = \Gamma'(x, f(x)).$$

Given a ξ-measurable set A in Ω, $u\in X$ is said to be *totally $\mathfrak{H}$-harmonic* (resp. *totally $\mathfrak{H}$-superharmonic*) on A if

$$\int_\Omega \nabla\Psi(u; g)d\xi + \int_\Omega \Gamma'(\cdot, u)gd\xi = 0 \qquad (\text{resp.} \geqq 0) \tag{1.2}$$

for any $g\in\mathbf{X}$ such that $Supp\, g$ is compact, $g\geqq 0$ on Ω and $g=0$ on $\Omega\setminus A$. In case $A=\Omega$, we shall omit the word "totally". The equality (1.2) gives the Euler equation for the variational integral

$$\int\Phi_{\mathfrak{H}}(f)d\xi = \int\Psi(f)d\xi + \int\Gamma(\cdot, f)d\xi.$$

Here

$$\Phi_{\mathfrak{H}}(f) = \Psi(f) + \Gamma(\cdot, f)$$

belongs to $L^1_{loc}(\Omega)$ for any $f\in\mathbf{X}$ by virtue of (Γ.3) and the equality $\Gamma(x, t)=\int_0^t \Gamma'(x, s)ds$.

LEMMA 1.1. (a) *$\Psi(f)\geqq 0$ for all $f\in\mathbf{X}$.*

(b) *$\Psi(f)=0$ if and only if $f=const.$*

(c) *$t\mapsto\Psi(tf)$ is monotone non-decreasing for $t\geqq 0$.*

(d) *$\Psi(f+c)=\Psi(f)$ for $f\in\mathbf{X}$ and constants c.*

(e) *$g\mapsto\nabla\Psi(f; g)$ is linear.*

(f) *$\nabla\Psi(f; f-g)\geqq\Psi(f)-\Psi(g)$; in particular $\nabla\Psi(f; f)\geqq\Psi(f)$.*

(g) *$\nabla\Psi(f; f-g)\geqq\nabla\Psi(g; f-g)$; the equality holds only when $f=g+const.$*

(h) *$\nabla\Psi(c; g)=\nabla\Psi(f; c)=0$ for f, $g\in\mathbf{X}$ and constants c.*

(i) *$\nabla\Psi(f; g)=0$ on the set $\{x\in\Omega | g(x)=0\}$.*

(j) *$\nabla\Psi(f_1; g)=\nabla\Psi(f_2; g)$ on the set $\{x\in\Omega | f_1(x)=f_2(x)+c\}$ for any constant c.*

PROOF. (a), (b) and (c) are easy consequences of (Ψ.1), (Ψ.2) and (Ψ.4); and (e), (f) and (g) follow from well-known properties of convex functions (cf. [4, Chap. I, §5]). By (Ψ.1) and (Ψ.4), $\Psi(f+c)\leqq t\Psi(t^{-1}f)$ for $0<t<1$. For any relatively compact ξ-measurable set A, $s\mapsto\int_A\Psi(sf)d\xi$ is a convex function on $\mathbf{R}$, and hence it is continuous. Hence, letting $t\to 1$, we obtain $\Psi(f+c)\leqq\Psi(f)$. Then (d) follows immediately. (h), (i) and (j) are consequences of (d) and (Ψ.3).

The next lemma is an immediate consequence of (Γ.1) and (Γ.2):

LEMMA 1.2. *For each* $x \in \Omega$, $\Gamma'(x, t)$ *is monotone non-decreasing in* $t \in \mathbf{R}$, $\Gamma'(x, t) \geqq 0$ *for* $t \geqq 0$ *and* $\Gamma'(x, t) \leqq 0$ *for* $t \leqq 0$.

PROPOSITION 1.1. *Let* A, A' *be* ξ*-measurable sets in* Ω.

(a) *If* u *is totally* $\mathfrak{H}$*-harmonic (resp.* $\mathfrak{H}$*-superharmonic) on* A *and* $v = u$ *on* $A' \subset A$, *then* v *is totally* $\mathfrak{H}$*-harmonic (resp.* $\mathfrak{H}$*-superharmonic) on* A'.

(b) *Non-negative constant functions are* $\mathfrak{H}$*-superharmonic on* Ω.

(c) *If* u *is totally* $\mathfrak{H}$*-superharmonic on* A, *then so is* $u + c$ *for any non-negative constant function* c.

PROOF. (a) is easily seen from the definition and Lemma 1.1 (j). (b) and (c) follow from Lemma 1.1 (h), (j) and Lemma 1.2.

PROPOSITION 1.2. *Let* A *be a relatively compact* ξ*-measurable set in* Ω. *If* u *and* $-v$ *are totally* $\mathfrak{H}$*-superharmonic on* A *and* $u \geqq v$ *on* $\Omega \setminus A$, *then* $u \geqq v$ *on* Ω.

PROOF. For simplicity, let $\Phi = \Phi_{\mathfrak{H}}$ and $\nabla\Phi(f; g) = \nabla\Psi(f; g) + \Gamma'(\cdot, f)g$. Take $g = v - \min(u, v)$. Then $g \in \mathbf{X}$, $g \geqq 0$ on Ω and $g = 0$ on $\Omega \setminus A$. Hence

$$\int_\Omega \nabla\Phi(u; g)d\xi \geqq 0 \quad \text{and} \quad \int_\Omega \nabla\Phi(v; g)d\xi \leqq 0.$$

On the other hand, by Lemma 1.1 (g), (i), (j) and Lemma 1.2,

$$0 \leqq \nabla\Phi(v; g) - \nabla\Phi(\min(u, v); g) = \nabla\Phi(v; g) - \nabla\Phi(u; g)$$

on Ω. Hence

$$0 \leqq \int_\Omega \{\nabla\Phi(v; g) - \nabla\Phi(\min(u, v); g)\}d\xi$$

$$= \int_\Omega \nabla\Phi(v; g)d\xi - \int_\Omega \nabla\Phi(u; g)d\xi \leqq 0,$$

so that $\nabla\Phi(v; g) = \nabla\Phi(\min(u, v); g)$. It follows that $\nabla\Psi(v; g) = \nabla\Psi(\min(u,v); g)$. Hence, by Lemma 1.1 (g), $v = \min(u, v) + c$ (const.). Since $v = \min(u, v)$ on $\Omega \setminus A$ and $\xi(\Omega \setminus A) > 0$, $c = 0$. Hence, $u \geqq v$ on Ω.

COROLLARY. *Let* A *be as in the above proposition. If* u, v *are totally* $\mathfrak{H}$*-harmonic on* A *and* $u = v$ *on* $\Omega \setminus A$, *then* $u = v$.

PROPOSITION 1.3. *Let* A, A' *be* ξ*-measurable subsets of* Ω *such that* $A \subset A'$. *If* u *is totally* $\mathfrak{H}$*-superharmonic on* A', v *is totally* $\mathfrak{H}$*-superharmonic on* A, $u = v$ *on* $A' \setminus A$ *and* $u \geqq v$ *on* A, *then* v *is totally* $\mathfrak{H}$*-superharmonic on* A'.

PROOF. Let $\nabla\Phi(f; g)$ be as in the proof of the previous proposition. Let $g \in \mathbf{X}$ be such that $Supp\, g$ is compact, $g \geqq 0$ on Ω, $g = 0$ on $\Omega \setminus A'$. For each $\rho > 0$, put $g_\rho = \min(g, \rho(u-v)^+)$, where $f^+ = \max(f, 0)$. Then $g_\rho \in \mathbf{X}$, $Supp\, g_\rho$ is compact, $g_\rho \geqq 0$ on Ω and $g_\rho = 0$ on $\Omega \setminus A$. Hence

$$\int_\Omega \nabla\Phi(v; g_\rho)d\xi \geqq 0. \tag{1.3}$$

Put $A_\rho = \{x \in \Omega \mid g(x) > \rho(u-v)^+(x)\}$. Then $A_\rho \subset A'$, $g = g_\rho$ on $\Omega \setminus A_\rho$ and $g_\rho = \rho(u-v)$ on A_ρ. Hence, using (1.3), Lemmas 1.1 and 1.2, we obtain

$$\begin{aligned}
\int_\Omega \nabla\Phi(v; g)d\xi &\geqq \int_\Omega \nabla\Phi(v; g)d\xi - \int_\Omega \nabla\Phi(v; g_\rho)d\xi \\
&= \int_{A_\rho} \nabla\Phi(v; g)d\xi - \rho \int_{A_\rho} \nabla\Phi(v; u-v)d\xi \\
&\geqq \int_{A_\rho} \nabla\Phi(v; g)d\xi - \rho \int_{A_\rho} \nabla\Phi(u; u-v)d\xi \\
&= \int_{A_\rho} \nabla\Phi(v; g)d\xi - \int_{A_\rho} \nabla\Phi(u; g_\rho)d\xi.
\end{aligned}$$

Since u is totally $\mathfrak{H}$-superharmonic on A_ρ, $g - g_\rho \geqq 0$ on Ω and $g - g_\rho = 0$ on $\Omega \setminus A_\rho$, we have

$$\int_{A_\rho} \nabla\Phi(u; g)d\xi \geqq \int_{A_\rho} \nabla\Phi(u; g_\rho)d\xi.$$

Therefore,

$$\begin{aligned}
\int_\Omega \nabla\Phi(v; g)d\xi &\geqq \int_{A_\rho} \{\nabla\Phi(v; g) - \nabla\Phi(u; g)\}d\xi \\
&= \int_{A_\rho \cap A^+} \{\nabla\Phi(v; g) - \nabla\Phi(u; g)\}d\xi,
\end{aligned}$$

where $A^+ = \{x \in A' \mid u(x) > v(x)\}$. Since $\nabla\Phi(v; g) - \nabla\Phi(u; g)$ is ξ-summable on $Supp\, g$ and $A_\rho \cap A^+ \downarrow \emptyset$ $(\rho \to \infty)$, the last integral tends to 0 as $\rho \to \infty$. Thus $\int_\Omega \nabla\Phi(v; g)d\xi \geqq 0$, and hence v is totally $\mathfrak{H}$-superharmonic on A'.

COROLLARY. *Let A be a ξ-measurable subset of Ω. If u, v are totally $\mathfrak{H}$-superharmonic on A, then so is* $\min(u, v)$.

PROOF. Put $w = \min(u, v)$ and $A_1 = \{x \in A \mid u(x) > v(x)\}$. Then $w = v$ on A_1, so that w is totally $\mathfrak{H}$-superharmonic on A_1. Since $w = u$ on $A \setminus A_1$ and $w \leqq u$ on A_1, the above proposition implies that w is totally $\mathfrak{H}$-superharmonic on A.

§2. Functional-harmonic spaces and classification I

Let $\mathfrak{H} = \{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\}$ be a functional space. A relatively compact ξ-

measurable set A in Ω will be said to be *resolutive* (with respect to $\mathfrak{H}$) if for any $f \in \mathbf{X}$ there exists $g_0 \in \mathbf{X}$ such that

$$g_0 = f \quad \text{on} \quad \Omega \setminus A, \quad \text{and} \tag{2.1}$$

$$\int_A \Phi_{\mathfrak{H}}(g_0) d\xi = \inf \left\{ \int_A \Phi_{\mathfrak{H}}(g) d\xi \,\middle|\, g \in \mathbf{X},\ g = f \text{ on } \Omega \setminus A \right\}. \tag{2.2}$$

The following proposition shows that $g_0 \in \mathbf{X}$ satisfying (2.1) and (2.2) is uniquely determined by f and A; we shall denote it by $R(f; A)$.

PROPOSITION 2.1. *Given a relatively compact ξ-measurable set A in Ω and $f \in \mathbf{X}$, there is at most one $g_0 \in \mathbf{X}$ satisfying* (2.1) *and* (2.2). *Furthermore, g_0 has the following properties* (*if it exists*):

(a) *g_0 is totally $\mathfrak{H}$-harmonic on A;*

(b) *If f is totally $\mathfrak{H}$-superharmonic on A, then $g_0 \leqq f$;*

(c) *If f is totally $\mathfrak{H}$-superharmonic on a ξ-measurable set A' containing A, then so is g_0 on A';*

(d) $\min(0, \inf_{\Omega \setminus A} f) \leqq g_0 \leqq \max(0, \sup_{\Omega \setminus A} f)$.

PROOF. Let $\Phi = \Phi_{\mathfrak{H}}$ and $\nabla\Phi(f; g) = \nabla\Psi(f; g) + \Gamma'(\cdot, f)g$. For any $g \in \mathbf{X}$ such that $g = 0$ on $\Omega \setminus A$ and for any $t \in \mathbf{R}$,

$$\int_A \Phi(g_0) d\xi \leqq \int_A \Phi(g_0 + tg) d\xi.$$

It follows that $\int_A \nabla\Phi(g_0; g) d\xi = 0$, or $\int_\Omega \nabla\Phi(g_0; g) d\xi = 0$. Hence g_0 is totally $\mathfrak{H}$-harmonic on A. Thus the uniqueness of g_0 follows from the corollary to Proposition 1.2. Property (b) is a consequence of Proposition 1.2, and property (c) follows from (b) and Proposition 1.3. To prove (d), put

$$m = \min(0, \inf_{\Omega \setminus A} f), \qquad M = \max(0, \sup_{\Omega \setminus A} f)$$

and

$$g_1 = \max(m, \min(g_0, M)).$$

Then $g_1 \in \mathbf{X}$, $g_1 = f$ on $\Omega \setminus A$ and $\int_A \Phi(g_1) d\xi \leqq \int_A \Phi(g_0) d\xi$. Hence, by the uniqueness of g_0, $g_1 = g_0$, so that (d) is valid.

Now, we consider the following conditions for $\mathfrak{H}$:

(R) There is an exhaustion $\{\Omega_n\}$ of Ω such that each Ω_n is a resolutive open set in Ω.

Here, an exhaustion means a sequence $\{\Omega_n\}$ of relatively compact open sets such that $\bar{\Omega}_n \subset \Omega_{n+1}$ for each n and $\cup \Omega_n = \Omega$.

(H.1) If $\{u_n\}$ is a locally uniformly bounded, monotone non-decreasing sequence of non-negative functions in $\mathbf{X}$ such that each u_n is totally $\mathfrak{H}$-harmonic on Ω_n for some exhaustion $\{\Omega_n\}$ of Ω, then $u = \lim_{n\to\infty} u_n$ is $\mathfrak{H}$-harmonic on Ω and

$$\int_K \Psi(u) d\xi \leqq \liminf_{n\to\infty} \int_K \Psi(u_n) d\xi$$

for any compact set K in Ω.

A functional space $\mathfrak{H}$ is called a *functional-harmonic space*, or simply an *FH-space* if it satisfies (R) and (H.1). The class of all FH-spaces will be denoted by $\mathscr{F}$.

Given $\mathfrak{H} = \{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\} \in \mathscr{F}$, we consider the following sets of functions:

$$SH(\mathfrak{H}) = \{u \in \mathbf{X} \mid u \text{ is } \mathfrak{H}\text{-superharmonic on } \Omega\},$$

$$H(\mathfrak{H}) = \{u \in \mathbf{X} \mid u \text{ is } \mathfrak{H}\text{-harmonic on } \Omega\},$$

$$SHP(\mathfrak{H}) = \{u \in SH(\mathfrak{H}) \mid u \geqq 0 \text{ on } \Omega\},$$

$$HP(\mathfrak{H}) = H(\mathfrak{H}) \cap SHP(\mathfrak{H}),$$

$$SHB(\mathfrak{H}) = \{u \in SH(\mathfrak{H}) \mid u \text{ is bounded on } \Omega\},$$

$$HB(\mathfrak{H}) = H(\mathfrak{H}) \cap SHB(\mathfrak{H}),$$

$$HD(\mathfrak{H}) = \left\{u \in H(\mathfrak{H}) \,\middle|\, \int_\Omega \Psi(u) d\xi < \infty\right\},$$

$$HDP(\mathfrak{H}) = HD(\mathfrak{H}) \cap HP(\mathfrak{H}),$$

$$HDB(\mathfrak{H}) = HD(\mathfrak{H}) \cap HB(\mathfrak{H}),$$

$$HE(\mathfrak{H}) = \left\{u \in H(\mathfrak{H}) \,\middle|\, \int_\Omega \Phi_{\mathfrak{H}}(u) d\xi < \infty\right\},$$

$$HEP(\mathfrak{H}) = HE(\mathfrak{H}) \cap HP(\mathfrak{H}),$$

$$HEB(\mathfrak{H}) = HE(\mathfrak{H}) \cap HB(\mathfrak{H}).$$

Let $Q(\mathfrak{H})$ be any one of the above sets and $\mathscr{G}$ be a subclass of $\mathscr{F}$. We denote by $O_Q(\mathscr{G})$ the class of all $\mathfrak{H} \in \mathscr{G}$ such that every element of $Q(\mathfrak{H})$ is a constant function. The following are trivial inclusion relations:

$$
(2.3)\qquad
\begin{array}{ccccccc}
O_{SHP}(\mathscr{F}) & \subset & O_{HP}(\mathscr{F}) & \subset & O_{HDP}(\mathscr{F}) & \subset & O_{HEP}(\mathscr{F}) \\
\cup & & \cup & & \cup & & \cup \\
O_{SH}(\mathscr{F}) & \subset & O_{H}(\mathscr{F}) & \subset & O_{HD}(\mathscr{F}) & \subset & O_{HE}(\mathscr{F}) \\
\cap & & \cap & & \cap & & \cap \\
O_{SHB}(\mathscr{F}) & \subset & O_{HB}(\mathscr{F}) & \subset & O_{HDB}(\mathscr{F}) & \subset & O_{HEB}(\mathscr{F}).
\end{array}
$$

THEOREM 2.1. $O_{SHP}(\mathscr{F})=O_{SHB}(\mathscr{F})$.

PROOF. Suppose $\mathfrak{H}\in O_{SHP}(\mathscr{F})$ and $u\in SHB(\mathfrak{H})$. Let $|u|\leqq M$. Then $u+M\in SHP(\mathfrak{H})$ by Proposition 1.1 (c), and hence $u+M$ is a constant, so that u is a constant. Hence $\mathfrak{H}\in O_{SHB}(\mathscr{F})$. Conversely, suppose $\mathfrak{H}\in O_{SHB}(\mathscr{F})$ and $v\in SHP(\mathfrak{H})$. If v is non-constant, then there is $c>0$ such that $\min(v, c)$ is non-constant. Since $\min(v, c)\in SHB(\mathfrak{H})$ by virtue of Proposition 1.1 (b) and the corollary to Proposition 1.3, this is a contradiction. Hence $\mathfrak{H}\in O_{SHP}(\mathscr{F})$.

REMARK. The above proof shows that this theorem remains valid for the class of functional spaces.

THEOREM 2.2. *$O_{HP}(\mathscr{F})\subset O_{HB}(\mathscr{F})$, $O_{HDP}(\mathscr{F})\subset O_{HDB}(\mathscr{F})$ and $O_{HEP}(\mathscr{F})\subset O_{HEB}(\mathscr{F})$.*

PROOF. Given $\mathfrak{H}\in\mathscr{F}$, let $\Phi=\Phi_{\mathfrak{H}}$ for simplicity. By condition (R) we can choose an exhaustion $\{\Omega_n\}$ of Ω such that each Ω_n is resolutive. Let $u\in HB(\mathfrak{H})$; $|u|\leqq M$. Put

$$v_n = R(\max(u, 0);\, \Omega_n) \quad\text{and}\quad w_n = R(\min(u, 0);\, \Omega_n),$$

$n=1, 2,\cdots$. Since $-\max(u, 0)$ and $\min(u, 0)$ are $\mathfrak{H}$-superharmonic on Ω, $-v_n$ and w_n are totally $\mathfrak{H}$-harmonic on Ω_n, $\mathfrak{H}$-superharmonic on Ω,

$$\max(u, 0)\leqq v_n\leqq M \quad\text{and}\quad -M\leqq w_n\leqq \min(u, 0)$$

by Proposition 2.1. It also follows from Propositions 2.1 and 1.2 that $\{v_n\}$ is monotone non-decreasing and $\{w_n\}$ is monotone non-increasing. Hence, by condition (H.1),

$$v=\lim_{n\to\infty} v_n \quad\text{and}\quad w=\lim_{n\to\infty} w_n$$

are $\mathfrak{H}$-harmonic on Ω, i.e., $v, -w\in HP(\mathfrak{H})$. Since $\int_{\Omega_n}\Phi(v_n)d\xi\leqq\int_{\Omega_n}\Phi(\max(u, 0))d\xi$ and $v_n=\max(u, 0)$ on $\Omega\setminus\Omega_n$,

$$
\begin{aligned}
&\int_{\Omega}\Psi(v_n)d\xi \\
&\leqq \int_{\Omega}\Psi(\max(u, 0))d\xi + \int_{\Omega_n}\{\Gamma(\cdot, \max(u, 0)) - \Gamma(\cdot, v_n)\}d\xi
\end{aligned}
$$

$$\leqq \int_\Omega \Psi(u)d\xi$$

and

$$\int_\Omega \Phi(v_n)d\xi \leqq \int_\Omega \Phi(\max(u, 0))d\xi \leqq \int_\Omega \Phi(u)d\xi.$$

Similarly, we obtain

$$\int_\Omega \Psi(w_n)d\xi \leqq \int_\Omega \Psi(u)d\xi \quad \text{and} \quad \int_\Omega \Phi(w_n)d\xi \leqq \int_\Omega \Phi(u)d\xi.$$

Hence by (H.1), we see that $u \in HDB(\mathfrak{H})$ (resp $HEB(\mathfrak{H})$) implies $v, w \in HDP(\mathfrak{H})$ (resp. $HEP(\mathfrak{H})$)

Now, suppose $\mathfrak{H} \in O_{HP}(\mathscr{F})$ (resp. $O_{HDP}(\mathscr{F})$, $O_{HEP}(\mathscr{F})$) and $u \in HB(\mathfrak{H})$ (resp. $HDB(\mathfrak{H})$, $HEB(\mathfrak{H})$). Then v and w are constant functions. Since

$$u - w = \max(u, 0) + \min(u, 0) - w \geqq \max(u, 0)$$

and $u-w$ is $\mathfrak{H}$-superharmonic on Ω, it follows from Proposition 1.2 that $u-w \geqq v_n$ for all n, so that $u-w \geqq v$. Similarly, we see that $-u+v \geqq -w$. Hence $u=v+w$ $=$const. Thus $\mathfrak{H} \in O_{HB}(\mathscr{F})$ (resp. $O_{HDB}(\mathscr{F})$, $O_{HEB}(\mathscr{F})$).

Combining (2.3), Theorems 2.1 and 2.2, we obtain

$$\begin{array}{ccccccc} O_{SH}(\mathscr{F}) & \subset & O_H(\mathscr{F}) & \subset & O_{HD}(\mathscr{F}) & \subset & O_{HE}(\mathscr{F}) \\ \cap & & \cap & & \cap & & \cap \\ O_{SHP}(\mathscr{F}) & \subset & O_{HP}(\mathscr{F}) & \subset & O_{HDP}(\mathscr{F}) & \subset & O_{HEP}(\mathscr{F}) \\ \| & & \cap & & \cap & & \cap \\ O_{SHB}(\mathscr{F}) & \subset & O_{HB}(\mathscr{F}) & \subset & O_{HDB}(\mathscr{F}) & \subset & O_{HEB}(\mathscr{F}) \end{array} \tag{2.4}$$

We shall see in § 6 and § 7 that all the above inclusion relations are strict and that other inclusion relations cannot be expected.

§ 3. Auxiliary conditions and their consequences

In order to obtain a class of FH-spaces for which $O_{HD}=O_{HDP}$ and $O_{HE}=O_{HEB}$ hold as in the classical case, we consider the following auxiliary conditions for $\mathfrak{H}=\{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\}$:

(X.3) For any compact set K in Ω and an open set $\omega \supset K$, there exists $h \in \mathbf{X}$ such that $Supp\, h$ is compact and contained in ω, $0 \leqq h \leqq 1$ on Ω and $h=1$ on K.
(D) $\mathbf{X}$ is an algebra and

$$\nabla\Psi(f; g_1g_2) = \nabla\Psi(f; g_1)g_2 + \nabla\Psi(f; g_2)g_1$$

for all $f, g_1, g_2 \in \mathbf{X}$.

(H.2) If $\{u_n\}$ is a monotone non-decreasing sequence of non-negative functions in $\mathbf{X}$ such that each u_n is totally $\mathfrak{H}$-harmonic on Ω_n for some exhaustion $\{\Omega_n\}$ of Ω, $\lim_{n\to\infty} u_n(x)<\infty$ on a set of positive ξ-measure and $\left\{\int_K \Psi(u_n)d\xi\right\}$ is bounded for any compact set K, then $\{u_n\}$ is locally uniformly bounded.

(Δ_2) There is a constant $C>2$ such that

$$\Psi(2f) \leqq C\Psi(f)$$

for all $f\in\mathbf{X}$.

Here we give some consequences of these conditions, which will be used in the next section.

LEMMA 3.1. *If* (Δ_2) *is satisfied, then*

(a) $\Psi(f+g) \leqq (C/2)\{\Psi(f)+\Psi(g)\}$ *for* $f, g\in\mathbf{X}$;

(b) $|\nabla\Psi(f; g)| \leqq (C-2)\Psi(f)+\Psi(g)$ *for* $f, g\in\mathbf{X}$;

(c) $|\nabla\Psi(f; g)| \leqq \rho^{-1}(C-2)\Psi(f)+C\rho^{p-1}\Psi(g)$ *for* $f, g\in\mathbf{X}$, *any function* $\rho\geqq 1$ *and any integer* p *such that* $C\leqq 2^p$.

PROOF. (a) follows immediately from the convexity of Ψ and condition (Δ_2). By Lemma 1.1 (f), we see that $\nabla\Psi(f;f)\leqq\Psi(2f)-\Psi(f)$. Hence, by ($\Delta_2$) and Lemma 1.1 ($f$) again, we obtain (b). To show (c), first suppose $\rho\geqq 1$ is a constant. If $2^{n-1}\leqq\rho<2^n$ (n: integer), then by (Δ_2)

$$\Psi(\rho f) \leqq \Psi(2^n f) \leqq C^n\Psi(f) \leqq C\rho^p\Psi(f).$$

Hence, by (b),

$$\rho|\nabla\Psi(f; g)| = |\nabla\Psi(f; \rho g)| \leqq (C-2)\Psi(f)+C\rho^p\Psi(g).$$

Then, we see easily that this inequality holds for any function $\rho\geqq 1$.

PROPOSITION 3.1. *Assume* (D) *and* (Δ_2). *If there is an increasing sequence* $\{f_n\}$ *of non-negative functions in* $\mathbf{X}$ *such that* $Supp f_n$ *is compact for each* n, $\lim_{n\to\infty} f_n(x)=\infty$ *a.e. on* Ω *and* $\left\{\int_\Omega \Psi(f_n)d\xi\right\}$ *is bounded, then* $\mathfrak{H}\in O_{HD}(\mathscr{F})$.

PROOF. Let $u\in HD(\mathfrak{H})$ and put

$$u_m = \max(-m, \min(u, m)) \qquad (m>0).$$

Then $u_m\in\mathbf{X}$, $|u_m|\leqq m$ and $\nabla\Psi(u; u_m)\geqq 0$ for each m by Lemma 1.1. By (D), $u_m f_n\in\mathbf{X}$. Since $Supp(u_m f_n)$ is compact,

$$\int\nabla\Psi(u; u_m f_n)d\xi + \int\Gamma'(\cdot, u)u_m f_n d\xi = 0.$$

Since $\Gamma'(\cdot, u)u_m \geqq 0$, the second integral is non-negative. Thus, using (D) and the above lemma, we have

$$\int_\Omega \nabla\Psi(u; u_m) f_n d\xi \leqq -\int_\Omega \nabla\Psi(u; f_n) u_m d\xi$$
$$\leqq m \int_\Omega |\nabla\Psi(u; f_n)| d\xi$$
$$\leqq m \left\{(C-2)\int_\Omega \Psi(u) d\xi + \int_\Omega \Psi(f_n) d\xi\right\}.$$

Since $f_n \uparrow \infty$ a.e. and $\left\{\int_\Omega \Psi(f_n) d\xi\right\}$ is bounded, it follows that $\nabla\Psi(u; u_m)=0$ on Ω, so that $\nabla\Psi(u; u)=0$ on the set $\{x \in \Omega \mid |u| \leqq m\}$. Since m is arbitrary, this means that $\nabla\Psi(u; u)=0$, so that $\Psi(u)=0$. Hence $u=$const. by Lemma 1.1 (b). Therefore $\mathfrak{H} \in O_{HD}(\mathscr{F})$.

LEMMA 3.2. *Assume* (Δ_2). *Let* A *be a* ξ*-measurable set and* $f_n, g \in \mathbf{X}$, $n=1, 2, \cdots$. *If*

$$\int_A \Psi(g) d\xi < \infty \quad \textit{and} \quad \lim_{n\to\infty} \int_A \Psi(f_n) d\xi = 0,$$

then

$$\lim_{n\to\infty} \int_A \nabla\Psi(f_n; g) d\xi = 0.$$

PROOF. By Lemma 3.1 (b),

$$|\nabla\Psi(f_n; g)| \leqq t^{-1}(C-2)\Psi(f_n) + t^{-1}\Psi(tg)$$

for each $t>0$. Hence

$$\limsup_{n\to\infty} \int_A |\nabla\Psi(f_n; g)| d\xi \leqq t^{-1}\int_A \Psi(tg) d\xi$$

for any $t>0$. Since $t^{-1}\Psi(tg) \leqq \Psi(g)$ for $0<t<1$ and $t^{-1}\Psi(tg) \to 0$ as $t \to 0$, Lebesgue's convergence theorem implies that $t^{-1}\int_A \Psi(tg) d\xi \to 0$ $(t \to 0)$. Hence we obtain the lemma.

LEMMA 3.3. *Assume* (X.3), (D) *and* (Δ_2) *and let* ω *be an open set in* Ω. *If* $\{u_n\}$ *is a monotone non-increasing sequence of non-negative functions in* $\mathbf{X}$ *such that each* u_n *is totally* $\mathfrak{H}$*-harmonic on* ω *and* $\lim_{n\to\infty} u_n = $ *const., then*

$$\lim_{n\to\infty} \int_K \Psi(u_n) d\xi = 0$$

for any compact set K in ω.

PROOF. Let $c=\lim_{n\to\infty} u_n$. Choose $h\in \mathbf{X}$ as in condition (X.3) for the above ω and a given $K\subset\omega$. Let p be an integer such that $C\leqq 2^p$. Then $(u_n-c)h^p\in \mathbf{X}$ by (D) and $(u_n-c)h^p=0$ on $\Omega\setminus\omega$. Hence, u_n being totally $\mathfrak{H}$-harmonic on ω,

$$\int_\omega \nabla\Psi(u_n;\,(u_n-c)h^p)d\xi+\int_\omega \Gamma'(\cdot\,,u_n)(u_n-c)h^p d\xi=0$$

for each n. Since $u_n-c\geqq 0$ and $u_n\geqq 0$, $\Gamma'(\cdot\,,u_n)(u_n-c)\geqq 0$. Hence, using (D), we have

$$\int_\omega \nabla\Psi(u_n;\,u_n)h^p d\xi\leqq -p\int_\omega \nabla\Psi(u_n;\,h)(u_n-c)h^{p-1}d\xi.$$

Therefore,

$$\begin{aligned}(3.1)\qquad \int_\omega \Psi(u_n)h^p d\xi &\leqq \int_\omega \nabla\Psi(u_n;\,u_n)h^p d\xi\\ &\leqq p\int_{A_h}|\nabla\Psi(u_n;\,h)|(u_n-c)h^{p-1}d\xi,\end{aligned}$$

where $A_h=\{x\in\omega|h(x)>0\}$. For $x\in A_h$, put

$$\rho_n(x)=\max\{1,\,2(C-2)p(u_n(x)-c)h(x)^{-1}\}.$$

By Lemma 3.1 (c),

$$|\nabla\Psi(u_n;\,h)|(x)\leqq \rho_n(x)^{-1}(C-2)\Psi(u_n)(x)+C\rho_n(x)^{p-1}\Psi(h)(x)$$

for $x\in A_h$. Thus, by (3.1), we have

$$(3.2)\qquad \int_\omega \Psi(u_n)h^p d\xi\leqq 2pC\int_{A_h}\rho_n^{p-1}\Psi(h)(u_n-c)h^{p-1}d\xi.$$

It is easy to see that $\{\rho_n^{p-1}\Psi(h)(u_n-c)h^{p-1}\}$ is uniformly bounded on A_h, which is relatively compact. Hence Lebesgue's convergence theorem implies that the right-hand side of (3.2) tends to zero as $n\to\infty$. Thus

$$\lim_{n\to\infty}\int_\omega \Psi(u_n)h^p d\xi=0,$$

which implies the assertion of the lemma.

§4. Classification II

Now, let $\mathscr{F}_1$ be the subclass of $\mathscr{F}$ consisting of all $\mathfrak{H}\in\mathscr{F}$ which satisfy conditions (X.3), (D), (H.2) and (Δ_2). The relations (2.4) are valid with $\mathscr{F}_1$ in the place of $\mathscr{F}$, since $\mathscr{F}_1 \subset \mathscr{F}$.

THEOREM 4.1. *$O_{HDP}(\mathscr{F}_1)=O_{HD}(\mathscr{F}_1)$ and $O_{HEP}(\mathscr{F}_1)=O_{HE}(\mathscr{F}_1)$.*

PROOF. We have to show

$$O_{HDP}(\mathscr{F}_1) \subset O_{HD}(\mathscr{F}_1) \quad \text{and} \quad O_{HEP}(\mathscr{F}_1) \subset O_{HE}(\mathscr{F}_1).$$

Suppose $\mathfrak{H} \in O_{HDP}(\mathscr{F}_1)$ (resp. $\in O_{HEP}(\mathscr{F}_1)$) and $\mathfrak{H} \notin O_{HD}(\mathscr{F}_1)$ (resp. $\notin O_{HE}(\mathscr{F}_1)$). Choose $u \in HD(\mathfrak{H})$ (resp. $HE(\mathfrak{H})$) which is non-constant. Let $\{\Omega_n\}$ be an exhaustion of Ω such that each Ω_n is resolutive and put

$$v_n = R(\max(u, 0); \Omega_n) \quad \text{and} \quad w_n = R(\min(u, 0); \Omega_n).$$

By Proposition 2.1, these are totally $\mathfrak{H}$-harmonic on Ω_n for each n, $\{v_n\}$ is monotone non-decreasing, $\{w_n\}$ is monotone non-increasing, $\max(u, 0) \leqq v_n$, $\min(u, 0) \geqq w_n$,

$$\int_{\Omega_n} \Phi_{\mathfrak{H}}(v_n) d\xi \leqq \int_{\Omega_n} \Phi_{\mathfrak{H}}(\max(u, 0)) d\xi$$

and

$$\int_{\Omega_n} \Phi_{\mathfrak{H}}(w_n) d\xi \leqq \int_{\Omega_n} \Phi_{\mathfrak{H}}(\min(u, 0)) d\xi.$$

Put $f_n = v_n - \max(u, 0)$. Then $f_n \in \mathbf{X}$, $f_n \geqq 0$, $\{f_n\}$ is monotone non-decreasing and each $Supp f_n$ is compact. By Lemma 3.1 (a),

$$\int_{\Omega} \Psi(f_n) d\xi \leqq \frac{C}{2} \left\{ \int_{\Omega} \Psi(v_n) d\xi + \int_{\Omega} \Psi(\max(u, 0)) d\xi \right\}.$$

As in the proof of Theorem 2.2, we see that

$$\int_{\Omega} \Psi(v_n) d\xi \leqq \int_{\Omega} \Psi(u) d\xi.$$

Hence

$$\int_{\Omega} \Psi(f_n) d\xi \leqq C \int_{\Omega} \Psi(u) d\xi < \infty.$$

Since $\mathfrak{H} \notin O_{HD}(\mathscr{F})$, Proposition 3.1 implies that

$$\xi(\{x \in \Omega \mid \lim_{n\to\infty} f_n(x) < \infty\}) > 0,$$

so that

$$\xi(\{x \in \Omega \mid \lim_{n\to\infty} v_n(x) < \infty\}) > 0.$$

Hence by (H.1) and (H.2), $v = \lim_{n\to\infty} v_n$ is $\mathfrak{H}$-harmonic and

$$\int_\Omega \Psi(v)d\xi \leqq \liminf_{n\to\infty} \int_\Omega \Psi(v_n)d\xi.$$

It follows that $\int_\Omega \Psi(v)d\xi \leqq \int_\Omega \Psi(u)d\xi < \infty$ (resp. $\int_\Omega \Phi_{\mathfrak{H}}(v)d\xi \leqq \int_\Omega \Phi_{\mathfrak{H}}(u)d\xi < \infty$), so that $v \in HDP(\mathfrak{H})$ (resp. $HEP(\mathfrak{H})$). Hence $v = \text{const}$. Similarly we see that $w = \lim_{n\to\infty} w_n$ is a constant. Then, by the same argument as in the last part of the proof of Theorem 2.2, we derive a contradiction that u is a constant.

THEOREM 4.2. $O_{HEB}(\mathscr{F}_1) = O_{HE}(\mathscr{F}_1)$.

PROOF. By virtue of the above theorem, it is enough to show that $O_{HEB}(\mathscr{F}_1) \subset O_{HEP}(\mathscr{F}_1)$. Let $\mathfrak{H} \in O_{HEB}(\mathscr{F}_1)$ and $u \in HEP(\mathfrak{H})$. Put $u_m = \min(u, m)$ for $m > 0$. Then $u_m \in \mathbf{X}$, $u_m \geqq 0$, u_m is $\mathfrak{H}$-superharmonic on Ω and

$$\int_\Omega \Phi_{\mathfrak{H}}(u_m)d\xi \leqq \int_\Omega \Phi_{\mathfrak{H}}(u)d\xi.$$

Let $\{\Omega_n\}$ be an exhaustion of Ω such that each Ω_n is resolutive and put

$$v_{m,n} = R(u_m; \Omega_n).$$

By Proposition 2.1, $0 \leqq v_{m,n} \leqq u_m$, each $v_{m,n}$ is $\mathfrak{H}$-superharmonic on Ω, totally $\mathfrak{H}$-harmonic on Ω_n and $\{v_{m,n}\}_n$ is monotone non-increasing. By (H.1), $w_m = \lim_{n\to\infty} v_{m,n}$ is $\mathfrak{H}$-harmonic on Ω and

$$\begin{aligned}\int_\Omega \Phi_{\mathfrak{H}}(w_m)d\xi &\leqq \liminf_{n\to\infty} \int_\Omega \Phi_{\mathfrak{H}}(v_{m,n})d\xi \\ &\leqq \int_\Omega \Phi_{\mathfrak{H}}(u_m)d\xi \leqq \int_\Omega \Phi_{\mathfrak{H}}(u)d\xi < \infty.\end{aligned}$$

Obviously, $0 \leqq w_m \leqq u_m \leqq m$. Hence $w_m \in HEB(\mathfrak{H})$, so that w_m is a constant for each m.

Since $v_{m,n}$ is totally $\mathfrak{H}$-harmonic on Ω_n and $v_{m,n} = u_m$ on $\Omega \setminus \Omega_n$, we have

$$(4.1) \qquad \int_\Omega \nabla\Psi(v_{m,n}; u_m - v_{m,n})d\xi + \int_\Omega \Gamma'(\cdot, v_{m,n})(u_m - v_{m,n})d\xi = 0.$$

Since w_m is a constant and $\mathfrak{H}$-harmonic on Ω,

$$\int_\Omega \Gamma'(\cdot, w_m)(u_m - v_{m,n})d\xi = 0,$$

which implies

$$\int_{\Omega} \Gamma'(\cdot, w_m)(u_m - w_m)d\xi = 0. \tag{4.2}$$

Now, by the convexity of $\Gamma(x, t)$ in t,

$$\begin{aligned} 0 \leqq \Gamma'(\cdot, v_{m,n})(u_m - v_{m,n}) &\leqq \Gamma(\cdot, u_m) - \Gamma(\cdot, v_{m,n}) \\ &\leqq \Gamma(\cdot, u_m) \leqq \Gamma(\cdot, u) \end{aligned}$$

and

$$\int_{\Omega} \Gamma(\cdot, u)d\xi < \infty$$

since $u \in HE(\mathfrak{H})$. Since $\Gamma'(\cdot, v_{m,n}) \to \Gamma'(\cdot, w_m)$ on Ω, it follows from Lebesgue's convergence theorem and (4.2) that

$$\lim_{n\to\infty} \int_{\Omega} \Gamma'(\cdot, v_{m,n})(u_m - v_{m,n})d\xi = 0. \tag{4.3}$$

On the other hand, since $\int_{\Omega} \Psi(u_m)d\xi \leqq \int_{\Omega} \Psi(u)d\xi < \infty$, given $\varepsilon > 0$ ($\varepsilon < 1$), there is a compact set K in Ω such that $\int_{\Omega\setminus K} \Psi(u_m)d\xi < \varepsilon$. Applying Lemma 3.1 (c) with $\rho = \varepsilon^{-1/p}$, we have

$$\begin{cases} \left| \int_{\Omega\setminus K} \nabla\Psi(v_{m,n}; u_m)d\xi \right| \\ \leqq \varepsilon^{1/p}(C-2)\int_{\Omega\setminus K} \Psi(v_{m,n})d\xi + \varepsilon^{-(p-1)/p}C\int_{\Omega\setminus K} \Psi(u_m)d\xi \\ \leqq \varepsilon^{1/p}\left\{(C-2)\int_{\Omega} \Phi_{\mathfrak{H}}(u)d\xi + C\right\}. \end{cases} \tag{4.4}$$

Since $v_{m,n}$ decreases to a constant w_m as $n \to \infty$, Lemma 3.3 implies that $\lim_{n\to\infty}\int_K \Psi(v_{m,n})d\xi = 0$, and hence by Lemma 3.2, we have

$$\lim_{n\to\infty} \int_K \nabla\Psi(v_{m,n}; u_m)d\xi = 0.$$

Hence, in view of (4.4), we obtain

$$\lim_{n\to\infty} \int_{\Omega} \nabla\Psi(v_{m,n}; u_m)d\xi = 0. \tag{4.5}$$

By (4.1), (4.3) and (4.5), we conclude that

$$\lim_{n\to\infty} \int_{\Omega} \nabla\Psi(v_{m,n}; v_{m,n})d\xi = 0,$$

or

$$\lim_{n\to\infty}\int_\Omega \Psi(v_{m,n})d\xi = 0 \tag{4.6}$$

for each $m>0$.

Since u is $\mathfrak{H}$-harmonic on Ω,

$$\int_\Omega \nabla\Psi(u;\, u_m - v_{m,n})d\xi + \int_\Omega \Gamma'(\cdot,\, u)(u_m - v_{m,n})d\xi = 0.$$

Noting that $\Gamma'(\cdot,\, u)(u_m - v_{m,n}) \geqq 0$, we have

$$\int_\Omega \nabla\Psi(u;\, u_m)d\xi \leqq \int_\Omega \nabla\Psi(u;\, v_{m,n})d\xi. \tag{4.7}$$

By (4.6), for sufficiently large n, $\int_\Omega \Psi(v_{m,n})d\xi \leqq 1$ (m being fixed). Applying Lemma 3.1 (c) with $\rho = \left\{\int_\Omega \Psi(v_{m,n})d\xi\right\}^{-1/p}\left(\text{if} \int_\Omega \Psi(v_{m,n})d\xi \neq 0\right)$, we obtain

$$\int_\Omega \nabla\Psi(u;\, v_{m,n})d\xi$$
$$\leqq \left\{\int_\Omega \Psi(v_{m,n})d\xi\right\}^{1/p}\left\{(C-2)\int_\Omega \Psi(u)d\xi + C\right\}.$$

Thus, by (4.6),

$$\lim_{n\to\infty}\int_\Omega \nabla\Psi(u;\, v_{m,n})d\xi = 0,$$

so that by (4.7),

$$\int_\Omega \nabla\Psi(u;\, u_m)d\xi = 0.$$

Hence $\nabla\Psi(u;\, u_m)=0$ on Ω (note that $\nabla\Psi(u;\, u_m)\geqq 0$ by Lemma 1.1). Since this is true for any $m>0$, it follows that $\nabla\Psi(u,\, u)=0$, which implies that $\Psi(u)=0$, i.e., $u=$const. Thus the theorem is proved.

Summing up, we have obtained

$$\begin{array}{ccccccc}
O_{SH}(\mathscr{F}_1) & \subset & O_H(\mathscr{F}_1) & & & & \\
\cap & & \cap & & & & \\
\left.\begin{array}{c} O_{SHP}(\mathscr{F}_1) \\ \| \\ O_{SHB}(\mathscr{F}_1) \end{array}\right\} & \subset & O_{HP}(\mathscr{F}_1) & \subset & \left\{\begin{array}{c} O_{HD}(\mathscr{F}_1) \\ \| \\ O_{HDP}(\mathscr{F}_1) \end{array}\right. & & \\
 & & \cap & & \cap & & \\
 & & O_{HB}(\mathscr{F}_1) & \subset & O_{HDB}(\mathscr{F}_1) & \subset & \left\{\begin{array}{c} O_{HE}(\mathscr{F}_1) \\ \| \\ O_{HEP}(\mathscr{F}_1) \\ \| \\ O_{HEB}(\mathscr{F}_1) \end{array}\right.
\end{array} \tag{4.8}$$

The special cases in § 6 and § 7 show that all inclusion relations in (4.8) are strict, except

$$O_{HDP}(\mathscr{F}_1) \subset O_{HDB}(\mathscr{F}_1).$$

We do not know whether this inclusion is strict or not; for the class of linear FH-spaces we have the equality (cf. [5], [6], [9]). In the next section, we shall consider a subclass $\mathscr{F}_2$ of $\mathscr{F}_1$, which contains all linear FH-spaces, and show that $O_{HDP}(\mathscr{F}_2) = O_{HDB}(\mathscr{F}_2)$.

§ 5. Classification III

We consider the following condition for Ψ, which is the dual of (Δ_2):

(Δ_2^*) There is a constant $C^* > 2$ such that

$$C^*\Psi(2f) \leqq \Psi(C^*f)$$

for all $f \in \mathbf{X}$.

We denote by $\mathscr{F}_2$ the class of all $\mathfrak{H} \in \mathscr{F}_1$ satisfying (Δ_2^*). Then we have

THEOREM 5.1. $O_{HDB}(\mathscr{F}_2) = O_{HD}(\mathscr{F}_2)$.

PROOF. It is enough to show that

$$O_{HDB}(\mathscr{F}_2) \subset O_{HDP}(\mathscr{F}_2)$$

by virtue of Theorem 4.1. Suppose $\mathfrak{H} \in O_{HDB}(\mathscr{F}_2)$ and $u \in HDP(\mathfrak{H})$. Let u_m, $\{\Omega_n\}$, $v_{m,n}$ and w_m be as in the proof of Theorem 4.2. Then

$$\int_\Omega \nabla\Psi(v_{m,n}; v_{m,n} - u_m)d\xi + \int_\Omega \Gamma'(\cdot, v_{m,n})(v_{m,n} - u_m)d\xi = 0 \tag{5.1}$$

and

$$\int_\Omega \nabla\Psi(u; u_m - v_{m,n})d\xi + \int_\Omega \Gamma'(\cdot, u)(u_m - v_{m,n})d\xi = 0. \tag{5.2}$$

Since $\int_\Omega \nabla\Psi(u; u_m)d\xi \geqq 0$ and

$$\Gamma'(\cdot, u)(u_m - v_{m,n}) \geqq \Gamma'(\cdot, v_{m,n})(u_m - v_{m,n}),$$

it follows from (5.1) and (5.2) that

$$\int_\Omega \nabla\Psi(v_{m,n}; v_{m,n})d\xi \leqq \int_\Omega \{\nabla\Psi(v_{m,n}; u_m) + \nabla\Psi(u; v_{m,n})\}d\xi. \tag{5.3}$$

Applying Lemma 3.1 (c) with $\rho=4C$, we have

$$(5.4)\qquad \begin{cases} \displaystyle\int_\Omega |\nabla\Psi(v_{m,n};\, u_m)|d\xi \\ \displaystyle\leqq 4^{-1}C^{-1}(C-2)\int_\Omega \Psi(v_{m,n})d\xi + 4^{p-1}C^p\int_\Omega \Psi(u_m)d\xi \\ \displaystyle\leqq 4^{-1}\int_\Omega \Psi(v_{m,n})d\xi + 4^{p-1}C^p\int_\Omega \Psi(u)d\xi. \end{cases}$$

Note that $\int_\Omega \Psi(v_{m,n})d\xi<\infty$, since $\int_\Omega \Psi(u_m)d\xi\leqq\int_\Omega \Psi(u)d\xi<\infty$ and $v_{m,n}=u_m$ on $\Omega\setminus\Omega_n$. Lemma 3.1 (b) and condition (Δ_2^*) imply

$$\begin{aligned} |\nabla\Psi(u;\, v_{m,n})| &= 2^{-1}C^*|\nabla\Psi(u;\, 2C^{*-1}v_{m,n})| \\ &\leqq 2^{-1}C^*\{(C-2)\Psi(u)+\Psi(2C^{*-1}v_{m,n})\} \\ &\leqq 2^{-1}C^*(C-2)\Psi(u)+2^{-1}\Psi(v_{m,n}), \end{aligned}$$

so that

$$(5.5)\qquad \begin{cases} \displaystyle\int_\Omega |\nabla\Psi(u;\, v_{m,n})|d\xi \\ \displaystyle\leqq 2^{-1}C^*(C-2)\int_\Omega \Psi(u)d\xi + 2^{-1}\int_\Omega \Psi(v_{m,n})d\xi. \end{cases}$$

From (5.3), (5.4) and (5.5), it follows that $\left\{\int_\Omega \Psi(v_{m,n})d\xi\right\}_{n,m}$ is bounded. Hence by (H.1),

$$\int_\Omega \Psi(w_m)d\xi \leqq \liminf_{n\to\infty}\int_\Omega \Psi(v_{m,n})d\xi < \infty,$$

so that $w_m\in HDB(\mathfrak{H})$, which implies $w_m=\text{const}$.

Given $\varepsilon>0$ $(\varepsilon<1)$, choose a positive integer l such that $\int_\Omega \Psi(v_{m,n})d\xi\leqq\varepsilon 2^{l-1}$ for all m, n. Since $\int_\Omega \Psi(u)d\xi<\infty$, there is a compact set K in Ω such that

$$\int_{\Omega\setminus K} \Psi(u)d\xi \leqq 2^{l-1}C^{*-l}(C-2)^{-1}\varepsilon.$$

By Lemma 3.1 (b) and the repeated use of condition (Δ_2^*) (cf. the computation above yielding (5.5)) we obtain

$$(5.6)\qquad \left|\int_{\Omega\setminus K} \nabla\Psi(u;\, v_{m,n})d\xi\right| \leqq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

On the other hand, by Lemma 3.3, $\lim_{n\to\infty}\int_K \Psi(v_{m,n})d\xi=0$. Applying Lemma 3.1 (c) with $\rho=\left(\int_K \Psi(v_{m,n})d\xi\right)^{-1/p}$ for large n, we have

$$(5.7)\qquad \begin{cases} \left|\int_K \nabla\Psi(u;\, v_{m,n})d\xi\,\right| \\ \leqq \left\{\int_K \Psi(v_{m,n})d\xi\right\}^{1/p}\left\{(C-2)\int_\Omega \Psi(u)d\xi + C\right\} \\ \to 0 \qquad (n\to\infty). \end{cases}$$

From (5.6) and (5.7) it follows that

$$\lim_{n\to\infty}\int_\Omega \nabla\Psi(u;\, v_{m,n})d\xi = 0.$$

Hence, by (5.2),

$$0 \leqq \int_\Omega \nabla\Psi(u;\, u_m)d\xi \leqq \int_\Omega \nabla\Psi(u;\, v_{m,n})d\xi \to 0 \qquad (n\to\infty),$$

so that $\nabla\Psi(u;\, u_m)=0$ on Ω for each m. It then follows that u is a constant.

REMARK 5.1. The above proof shows that the equality $O_{HDB}=O_{HD}$ can be proved without using Royden boundary or Green potentials in the linear case (cf. [6], [9]).

By (4.8) and Theorem 5.1, we obtain

$$(5.8)\qquad \begin{array}{ccccccccc} O_{SH}(\mathscr{F}_2) & \subset & O_H(\mathscr{F}_2) & & & & & & \\ \cap & & \cap & & & & \left\{\begin{array}{c} O_{HD}(\mathscr{F}_2) \\ \Vert \\ O_{HDP}(\mathscr{F}_2) \\ \Vert \\ O_{HDB}(\mathscr{F}_2) \end{array}\right\} & \subset & \left\{\begin{array}{c} O_{HE}(\mathscr{F}_2) \\ \Vert \\ O_{HEP}(\mathscr{F}_2) \\ \Vert \\ O_{HEB}(\mathscr{F}_2) \end{array}\right. . \\ \left.\begin{array}{c} O_{SHP}(\mathscr{F}_2) \\ \Vert \\ O_{SHB}(\mathscr{F}_2) \end{array}\right\} & \subset & O_{HP}(\mathscr{F}_2) & \subset & O_{HB}(\mathscr{F}_2) & \subset & & & \end{array}$$

All inclusion relations (5.8) are known to be strict in the linear case (cf. the next two sections).

REMARK 5.2. If we consider the class

$$\mathscr{F}_1' = \left\{\mathfrak{H}\in\mathscr{F}_1 \,\middle|\, \int_\Omega \Gamma(x,\, t)d\xi(x) < \infty \text{ for every } t\in\mathbf{R}\right\},$$

then, in almost the same way as in the proof of Theorem 4.2, we see that

$$O_{HEB}(\mathscr{F}_1') \subset O_{HDP}(\mathscr{F}_1').$$

Thus, in view of (4.8), we have

$$
(5.9)\quad
\begin{array}{ccccccc}
O_{SH}(\mathscr{F}'_1) & \subset & O_H(\mathscr{F}'_1) & & & & \\
\cap & & & & & & \left\{\begin{array}{ccc} O_{HD}(\mathscr{F}'_1) & = & O_{HE}(\mathscr{F}'_1) \\ \| & & \| \end{array}\right. \\
\left.\begin{array}{c} O_{SHP}(\mathscr{F}'_1) \\ \| \\ O_{SHB}(\mathscr{F}'_1) \end{array}\right\} & \subset & \begin{array}{c}\cap \\ O_{HP}(\mathscr{F}'_1)\end{array} & \subset & O_{HB}(\mathscr{F}'_1) & \subset & \begin{array}{ccc} O_{HDP}(\mathscr{F}'_1) & = & O_{HEP}(\mathscr{F}'_1) \\ \| & & \| \\ O_{HDB}(\mathscr{F}'_1) & = & O_{HEB}(\mathscr{F}'_1)\,. \end{array}
\end{array}
$$

§6. Quasi-linear networks

In this section, we consider a special class of FH-spaces, namely, the class of quasi-linear networks.

Let X and Y be countable (infinite) sets and let K be a function on $X \times Y$ satisfying the following conditions:

(K.1) The range of K is $\{-1, 0, 1\}$;
(K.2) For each $y \in Y$, $e(y) \equiv \{x \in X \mid K(x, y) \neq 0\}$ consists of exactly two points x_1 and x_2 and $K(x_1, y)K(x_2, y) = -1$;
(K.3) For each $x \in X$, $Y(x) \equiv \{y \in Y \mid K(x, y) \neq 0\}$ is a non-empty finite set;
(K.4) For each $x, x' \in X$, there are $x_1, \cdots, x_k \in X$ and $y_1, \cdots, y_{k+1} \in Y$ such that $e(y_j) = \{x_{j-1}, x_j\}$, $j = 1, \cdots, k+1$, with $x_0 = x$ and $x_{k+1} = x'$.

Then $G = \{X, Y, K\}$ is called a (connected, locally finite) infinite graph (cf. [13]).

For each $y \in Y$, we consider a set S_y and a bijection j_y of S_y onto the open unit interval (0, 1) and let

$$\Omega = \Omega_{\{X,Y\}} = X \cup \bigcup_{y \in Y} S_y$$

be a disjoint union. A topology is introduced on Ω as follows: $\omega \subset \Omega$ is open if (and only if) $j_y(\omega \cap S_y)$ is open in (0, 1) for each y, $j_y(\omega \cap S_y)$ contains an interval of the form $(0, \varepsilon)$ $(\varepsilon > 0)$ in case $x \in \omega$ and $K(x, y) = -1$ and it contains an interval of the form $(1 - \varepsilon', 1)$ $(\varepsilon' > 0)$ in case $x \in \omega$ and $K(x, y) = 1$. Then Ω is a connected, non-compact, σ-compact, locally compact Hausdorff space. For each $y \in Y$, j_y is extended to be a homeomorphism of $\bar{S}_y = S_y \cup e(y)$ onto [0, 1].

Let μ_y be the measure on S_y induced by j_y from the Lebesgue measure on (0, 1) and let ν be the counting measure on X. We define $\xi = \xi_G$ by

$$\xi = \nu + \sum_{y \in Y} \mu_y,$$

which is a positive Radon measure on Ω whose support is the whole space Ω.

Let

$$\mathbf{X} = \mathbf{X}_G = \left\{ f\colon \Omega \to \mathbf{R} \,\middle|\, \begin{array}{l} f \text{ is continuous on } \Omega,\, f\circ j_y^{-1} \text{ is} \\ \text{Lipschitz continuous on } (0,\, 1) \text{ for each } y \end{array} \right\}.$$

It is easy to see that this $\mathbf{X}$ satisfies conditions (X.1) and (X.2) in § 1, and also (X.3) in §3. If $f \in \mathbf{X}$, then $(f\circ j_y^{-1})'$ exists a.e. on (0, 1). For simplicity, we write $f'(z)$ for $(f\circ j_y^{-1})'(j_y(z))$ in case $z \in S_y$.

Next, we consider two functions $\phi\colon Y\times\mathbf{R}\to\mathbf{R}$ and $\gamma\colon X\times\mathbf{R}\to\mathbf{R}$ satisfying the following conditions:

(ϕ.1) $\phi(y, t) = -\phi(y, -t)$ for all $y \in Y$ and $t \in \mathbf{R}$;
(ϕ.2) For each $y \in Y$, $\phi(y, t)$ is continuous and strictly increasing in t;
(γ.1) $\gamma(x, t) = -\gamma(x, -t)$ for all $x \in X$ and $t \in \mathbf{R}$;
(γ.2) For each $x \in X$, $\gamma(x, t)$ is continuous and monotone non-decreasing in t.

Put $\psi(y, t) = \int_0^t \phi(y, s)ds$,

$$\Psi(f)(z) = \begin{cases} 0, & \text{if } \; z \in X \\ \psi(y, f'(z)), & \text{if } \; z \in S_y \end{cases}$$

for $f \in \mathbf{X}$ and

$$\Gamma(z, t) = \begin{cases} \int_0^t \gamma(x, s)ds, & \text{if } \; z = x \in X \\ 0, & \text{if } \; z \notin X \end{cases}$$

for $z \in \Omega$, $t \in \mathbf{R}$.

We see that $\Psi(f)$ is defined ξ-a.e. on Ω and $\Psi(f) \in L^1_{\mathrm{loc}}(\Omega)$ for any $f \in \mathbf{X}$. It is easy to verify that this Ψ satisfies conditions (Ψ.1)$\sim$(Ψ.5) in § 1 and

$$\nabla\Psi(f; g)(z) = \begin{cases} 0, & \text{if } \; z \in X \\ \phi(y, f'(z))g'(z), & \text{if } \; z \in S_y. \end{cases}$$

From this, we see that condition (D) in § 3 is also valid. Obviously, Γ defined above satisfies (Γ.1)$\sim$(Γ.3) in § 1 with

$$\Gamma'(z, t) = \begin{cases} \gamma(x, t), & \text{if } \; z = x \in X \\ 0, & \text{if } \; z \notin X. \end{cases}$$

Thus $\mathfrak{H} = \{\Omega_{\{X,Y\}}, \xi_G, \mathbf{X}_G, \Psi, \Gamma\}$ is a functional space, which we shall call a *quasi-linear network*. It will be often denoted by $\mathfrak{H} = [G, \phi, \gamma]$. We denote by $\mathscr{N}$ the class of all quasi-linear networks.

For an open set $\omega \subset \Omega$, let

$$X(\omega) = \{x \in X \cap \omega \mid S_y \subset \omega \qquad \text{for all} \quad y \in Y(x)\}.$$

An open set ω will be said to be regular if $X(\omega)=X \cap \omega$. Obviously, Ω is regular. We shall say that a function f on Ω is linear on S_y if $f \circ j_y^{-1}$ is linear on (0, 1).

LEMMA 6.1. *Let* $\mathfrak{H}=[G, \phi, \gamma] \in \mathscr{N}$.

(a) *If* ω *is a regular open set and* $u \in \mathbf{X}$ *is totally* $\mathfrak{H}$*-harmonic on* ω, *then* u *is linear on* S_y *for every* $y \in Y$ *such that* $S_y \subset \omega$ *and*

$$(6.1) \qquad \sum_{y \in Y(x)} K(x, y)\phi(y, \sum_{x' \in X} K(x', y)u(x')) + \gamma(x, u(x)) = 0$$

for all $x \in X \cap \omega$.

(b) *If* $u \in \mathbf{X}$ *is linear on every* S_y, $y \in Y$, *and satisfies* (6.1) *for all* $x \in X$, *then* u *is* $\mathfrak{H}$*-harmonic on* Ω.

PROOF. Obviously, u is totally $\mathfrak{H}$-harmonic on S_y if and only if $\phi(y, u'(z)) =$ const. on S_y, that is, $u'(z)=$const. on S_y, or equivalently, u is linear on S_y.

For $x \in X$, let $U(x)=\{x\} \cup \bigcup_{y \in Y(x)} S_y$. Suppose $u \in \mathbf{X}$ is linear on each S_y, $y \in Y(x)$. Then we see that u is totally $\mathfrak{H}$-harmonic on $U(x)$ if and only if

$$\sum_{y \in Y(x)} K(x, y)\phi(y, u'(z_y)) + \gamma(x, u(x)) = 0,$$

where z_y is any point on S_y. Since $u'(z_y)=\sum_{x' \in X} K(x', y)u(x')$, this equality is nothing but (6.1). Hence our lemma follows.

PROPOSITION 6.1. $\mathscr{N} \subset \mathscr{F}$, *i.e., every quasi-linear network is an FH-space*; *each* $\mathfrak{H} \in \mathscr{N}$ *satisfies conditions* (X.3), (D) *and* (H.2).

PROOF. We have already seen that each $\mathfrak{H} \in \mathscr{N}$ is a functional space and satisfies (X.3) and (D).

Since there is an exhaustion of Ω consisting of regular open sets, to show (H.1) and (H.2) we may assume that each Ω_n is regular. Then (H.1) is easily seen from Lemma 6.1. Let $\{u_n\}$ be a sequence as described in condition (H.2). Then there is $x_0 \in X$ such that $\{u_n(x_0)\}$ is bounded. For each $x \in X$, we find by (K.4) $x_1, \cdots, x_k \in X$ and $y_1, \cdots, y_{k+1} \in Y$ such that $e(y_j)=\{x_{j-1}, x_j\}$, $j=1, \cdots, k+1$ with $x_{k+1}=x$. Let $F=\bigcup_{j=1}^{k+1} \bar{S}_{y_j}$. Then F is a compact set in Ω and

$$\int_F \Psi(u_n) d\xi = \sum_{j=1}^{k+1} \psi(y_j, u_n(x_j) - u_n(x_{j-1})).$$

Since $\left\{\int_F \Psi(u_n) d\xi\right\}$ is bounded and $\psi(y, t) \to \infty$ as $|t| \to \infty$ for each t, it follows that $\{u_n(x)\}$ is bounded. Hence $\{u_n\}$ is locally uniformly bounded on Ω. Thus (H.2) is satisfied.

Finally, we shall verify (R). Let ω be a relatively compact regular open set in Ω and let $f \in \mathbf{X}$. Then $M=\max_{z \in \bar{\omega}} |f(z)|$ is finite. Let

$$\mathbf{D} = \{g \in \mathbf{X} \,|\, g = f \text{ on } \Omega \setminus \omega\}$$

and

$$\mathbf{D}^* = \left\{ g \in \mathbf{D} \;\middle|\; \begin{array}{l} g \text{ is linear on each } S_y \subset \omega \\ \text{and } |g| \leqq M \text{ on } \omega \end{array} \right\}.$$

For each $g \in \mathbf{D}$, we can find $g^* \in \mathbf{D}^*$ such that

$$g^*(x) = \max(-M, \min(g(x), M))$$

for all $x \in X \cap \omega$. It is easy to see that

$$\int_\omega \Phi_{\mathfrak{H}}(g^*)d\xi \leqq \int_\omega \Phi_{\mathfrak{H}}(g)d\xi.$$

Hence,

$$\alpha \equiv \inf\left\{\int_\omega \Phi_{\mathfrak{H}}(g)d\xi \,\middle|\, g \in \mathbf{D}\right\} = \inf\left\{\int_\omega \Phi_{\mathfrak{H}}(g)d\xi \,\middle|\, g \in \mathbf{D}^*\right\}.$$

Since $X \cap \omega$ is a finite set and $\{g(x) | g \in \mathbf{D}^*\}$ is bounded for each $x \in X \cap \omega$, we can find a sequence $\{g_n\} \subset \mathbf{D}^*$ such that $\{g_n(x)\}$ is convergent for every $x \in X \cap \omega$ and

$$\lim_{n\to\infty} \int_\omega \Phi_{\mathfrak{H}}(g_n)d\xi = \alpha.$$

Then $g_0 = \lim_{n\to\infty} g_n$ exists and belongs to $\mathbf{D}^*$. We see easily that $g_0 = R(f; \omega)$. Hence (R) is satisfied.

By virtue of this proposition, inclusion relations (2.4) hold with $\mathscr{N}$ in the place of $\mathscr{F}$. Furthermore, if we put

$$\mathscr{N}_1 = \mathscr{N} \cap \mathscr{F}_1 = \{\mathfrak{H} \in \mathscr{N} \,|\, \mathfrak{H} \text{ satisfies } (\Delta_2)\}$$

and

$$\mathscr{N}_2 = \mathscr{N} \cap \mathscr{F}_2 = \{\mathfrak{H} \in \mathscr{N} \,|\, \mathfrak{H} \text{ satisfies } (\Delta_2) \text{ and } (\Delta_2^*)\},$$

then inclusion relations (4.8) hold with $\mathscr{N}_1$ in the place of $\mathscr{F}_1$ and (5.8) hold with $\mathscr{N}_2$ in the place of $\mathscr{F}_2$. Note that conditions (Δ_2) and (Δ_2^*) for $\mathfrak{H} \in \mathscr{N}$ may be written as follows:

$(\Delta_2)_{\mathscr{N}}$: There is a constant $c > 1$ such that

$$\phi(y, 2t) \leqq c\phi(y, t) \qquad \text{for all} \quad y \in Y,\ t \geqq 0.$$

$(\Delta_2^*)_{\mathscr{N}}$: There is a constant $c^* > 1$ such that

$$2\phi(y, t) \leqq \phi(y, c^*t) \qquad \text{for all} \quad y \in Y,\ t \geqq 0.$$

Note that the network considered in [13] belongs to $\mathscr{N}_2$ ($\phi(y, t) = r(y)|t|^{p-2}t$, $r(y) > 0$, $1 < p < \infty$ and $\gamma(x, t) \equiv 0$).

Now, we shall show by special quasi-linear networks that inclusion relations in (2.4) and (5.8) are all strict.

PROPOSITION 6.2. $O_H(\mathscr{N}_2) \not\subset O_{SHP}(\mathscr{N}_2)$.

PROOF. Let $X = \{x_0, x_1, \cdots\}$, $Y = \{y_1, y_2, \cdots\}$, $K(x_n, y_n) = 1$ and $K(x_{n-1}, y_n) = -1$, $n = 1, 2, \cdots$, $K(x_n, y_m) = 0$ if $m \neq n+1, n$. Then $G = \{X, Y, K\}$ is an infinite graph. Let

$$\phi(y_n, t) = n^2 t, \qquad n = 1, 2, \cdots, t \in \mathbf{R}$$

and $\gamma \equiv 0$. Then $\mathfrak{H} = [G, \phi, \gamma]$ belongs to $\mathscr{N}_2$. Since $u \in H(\mathfrak{H})$ if and only if u is linear on each S_{y_n} and

$$0 = \phi(y_1, u(x_0) - u(x_1)) = \cdots = \phi(y_n, u(x_{n-1}) - u(x_n)) = \cdots,$$

$H(\mathfrak{H})$ consists only of constant functions, i.e., $\mathfrak{H} \in O_H(\mathscr{N}_2)$. On the other hand, if we define v to be linear on each S_{y_n} and

$$v(x_n) = 2 - \sum_{k=1}^{n} k^{-2}, \qquad n = 1, 2, \cdots,$$

then $v \in SHP(\mathfrak{H})$. Hence $\mathfrak{H} \notin O_{SHP}(\mathscr{N}_2)$.

COROLLARY. *$O_{SH}(\mathscr{N}_2) \neq O_H(\mathscr{N}_2)$ and $O_{SHP}(\mathscr{N}_2) \neq O_{HP}(\mathscr{N}_2)$.*

PROPOSITION 6.3. $O_{HD}(\mathscr{N}_2) \not\subset O_{HB}(\mathscr{N}_2)$.

PROOF. Let $X = X_1 \cup X_1'$ with

$$X_1 = \{x_n \mid n \in \mathbf{Z}\} \quad \text{and} \quad X_1' = \{x_n' \mid n \in \mathbf{Z}\},$$

where $\mathbf{Z}$ is the set of all integers, and let $Y = Y_1 \cup Y_1' \cup Y_2$ with

$$Y_1 = \{y_n \mid n \in \mathbf{Z}\},\ Y_1' = \{y_n' \mid n \in \mathbf{Z}\} \quad \text{and} \quad Y_2 = \{z_n \mid n = 0, 1, \cdots\}.$$

We define $K(x, y)$ on $X \times Y$ as follows:

$$K(x_n, y_n) = K(x_n', y_n') = 1,\ K(x_{n-1}, y_n) = K(x_{n-1}', y_n') = -1 \quad (n \in \mathbf{Z}),$$

$$K(x_n, z_n) = 1,\ K(x_n', z_n) = -1 \quad (n = 0, 1, \cdots) \quad \text{and}$$

$$K(x, y) = 0 \text{ for any other pair } (x, y) \in X \times Y.$$

Then $G = \{X, Y, K\}$ is an infinite graph. Let

$$\phi(y_n, t) = \phi(y'_n, t) = 2^n t, \qquad n = 1, 2, \cdots,$$

$$\phi(y_{-n}, t) = \phi(y'_{-n}, t) = \phi(z_n, t) = t, \qquad n = 0, 1, \cdots$$

and $\gamma \equiv 0$. Then $\mathfrak{H} = [G, \phi, \gamma] \in \mathcal{N}_2$. If $u \in HB(\mathfrak{H}) \cup HD(\mathfrak{H})$, then $u(x_{-n}) = u(x_0)$ and $u(x'_{-n}) = u(x'_0)$ for all $n = 1, 2, \cdots$. If we put $a_n = u(x_n) - u(x'_n)$, $n = 0, 1, \cdots$, then

$$(6.2) \qquad 2^n(a_n - a_{n-1}) - 2^{n+1}(a_{n+1} - a_n) + 2a_n = 0 \qquad (n = 0, 1, \cdots).$$

Hence

$$(6.3) \qquad a_{n+1} = a_n + 2^{-n} \sum_{k=0}^{n} a_k, \qquad n = 0, 1, \cdots.$$

Any sequence $\{a_n\}$ satisfying (6.3) is bounded. Since

$$u(x_n) = \frac{1}{2}\{u(x_0) + u(x'_0) + a_n\} \quad \text{and} \quad u(x'_n) = \frac{1}{2}\{u(x_0) + u(x'_0) - a_n\},$$

$n = 0, 1, \cdots$, we see that $HB(\mathfrak{H})$ contains non-constant functions, i.e., $\mathfrak{H} \notin O_{HB}(\mathcal{N}_2)$. On the other hand, if $a_0 \neq 0$, then $|a_n| \geqq |a_0|$ for all $n = 1, 2, \cdots$, so that

$$\int_\Omega \Psi(u) d\xi \geqq \sum_{n=0}^{\infty} \psi(z_n, u(x_n) - u(x'_n)) = \frac{1}{2} \sum_{n=0}^{\infty} a_n^2 = \infty.$$

Hence $u \in HD(\mathfrak{H})$ implies $a_0 = 0$, i.e., $u = \text{const}$. Therefore $\mathfrak{H} \in O_{HD}(\mathcal{N}_2)$.

COROLLARY. *$O_H(\mathcal{N}) \neq O_{HD}(\mathcal{N})$, $O_{HP}(\mathcal{N}_1) \neq O_{HDP}(\mathcal{N}_1)$ and $O_{HB}(\mathcal{N}_1) \neq O_{HDB}(\mathcal{N}_1)$.*

PROPOSITION 6.4. *$O_{HE}(\mathcal{N}_2) \not\subset O_{HDB}(\mathcal{N}_2)$.*

PROOF. Let G be as in the proof of Proposition 6.2 and let $\phi(y_n, t) = 2^{n-1}t$, $n = 1, 2, \cdots$ and $\gamma(x_n, t) = t$, $n = 0, 1, \cdots$. Then $\mathfrak{H} = [G, \phi, \gamma] \in \mathcal{N}_2$. For $u \in H(\mathfrak{H})$, put $a_{-1} = 0$ and $a_n = u(x_n)$, $n = 0, 1, \cdots$. Then $\{a_n\}$ satisfies (6.2) and hence (6.3) in the proof of the previous proposition. Thus, any $u \in H(\mathfrak{H})$ is bounded. Furthermore,

$$\int_\Omega \Psi(u) d\xi = \sum_{n=1}^{\infty} 2^{n-2}(a_n - a_{n-1})^2 = \sum_{n=1}^{\infty} 2^{-n}\left(\sum_{k=0}^{n-1} a_k\right)^2$$

$$\leqq (\sup_n |a_n|^2) \sum_{n=1}^{\infty} 2^{-n} n^2 < \infty.$$

Hence $H(\mathfrak{H}) = HDB(\mathfrak{H})$, which contains non-constant functions. Therefore $\mathfrak{H} \notin O_{HDB}(\mathcal{N}_2)$.

On the other hand, if $a_0 \neq 0$, then $|a_n| \geqq |a_0|$ for all n, so that

$$\int_{\Omega} \Gamma(\cdot, u) d\xi = \frac{1}{2} \sum_{n=0}^{\infty} a_n{}^2 = \infty.$$

Hence $HE(\mathfrak{H}) = \{0\}$, so that $\mathfrak{H} \in O_{HE}(\mathcal{N}_2)$.

COROLLARY. *$O_{HD}(\mathcal{N}) \neq O_{HE}(\mathcal{N})$, $O_{HDP}(\mathcal{N}) \neq O_{HEP}(\mathcal{N})$ and $O_{HDB}(\mathcal{N}_1) \neq O_{HEB}(\mathcal{N}_1)$.*

PROPOSITION 6.5. *$O_{SHP}(\mathcal{N}_2) \not\subset O_H(\mathcal{N}_2)$ and $O_{HB}(\mathcal{N}_2) \not\subset O_{HP}(\mathcal{N}_2)$.*

PROOF. Let $X = \{x_n \mid n \in Z\}$, $Y = \{y_n \mid n \in Z\}$, $K(x_n, y_n) = 1$ and $K(x_{n-1}, y_n) = -1$ for $n \in Z$ and $K(x_n, y_m) = 0$ if $m \neq n+1, n$. Then $G = \{X, Y, K\}$ is an infinite graph. Let $\phi(y_n, t) = c_n t$ $(c_n > 0)$, $n \in Z$, and $\gamma \equiv 0$. Then $\mathfrak{H} = [G, \phi, \gamma] \in \mathcal{N}_2$. We easily see that

(i) $\mathfrak{H} \notin O_H(\mathcal{N}_2)$;

(ii) $\mathfrak{H} \in O_{SHP}(\mathcal{N}_2)$ as well as $\mathfrak{H} \in O_{HP}(\mathcal{N}_2)$ if and only if

$$\sum_{n=1}^{\infty} c_n^{-1} = \sum_{n=-1}^{-\infty} c_n^{-1} = \infty;$$

(iii) $\mathfrak{H} \in O_{HB}(\mathcal{N}_2)$ if and only if $\sum_{n=-\infty}^{\infty} c_n^{-1} = \infty$.

Then the assertions of the proposition immediately follow.

COROLLARY. *$O_{SH}(\mathcal{N}_2) \neq O_{SHP}(\mathcal{N}_2)$ and $O_H(\mathcal{N}_2) \neq O_{HP}(\mathcal{N}_2)$.*

REMARK. The quasi-linear networks given in the proofs of Propositions 6.2, 6.3, 6.5 all belong to $\mathscr{F}'_1$ (see, Remark 5.2), and hence provide examples to show that all inclusion relations in (5.9) are strict.

PROPOSITION 6.6. *$O_{SHP}(\mathcal{N}) \not\subset O_{HE}(\mathcal{N})$.*

PROOF. Let G be as in the proof of the previous proposition and let $\phi(y_n, t) = |t|^{n^2} t$, $n \in \mathbf{Z}$, $\gamma \equiv 0$. Then $\mathfrak{H} = [G, \phi, \gamma] \in \mathcal{N}$ (but $\mathfrak{H} \notin \mathcal{N}_1$). Let $v \in SHP(\mathfrak{H})$ and put

$$b_n = |v(x_n) - v(x_{n-1})|^{n^2} \{v(x_n) - v(x_{n-1})\}, \qquad n \in \mathbf{Z}.$$

Then $b_n \geqq b_{n+1}$ for all $n \in \mathbf{Z}$. It follows that v cannot be non-negative unless b_n are all zero. Hence $\mathfrak{H} \in O_{SHP}(\mathcal{N})$. On the other hand, if $u(x_n) = n$ for all $n \in \mathbf{Z}$ and u is linear on each S_{y_n}, then $u \in HD(\mathfrak{H}) = HE(\mathfrak{H})$. Hence $\mathfrak{H} \notin O_{HE}(\mathcal{N})$.

COROLLARY. *$O_{HD}(\mathcal{N}) \neq O_{HDP}(\mathcal{N})$ and $O_{HE}(\mathcal{N}) \neq O_{HEP}(\mathcal{N})$.*

PROPOSITION 6.7. *$O_{HB}(\mathcal{N}) \not\subset O_{HEP}(\mathcal{N})$.*

PROOF. Let G be as in the proof of Proposition 6.5 (and Proposition 6.6), let

$$\phi(y_n, t) = \begin{cases} |t|^{n^2}t, & t \in \mathbf{R}, \quad n = 0, 1, 2, \cdots \\ n^2 t, & t \in \mathbf{R}, \quad n = -1, -2, \cdots \end{cases}$$

and $\gamma \equiv 0$. Then $\mathfrak{H} = [G, \phi, \gamma] \in \mathscr{N}$ ($\mathfrak{H} \notin \mathscr{N}_1$). If $u \in H(\mathfrak{H})$, then

$$\begin{aligned} |u(x_n) - u(x_{n-1})|^{n^2}\{u(x_n) - u(x_{n-1})\} &= u(x_0) - u(x_{-1}) \\ &= m^2\{u(x_m) - u(x_{m-1})\} \end{aligned}$$

for all $n = 1, 2, \cdots$ and $m = -1, -2, \cdots$. If $u \in HB(\mathfrak{H})$, then $u(x_0) - u(x_{-1}) = 0$, and hence $u = \text{const}$. Therefore $\mathfrak{H} \in O_{HB}(\mathscr{N})$. On the other hand, if we define u_0 by

$$u_0(x_n) = \begin{cases} 3 + n, & n = -1, 0, 1, \cdots \\ 2 - \sum_{k=1}^{|n+1|} k^{-2}, & n = -2, -3, \cdots, \end{cases}$$

then $u_0 \in HDP(\mathfrak{H}) = HEP(\mathfrak{H})$. Hence $\mathfrak{H} \notin O_{HEP}(\mathscr{N})$.

COROLLARY. *$O_{HDP}(\mathscr{N}) \neq O_{HDB}(\mathscr{N})$ and $O_{HEP}(\mathscr{N}) \neq O_{HEB}(\mathscr{N})$.*

§7. FH-spaces on differentiable manifolds

In this section, we are concerned with FH-spaces defined on C^1-manifolds.

Let Ω be a connected, σ-compact (or, equivalently, para-compact), non-compact C^1-manifold of dimension d ($\geqq 1$) and let $\{(V_\lambda, \chi_\lambda)\}_{\lambda \in \Lambda}$ be a locally finite system of coordinate neighborhoods such that each V_λ is relatively compact and $\bar{V}_\lambda \subset U_\lambda$ for some coordinate neighborhood $(U_\lambda, \tilde{\chi}_\lambda)$ such that $\tilde{\chi}_\lambda | V_\lambda = \chi_\lambda$. Let ξ be a positive Radon measure on Ω such that $d\xi = h_\lambda d\mu_\lambda$ on U_λ for each $\lambda \in \Lambda$ with a positive C^1-function h_λ on U_λ, where μ_λ is the measure on U_λ induced by $\tilde{\chi}_\lambda$ from the Lebesgue measure on $\mathbf{R}^d$. Next, we consider a system $\{\psi_\lambda\}_{\lambda \in \Lambda}$ of functions ψ_λ: $\chi_\lambda(V_\lambda) \times \mathbf{R}^d \to \mathbf{R}$ satisfying the following conditions:

(ψ.0) If $V_\lambda \cap V_{\lambda'} \neq \phi$, then for each $z \in V_\lambda \cap V_{\lambda'}$ and $\tau \in \mathbf{R}^d$,

$$\psi_\lambda(\chi_\lambda(z), \tau) = \psi_{\lambda'}(\chi_{\lambda'}(z), J_\lambda^{\lambda'}(z)\tau),$$

where $J_\lambda^{\lambda'}(z)$ is the Jacobian matrix of the transformation $\chi_\lambda \circ \chi_{\lambda'}^{-1}$ at $\chi_{\lambda'}(z)$. (This means that $\{\psi_\lambda\}_{\lambda \in \Lambda}$ defines a real function on the cotangent bundle over Ω.)

(ψ.1) $\psi_\lambda(x, \tau) \geqq 0$, $\psi_\lambda(x, 0) = 0$ and $\psi_\lambda(x, \tau) = \psi_\lambda(x, -\tau)$ for all $\lambda \in \Lambda$, $x \in \chi_\lambda(V_\lambda)$, $\tau \in \mathbf{R}^d$.

(ψ.2) For each $\lambda \in \Lambda$ and $x \in \chi_\lambda(V_\lambda)$, $\psi_\lambda(x, \tau)$ is strictly convex and continuously differentiable in $\tau \in \mathbf{R}^d$.

(ψ.3) For each $\lambda \in \Lambda$ and $\tau \in \mathbf{R}^d$, $\nabla_\tau \psi_\lambda(\cdot, \tau)$ is measurable on $\chi_\lambda(V_\lambda)$.

(ψ.4) With some $p > 1$, the following holds: for each $\lambda \in \Lambda$ there are constants

$\alpha_\lambda>0$, $\beta_\lambda>0$ and functions $a_\lambda\in L^{p'}(\chi_\lambda(V_\lambda))$, $b_\lambda\in L^{p''}(\chi_\lambda(V_\lambda))$ with

$$p'=\begin{cases}\max(d,p)/(p-1) & \text{if } p\neq d\\ d/(d-1)+\varepsilon & \text{if } p=d\ (\varepsilon>0)\end{cases};\quad p''=\begin{cases}d/p+\varepsilon' & \text{if } p\leqq d\ (\varepsilon'>0)\\ 1 & \text{if } p>d,\end{cases}$$

such that

$$|\nabla_\tau\psi_\lambda(x,\tau)|\leqq\alpha_\lambda|\tau|^{p-1}+a_\lambda(x),$$

$$\langle\nabla_\tau\psi_\lambda(x,\tau),\tau\rangle\geqq\beta_\lambda|\tau|^p-b_\lambda(x)$$

for all $\lambda\in\Lambda$, $x\in\chi_\lambda(V_\lambda)$ and $\tau\in\mathbf{R}^d$, where $\langle\ ,\ \rangle$ denotes the ordinary inner product in $\mathbf{R}^d$.

(ψ.5) For each $\lambda\in\Lambda$ and for any positive numbers δ, ρ such that $0<\delta<1<\rho$, there are $r=r(\lambda,\delta,\rho)>1$ and $\eta=\eta(\lambda,\delta,\rho)>0$ such that if $\delta\leqq\max(|\tau|,|\tau'|)\leqq\rho$ then

$$\langle\nabla_\tau\psi_\lambda(x,\tau)-\nabla_\tau\psi_\lambda(x,\tau'),\tau-\tau'\rangle\geqq\eta|\tau-\tau'|^r$$

for all $x\in\chi_\lambda(V_\lambda)$.

Finally, let Γ: $\Omega\times\mathbf{R}\to\mathbf{R}$ satisfy (Γ.1)$\sim$(Γ.3) in § 1 and

(Γ.4) With $p>1$ and p'' given in (ψ.4), for each $\lambda\in\Lambda$ there is $e_\lambda\in L^{p''}(\chi_\lambda(V_\lambda))$ such that

$$|\Gamma'(\chi_\lambda^{-1}(x),t)|\leqq e_\lambda(x)(|t|^{p-1}+1)$$

for all $\lambda\in\Lambda$, $x\in\chi_\lambda(V_\lambda)$ and $t\in\mathbf{R}$.

With $p>1$ given in (ψ.4), let $\mathbf{X}=W^{1,p}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$, i.e.,

$$\mathbf{X}=\{f\in L^\infty_{\mathrm{loc}}(\Omega)\,|\,|\nabla(f\circ\chi_\lambda^{-1})|\in L^p(\chi_\lambda(V_\lambda))\ \text{for every}\ \lambda\in\Lambda\}.$$

By (ψ.0)$\sim$(ψ.4), we see that

$$\Psi(f)(z)=\psi_\lambda(\chi_\lambda(z),\nabla(f\circ\chi_\lambda^{-1})(\chi_\lambda(z)))\qquad\text{for}\quad z\in V_\lambda \tag{7.1}$$

defines a function belonging to $L^1_{\mathrm{loc}}(\Omega)$ for each $f\in\mathbf{X}$.

The class of $\mathfrak{H}=\{\Omega,\xi,\mathbf{X},\Psi,\Gamma\}$ defined as above will be denoted by $\mathscr{V}$. Then we have

PROPOSITION 7.1. *$\mathscr{V}\subset\mathscr{F}$, i.e., each $\mathfrak{H}\in\mathscr{V}$ is an FH-space. Furthermore, each $\mathfrak{H}\in\mathscr{V}$ satisfies* (X.3), (D) *and* (H.2) *in* § 3.

PROOF. Conditions (X.1) and (X.3) for $\mathbf{X}$ are obviously satisfied. Conditions (Ψ.1) and (Ψ.2) for Ψ are immediate consequences of (ψ.1) and (7.1); and (Ψ.4) follows from (ψ.2). Since $\nabla(f\circ\chi_\lambda^{-1})=0$ a.e. on the set $\{x\in V_\lambda|f(x)=0\}$

if $f \in \mathbf{X}$ (cf., e.g., [3, Théorème 3.2]), (Ψ.3) and (X.2) are seen to be valid. By virtue of (ψ.3) and (ψ.4), we see that (Ψ.5) is satisfied with

$$\nabla \Psi(f; g)(z) = \langle \nabla_\tau \psi_\lambda(\chi_\lambda(z), \nabla(f \circ \chi_\lambda^{-1})(\chi_\lambda(z))), \nabla(g \circ \chi_\lambda^{-1})(\chi_\lambda(z)) \rangle \tag{7.2}$$

for $z \in V_\lambda$. From the definition of $\mathbf{X}$ and (7.2), condition (D) is easily verified.

By applying the standard variational method (see, e.g., [8, Chap. 5, Theorem 2.1]), we can show that any relatively compact open set is resolutive. Thus condition (R) is satisfied. Condition (H.2) follows from [12, Theorems 5, 6 and 9] in view of (ψ.4) and (Γ.4).

Thus, what remains to show is the verification of (H.1), which will be given in the Appendix.

Conditions (Δ_2) and (Δ_2^*) for $\mathfrak{H} \in \mathscr{V}$ may be written as follows:

$(\Delta_2)_\psi$ There is $C>2$ such that

$$\psi_\lambda(x, 2\tau) \leqq C\psi_\lambda(x, \tau)$$

for all $\lambda \in \Lambda$, $x \in \chi_\lambda(V_\lambda)$ and $\tau \in \mathbf{R}^d$.

$(\Delta_2^*)_\psi$ There is $C^*>2$ such that

$$C^*\psi_\lambda(x, 2\tau) \leqq \psi_\lambda(x, C^*\tau)$$

for all $\lambda \in \Lambda$, $x \in \chi_\lambda(V_\lambda)$ and $\tau \in \mathbf{R}^d$.

Thus if we put

$$\mathscr{V}_1 = \{\mathfrak{H} \in \mathscr{V} \mid \mathfrak{H} \text{ satisfies } (\Delta_2)_\psi\}$$

and

$$\mathscr{V}_2 = \{\mathfrak{H} \in \mathscr{V}_1 \mid \mathfrak{H} \text{ satisfies } (\Delta_2^*)_\psi\},$$

then $\mathscr{V}_1 = \mathscr{V} \cap \mathscr{F}_1$ and $\mathscr{V}_2 = \mathscr{V} \cap \mathscr{F}_2$ by virtue of Proposition 7.1. Hence, inclusion relations (2.4), (4.8) and (5.8) are valid with $\mathscr{V}$, $\mathscr{V}_1$ and $\mathscr{V}_2$ in the place of $\mathscr{F}$, $\mathscr{F}_1$ and $\mathscr{F}_2$, respectively.

REMARK. If Ω is a Riemannian manifold with Riemannian metric (g_{ij}), ξ is the corresponding volume element, $\mathbf{X} = W_{loc}^{1,2}(\Omega) \cap L_{loc}^{\infty}(\Omega)$, $\psi_\lambda(x, \tau) = \Sigma g^{ij}(x)\tau_i\tau_j$ on V_λ and $\Gamma(x, t) = P(x)t^2$ with $P \in L_{loc}^q(\Omega)$ ($q>d/2$, $q \geqq 1$), $P \geqq 0$, then $\mathfrak{H} = \{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\} \in \mathscr{V}_2$, where Ψ is defined by (7.1) from the above $\{\psi_\lambda\}_{\lambda \in \Lambda}$. In this case, $H(\mathfrak{H})$ is the space of weak solutions of $\Delta u = Pu$ (Δ: the Laplace-Beltrami operator), and thus the classification theory given in [5], [9] as well as the classical classification theory of Riemann surfaces are included in the classification theory for $\mathscr{V}_2$. In particular, non-inclusion relations

$$O_{HD}(\mathscr{V}_2) \not\subset O_{HB}(\mathscr{V}_2), \tag{7.3}$$

$$O_{HE}(\mathscr{V}_2) \not\subset O_{HDB}(\mathscr{V}_2), \tag{7.4}$$

$$O_{SHP}(\mathscr{V}_2) \not\subset O_{H}(\mathscr{V}_2), \tag{7.5}$$

$$O_{HB}(\mathscr{V}_2) \not\subset O_{HP}(\mathscr{V}_2) \tag{7.6}$$

are known; in fact, (7.3), (7.5) and (7.6) are classical (see [11, Chap. III, 4H] for (7.3), [11, Chap. IV, 3C] and [11, Appendix 3A] for (7.6); also see [10]) and (7.4) is shown in [10].

As for $\mathscr{V}$, modifying the proofs of Propositions 6.6 and 6.7, we obtain

PROPOSITION 7.2. *$O_{SHP}(\mathscr{V}) \not\subset O_{HE}(\mathscr{V})$ and $O_{HB}(\mathscr{V}) \not\subset O_{HEP}(\mathscr{V})$.*

PROOF. Let $\Omega = \mathbf{R}$, ξ be the Lebesgue measure on $\mathbf{R}$,

$$\mathbf{X} = \{f\colon \mathbf{R} \to \mathbf{R} \mid \text{locally absolutely continuous and } f' \in L^2_{\mathrm{loc}}(\mathbf{R})\},$$

$\Gamma(x, t) \equiv 0$,

$$\psi_0(x, \tau) = \begin{cases} (2 + x^2)^{-1}|\tau|^{2+x^2}, & \text{if } |\tau| \leqq 1,\ x \in \mathbf{R} \\ 2^{-1}|\tau|^2 + (2 + x^2)^{-1} - 2^{-1}, & \text{if } |\tau| > 1,\ x \in \mathbf{R} \end{cases}$$

and

$$\psi_1(x, \tau) = \begin{cases} \psi_0(x, \tau), & \text{if } x \geqq 0,\ \tau \in \mathbf{R} \\ 2^{-1}(1 + x^2)|\tau|^2, & \text{if } x < 0,\ \tau \in \mathbf{R}. \end{cases}$$

Then, $\{\Omega, \xi, \mathbf{X}, \Psi_0, \Gamma\} \in O_{SHP}(\mathscr{V}) \setminus O_{HE}(\mathscr{V})$ and $\{\Omega, \xi, \mathbf{X}, \Psi_1, \Gamma\} \in O_{HB}(\mathscr{V}) \setminus O_{HEP}(\mathscr{V})$, where $\Psi_0(f) = \psi_0(\cdot, f')$ and $\Psi_1(f) = \psi_1(\cdot, f')$. Note that these spaces satisfy (ψ.4) and (ψ.5) with $p = r = 2$.

APPENDIX. In order to verify (H.1) for $\mathfrak{H} \in \mathscr{V}$, it is enough to prove the following theorem.

THEOREM A. *Let Ω be a bounded open set in $\mathbf{R}^d$ and ξ be the Lebesgue measure on $\mathbf{R}^d$. Suppose $\psi\colon \Omega \times \mathbf{R}^d \to \mathbf{R}$ satisfies conditions (ψ.1)$\sim$(ψ.5) with $\{\psi_\lambda\}_{\lambda \in \Lambda} = \{\psi\}$ ($V_\lambda = \Omega$, $\chi_\lambda =$ the identity mapping) and $\Gamma\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfies (Γ.1)$\sim$(Γ.4). Let $\mathfrak{H} = \{\Omega, \xi, \mathbf{X}, \Psi, \Gamma\}$, where $\mathbf{X} = W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $p > 1$ given in (ψ.4) and $\Psi(f) = \psi(\cdot, \nabla f)$ for $f \in \mathbf{X}$. If $\{u_n\}$ is a uniformly bounded convergent sequence of $\mathfrak{H}$-harmonic functions on Ω, then $u = \lim_{n\to\infty} u_n$ is $\mathfrak{H}$-harmonic on Ω and $\int_K \Psi(u)d\xi \leqq \liminf_{n\to\infty} \int_K \Psi(u_n)d\xi$ for any compact set K in Ω.*

A similar result is obtained in B. Calvert [2]. But our assumptions, and hence proofs, are slightly different from those in [2]. We prove Theorem A in four steps.

PROOF OF THEOREM A:

(I) $\left\{\int_K |\nabla u_n(x)|^p dx\right\}$ *is bounded for any compact set* K *in* Ω.

This can be proved in the same way as [2, Lemma 2], and we omit the proof.

(II) *For any compact set* K *in* Ω,

$$\int_K \langle \nabla_\tau \psi(\cdot, \nabla u_n) - \nabla_\tau \psi(\cdot, \nabla u_m), \nabla u_n - \nabla u_m \rangle dx \to 0 \qquad (n, m \to \infty).$$

PROOF. Let ϕ be a C^1-function with compact support in Ω such that $\phi \geqq 0$ and $\phi = 1$ on K. Then,

$$\int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u_k), \nabla[(u_n - u_m)\phi] \rangle dx + \int_\Omega \Gamma'(\cdot, u_k)(u_n - u_m)\phi dx = 0$$

for any k, n, m. Thus

$$\begin{aligned} I_{n,m} &\equiv \int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u_n) - \nabla_\tau \psi(\cdot, \nabla u_m), \nabla u_n - \nabla u_m \rangle \phi dx \\ &= -\int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u_n) - \nabla_\tau \psi(\cdot, \nabla u_m), \nabla \phi \rangle (u_n - u_m) dx \\ &\quad - \int_\Omega \{\Gamma'(\cdot, u_n) - \Gamma'(\cdot, u_m)\}(u_n - u_m)\phi dx. \end{aligned}$$

The last integral is non-negative. Hence, by (ψ.4) ($\alpha = \alpha_\lambda$, $a = a_\lambda$),

$$\begin{aligned} I_{n,m} &\leqq \alpha \int_\Omega (|\nabla u_n|^{p-1} + |\nabla u_m|^{p-1}) |\nabla \phi| \, |u_n - u_m| dx \\ &\qquad + 2\int_\Omega a |\nabla \phi| \, |u_n - u_m| dx \\ &\leqq \alpha(J_n + J_m) \left\{\int_{K'} |\nabla \phi|^p |u_n - u_m|^p dx\right\}^{1/p} \\ &\qquad + 2\int_{K'} a |\nabla \phi| \, |u_n - u_m| dx, \end{aligned}$$

where $K' = Supp\,\phi$ and $J_n = \left\{\int_{K'} |\nabla u_n|^p dx\right\}^{1/p^*}$, $p^* = p/(p-1)$. Hence, by (I) and Lebesgue's convergence theorem, we conclude that $I_{n,m} \to 0$ $(n, m \to \infty)$, from which (II) follows immediately.

(III) *For any compact set* K *in* Ω,

$$\int_K |\nabla u_n - \nabla u_m| dx \to 0 \qquad (n, m \to \infty).$$

PROOF. Let $0<\delta<1<\rho$. Fix n and m for the time being and put

$$E_0 = \{x \in K \mid |\nabla u_n(x)| \leqq \delta,\ |\nabla u_m(x)| \leqq \delta\},$$

$$E_1 = \{x \in K \mid |\nabla u_n(x)| > \rho\},\ E_1' = \{x \in K \mid |\nabla u_m(x)| > \rho\},$$

$$E_2 = K \setminus (E_0 \cup E_1 \cup E_1').$$

Obviously,

$$\int_{E_0} |\nabla u_n - \nabla u_m| dx \leqq 2\delta\xi(K). \tag{A.1}$$

By (I), there is $M>0$ such that $\int_K |\nabla u_k|^p dx \leqq M$ for all k. Then,

$$\rho\xi(E_1) \leqq \int_{E_1} |\nabla u_n| dx \leqq M^{1/p}\xi(E_1)^{1/p^*},$$

so that $\xi(E_1) \leqq \rho^{-p}M$. Similarly, $\xi(E_1') \leqq \rho^{-p}M$. Hence

$$\text{(A.2)}\quad \left\{\begin{aligned} \int_{E_1 \cup E_1'} |\nabla u_n - \nabla u_m| dx &\leqq \int_{E_1 \cup E_1'} |\nabla u_n| dx + \int_{E_1 \cup E_1'} |\nabla u_m| dx \\ &\leqq 2M^{1/p}\{\xi(E_1) + \xi(E_1')\}^{1/p^*} \leqq 4M\rho^{1-p}. \end{aligned}\right.$$

By (ψ.5),

$$\begin{aligned} &\int_{E_2} |\nabla u_n - \nabla u_m| dx \\ &\leqq \eta^{-1/r} \int_{E_2} \langle \nabla_\tau \psi(\cdot, \nabla u_n) - \nabla_\tau \psi(\cdot, \nabla u_m), \nabla u_n - \nabla u_m \rangle^{1/r} dx \\ &\qquad\qquad \leqq \eta^{-1/r}\xi(K)^{(r-1)/r}(I_{m,n})^{1/r}, \end{aligned}$$

where

$$I_{n,m} = \int_K \langle \nabla_\tau \psi(\cdot, \nabla u_n) - \nabla_\tau \psi(\cdot, \nabla u_m), \nabla u_n - \nabla u_m \rangle dx.$$

Hence, together with (A.1) and (A.2), we have

$$\int_K |\nabla u_n - \nabla u_m| dx \leqq 2\delta\xi(K) + 4M\rho^{1-p} + \eta^{-1/r}\xi(K)^{(r-1)/r}(I_{n,m})^{1/r}$$

for any n, m. Since $I_{n,m} \to 0$ $(n, m \to \infty)$ by (II),

$$\limsup_{n,m\to\infty} \int_K |\nabla u_n - \nabla u_m| dx \leqq 2\delta\xi(K) + 2M\rho^{1-p}.$$

Letting $\delta \to 0$ and $\rho \to \infty$, we obtain (III).

(IV) By (I) and pointwise convergence of $\{u_n\}$, we see that $u \in W^{1,p}_{loc}(\Omega) \cap L^{\infty}_{loc}(\Omega)$. By (III), we can choose a subsequence $\{u_{n_j}\}$ such that $\nabla u_{n_j} \to \nabla u$ a.e. on Ω. Then

$$\nabla_\tau \psi(x, \nabla u_{n_j}(x)) \to \nabla_\tau \psi(x, \nabla u(x)) \qquad \text{a.e. on} \quad \Omega$$

by (ψ.2). On the other hand, by (I) and (ψ.4), $\{\nabla_\tau \psi(\cdot, \nabla u_{n_j})\}_j$ is bounded in $(L^{p^*}(\omega))^d$ for any relatively compact open set ω such that $\bar{\omega} \subset \Omega$. Hence, there is another subsequence $\{u_{m_j}\}$ of $\{u_{n_j}\}$ such that $\nabla_\tau \psi(\cdot, \nabla u_{m_j})|\omega \to \nabla_\tau \psi(\cdot, \nabla u)|\omega$ weakly in $(L^{p^*}(\omega))^d$ for any ω as above. Hence

$$\text{(A.3)} \qquad \int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u_{m_j}), \nabla g \rangle dx \to \int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u), \nabla g \rangle dx$$

for any $g \in W^{1,p}(\Omega)$ with compact support in Ω. On the other hand,

$$\text{(A.4)} \qquad \int_\Omega \Gamma'(\cdot, u_n) g dx \to \int_\Omega \Gamma'(\cdot, u) g dx$$

for any $g \in L^\infty(\Omega)$ with compact support in Ω by Lebesgue's convergence theorem. Since

$$\int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u_n), \nabla g \rangle dx + \int_\Omega \Gamma'(\cdot, u_n) g dx = 0$$

for any $g \in \mathbf{X}$ with compact support in Ω, it follows from (A.3) and (A.4) that

$$\int_\Omega \langle \nabla_\tau \psi(\cdot, \nabla u), \nabla g \rangle dx + \int_\Omega \Gamma'(\cdot, u) g dx = 0$$

for any $g \in \mathbf{X}$ as above, i.e., u is $\mathfrak{H}$-harmonic on Ω.

Furthermore, given a compact set K, we could choose $\{u_{n_j}\}$ to satisfy

$$\lim_{j\to\infty} \int_K \Psi(u_{n_j}) d\xi = \liminf_{n\to\infty} \int_K \Psi(u_n) d\xi.$$

Since $\Psi(u_{n_j})(x) = \psi(x, \nabla u_{n_j}(x)) \to \psi(x, \nabla u(x)) = \Psi(u)(x)$ a.e. on Ω, Fatou's lemma implies

$$\int_K \Psi(u) d\xi \leqq \liminf_{n\to\infty} \int_K \Psi(u_n) d\xi.$$

Added in proof: It is possible to prove Theorem *A* in the appendix without condition (ψ. 5), so that this condition is not necessary for the discussions in §7.

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