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Some results on the normalization and normal flatness

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Introduction

In this paper, we shall give a sufficient condition that properties for a reduced noetherian scheme X to be Cohen-Macaulay or Gorenstein can be ascended to or can be descended from the same properties on the normalization $\bar{X}$ of X. It is well-known that the condition of flatness plays an important role in the study of many properties on an extension of a noetherian rings (e.g. [21]). But the normalization of a reduced noetherian ring is an integral extension which is far from a flat one. Therefore it seems to the author that we need a "flatness" condition on X, in some sense, in order to give the above sufficient condition. Fortunately, in his famous paper [11], H. Hironaka defined the notion of normal flatness in 1964 (see Def. 2 in this paper). From that time, many mathematicians have studied properties on normal flatness and have obtained many results on it (e.g. [9], [10]). Let Y be the closed subscheme of X defined by the conductor of X in $\bar{X}$. By the definition of normal flatness, if X is normally flat along Y, that is, if the normal cone N of X along Y is flat over Y, then $X' \times_X Y$ is flat over Y where X' is the blowing up of X along Y. On the other hand, there is a canonical morphism from X' to $\bar{X}$ (see Prop. 3 in this paper) and P. H. Wilson showed, in the case where X is a hypersurface, that a necessary and sufficient condition for this canonical morphism to be an isomorphism can be spoken by a "flatness" condition (cf. Theorem 2.7 in his paper [22]). The author believes that, under the condition that X is normally flat along Y, the fibres of N along Y and hence the fibres of X' along Y are well parametrized. In this point of view, we shall study the structure of N and show that *if X is normally flat along Y and Y is of pure codimension* 1 *in X, then*

(i) *X' is naturally isomorphic to $\bar{X}$.*

(ii) *$\bar{X}$ is a Cohen-Macaulay scheme if and only if so is X.*

(iii) *$\bar{X}$ is a Gorenstein scheme if so is X.*

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§ 1. Normal flatness (1)

In this section, we shall consider a noetherian scheme S and a closed subscheme T of S. We refer scheme theoretic languages to [5], [6] and [7].

Let $\mathscr{M}$ be an $\mathcal{O}_S$-module. For any point s of S, we denote the stalk of $\mathscr{M}$ at s by $\mathscr{M}_s$ and the maximal ideal of $\mathcal{O}_{S,s}$ by $\mathfrak{m}_s$. We put $\mathscr{M}(s)=\mathscr{M}_s/\mathfrak{m}_s\mathscr{M}_s$ and $\kappa(s)=\mathcal{O}_S(s)=\mathcal{O}_{S,s}/\mathfrak{m}_s$ as usual.

Let $\mathscr{I}$ be the sheaf of ideals of $\mathcal{O}_S$ which defines T. We denote the normal cone of S along T by $N_{T,S}$. Hence by the definition of normal cones,

$$N_{T,S} = \mathscr{S}pec_T(\mathscr{G}r_{\mathscr{I}}(\mathcal{O}_S))$$

where $\mathscr{G}r_{\mathscr{I}}(\mathcal{O}_S)=\mathcal{O}_T\oplus(\oplus_{n\geqq 1}\mathscr{I}^n/\mathscr{I}^{n+1})$ is the graded $\mathcal{O}_T$-algebra associated with $\mathscr{I}$. For any point t of T, we put

$$N_{T,S}(t) = \mathrm{Spec}\,(\mathscr{G}r_{\mathscr{I}}(\mathcal{O}_S)(t)) = \mathrm{Spec}\,(\kappa(t)\oplus(\oplus_{n\geqq 1}\mathscr{I}_t^n/\mathfrak{m}_t\mathscr{I}_t^n))$$

and $H_{T,S}(t;\, n)=\dim_{\kappa(t)}(\mathscr{I}_t^n/\mathfrak{m}_t\mathscr{I}_t^n)$. Then we have a well-known proposition.

PROPOSITION 1. *There exists the numerical polynomial P with coefficients in the field of rational numbers such that $H_{T,S}(t;\, n)=P(n)$ for every sufficiently large n and the dimension of $N_{T,S}(t)$ is equal to* $\deg(P)+1$.

PROOF. The assertion follows from Th. 20.5 in [15] and Th. 19 of § 7.10 in [18].

DEFINITION 1. We define *the degree of* $H_{T,S}(t;\, n)$ by one of the above polynomial P.

We now give the definition of normal flatness.

DEFINITION 2. For any point t of T, we say that S *is normally flat along T at t* if $\mathscr{G}r_{\mathscr{I}}(\mathcal{O}_S)_t$ is a flat $\mathcal{O}_{T,t}$-module, that is to say, $\mathscr{I}_t^n/\mathscr{I}_t^{n+1}$ is a free $\mathcal{O}_{T,t}$-module for any n. *S is said to be normally flat along T* if S is normally flat along T at any point of T, in other wards, if $\mathscr{I}^n/\mathscr{I}^{n+1}$ is a locally free $\mathcal{O}_T$-module for any n. This is equivalent to the condition that $N_{T,S}$ is flat over T.

We denote the blowing up of S along T by $\mathscr{B}\ell_T(S)$. Hence by the definition of a blowing up,

$$\mathscr{B}\ell_T(S) = \mathscr{P}roj_T(\mathscr{R}_{\mathscr{I}}(\mathcal{O}_S))$$

where $\mathscr{R}_{\mathscr{I}}(\mathcal{O}_S)=\mathcal{O}_S\oplus(\oplus_{n\geqq 1}\mathscr{I}^n)$ is the Rees $\mathcal{O}_S$-algebra defined by $\mathscr{I}$. Then we know that for any point t of T,

$$\mathscr{B}\ell_T(S)\times_T \mathrm{Spec}(\kappa(t)) = \mathrm{Proj}(\kappa(t)\oplus(\oplus_{n\geqq 1}\mathscr{I}_t^n/\mathfrak{m}_t\mathscr{I}_t^n)).$$

While many results on normal flatness have been obtained, we need the following results in this paper.

PROPOSITION 2. *Suppose that S is normally flat along T.*

(i) $H_{T,S}(t_1; n) = H_{T,S}(t_2; n)$ *for any two points t_1, t_2 of a connected component of T. In particular,*

$$\dim(N_{T,S}(t_1)) = \dim(N_{T,S}(t_2)).$$

(ii) *For any point t of T, $\dim(N_{T,S}(t)) = \operatorname{codim}(Z, S)$ where Z is any irreducible component of T which passes t.*

(iii) *For any point t of T, $\dim(\mathcal{O}_{S,t}) = \dim(\mathcal{O}_{T,t}) + \dim(\mathcal{O}_{S,z})$ where z is the generic point of an irreducible component of T which passes t.*

PROOF. The assertions follow from (6.10.5) in [7] and Korollar 1.52. in [9].

COROLLARY 1. *If S is normally flat along T and is connected, then T is of pure codimension in S.*

PROOF. The assertion follows from (i), (ii) in Prop. 2.

COROLLARY 2. *If S is normally flat along T and T is of pure codimension 1 in S, then $\mathcal{B}\ell_T(S)$ is finite over S.*

PROOF. The assertion follows from (ii) in Prop. 2 and (4.4.2) in [6].

COROLLARY 3. *Suppose that S is normal and T is of pure codimension 1 in S. If S is normally flat along T, then $\mathcal{B}\ell_T(S) = S$ and therefore $\mathcal{I}$ is an invertible sheaf of ideals of $\mathcal{O}_S$.*

PROOF. The assertions follow from the above corollary.

§ 2. Normal flatness (2)

From now on, we shall consider a reduced noetherian scheme X of which the normalization, denoted by $\bar{X}$, is finite over X. Let π be the canonical morphism from $\bar{X}$ to X. By the conductor $\mathcal{C}$ of X in $\bar{X}$ we mean the largest sheaf of ideals of $\mathcal{O}_X$ which is also a sheaf of ideals of $\pi_*(\mathcal{O}_{\bar{X}})$. Therefore $\mathcal{C} = \mathcal{A}nn_{\mathcal{O}_X}(\pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X)$. Since $\bar{X} = \mathcal{S}pec_X(\pi_*(\mathcal{O}_{\bar{X}}))$, we may consider $\mathcal{C}$ as a sheaf of ideals of $\mathcal{O}_{\bar{X}}$. We denote by Y and $\bar{Y}$ the closed subschemes of X and $\bar{X}$ defined by $\mathcal{C}$ respectively. We now put shortly

$$N = N_{Y,X}, \quad \bar{N} = N_{\bar{Y},\bar{X}}, \quad X' = \mathcal{B}\ell_Y(X), \quad \text{and} \quad \bar{X}' = \mathcal{B}\ell_{\bar{Y}}(\bar{X}).$$

Then we have the following commutative diagrams.

$$\begin{array}{ccccc} \bar{N} & \xrightarrow{\bar{\tau}} & \bar{Y} & \xrightarrow{j} & \bar{X} \\ {\scriptstyle \upsilon}\downarrow & & {\scriptstyle \bar{\pi}}\downarrow & & {\scriptstyle \pi}\downarrow \\ N & \xrightarrow{\tau} & Y & \xrightarrow{i} & X \end{array} \qquad \begin{array}{ccc} \bar{X}' & \xrightarrow{\pi'} & X' \\ {\scriptstyle \rho'}\downarrow & & {\scriptstyle \rho}\downarrow \\ \bar{X} & \xrightarrow{\pi} & X \end{array}$$

where i and j are natural injections and the others are canonical morphisms induced by π.

P. H. Wilson obtained that π' is an isomorphism in case that X is an irreducible variety (cf. Theorem 1.2 in [22]). More generally we have the following proposition.

PROPOSITION 3. *π' is an isomorphism.*

PROOF. The assertion follows the fact that $(\mathscr{R}_{\mathscr{I}}(\mathcal{O}_X))_+ = (\mathscr{R}_{\mathscr{I}}(\mathcal{O}_{\bar{X}}))_+$ and from the construction of $\mathscr{P}roj$ (cf. (2.4) in [5]).

The following theorem is important in this paper.

THEOREM 1. *The following conditions are equivalent.*

(i) *X is normally flat along Y and Y is of pure codimension 1 in X.*

(ii) (1) *$\bar{Y}$ is flat over Y.*

(2) *$\mathscr{C}$ is an invertible sheaf of ideals of $\mathcal{O}_{\bar{X}}$.*

PROOF. (i)⇒(ii): By Cor. (ii) of Prop. 2 and Prop. 3, we have $X'=\bar{X}'=\bar{X}$. Hence $\mathscr{C}$ is an invertible sheaf of ideals of $\mathcal{O}_{\bar{X}}$. Since $\bar{Y}=\bar{X}\times_X Y=X'\times_X Y=\mathscr{P}roj_Y(\tau_*(\mathcal{O}_N))$ and τ is flat by the definition of normal flatness, $\bar{Y}$ is flat over Y by using (2.2.1) in [5].

(ii)⇒(i): Since $\mathscr{C}$ is an invertible sheaf of ideals of $\mathcal{O}_{\bar{X}}$ by our assumption (2), $\mathscr{C}$ is an invertible one of $\pi_*(\mathcal{O}_{\bar{X}})$. For any $n\geqq 1$, $\mathscr{C}^n/\mathscr{C}^{n+1}\cong\mathscr{C}^n\otimes_{\pi_*(\mathcal{O}_{\bar{X}})}\bar{\pi}_*(\mathcal{O}_{\bar{Y}})$ and therefore we conclude that $\mathscr{C}^n/\mathscr{C}^{n+1}$ is a locally free $\bar{\pi}_*(\mathcal{O}_{\bar{Y}})$-module of rank 1. On the other hand, $\bar{\pi}_*(\mathcal{O}_{\bar{Y}})$ is a locally free $\mathcal{O}_Y$-module by the assumption (1). Hence $\mathscr{C}^n/\mathscr{C}^{n+1}$ is a locally free one for any n. In other words, X is normally flat along Y.

Let z be the generic point of an irreducible component of Y. Since $\bar{X}$ is finite over X, there exists a point $\bar{z}$ of $\bar{Y}$ such that $\pi(\bar{z})=z$ and $\dim(\mathcal{O}_{\bar{X},\bar{z}})=\dim(\mathcal{O}_{X,z})$. From the assumption (1), it follows that

$$\dim(\mathcal{O}_{\bar{Y},\bar{z}})=\dim(\mathcal{O}_{Y,z})=0 \quad \text{(cf. Theorem 20 in [14]).}$$

Therefore $\bar{z}$ is the generic point of some irreducible component of $\bar{Y}$. By the assumption (2), we have $\dim(\mathcal{O}_{\bar{X},\bar{z}})=1$ and hence $\dim(\mathcal{O}_{X,z})=1$. Therefore we conclude that Y is of pure codimension 1 in X.

COROLLARY 1. *Under the equivalent conditions of the above theorem, we* *conclude that the blowing up X' of X along Y is the normalization $\bar{X}$ of X.*

We have already shown the assertion in the above proof of Theorem 1. Hence we omit the proof.

COROLLARY 2. *Let X be an affine scheme. Suppose that $\Gamma(X, \mathcal{O}_X)$ is local. Then the following conditions are equivalent.*

(i) *X is normally flat along Y and* codim$(Y, X)=1$.

(ii) (1) *$\Gamma(\bar{Y}, \mathcal{O}_{\bar{Y}})$ is a free $\Gamma(Y, \mathcal{O}_Y)$-module.*

(2) *$\Gamma(\bar{X}, \mathscr{C})$ is a principal ideal of $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}})$ generated by a regular element.*

PROOF. The assertion can be easily seen by Cor. 1 of Prop. 2 and Th. 1.

As for the dimension of the local ring of $\bar{X}$ at any point, we have the following theorem.

THEOREM 2. *Suppose that X is normally flat along Y and Y is of pure codimension 1 in X. Let $\bar{x}$ be any point of $\bar{X}$ and let x be the point $\pi(\bar{x})$ of X. Then we have*

$$\dim(\mathcal{O}_{\bar{X},\bar{x}}) = \dim(\mathcal{O}_{X,x}).$$

PROOF. We may assume that $\bar{x}$ is contained in $\bar{Y}$. By (iii) of Prop. 2 and Th. 1, we have

$$\dim(\mathcal{O}_{\bar{X},\bar{x}}) = \dim(\mathcal{O}_{\bar{Y},\bar{x}}) + 1,$$

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,x}) + 1.$$

Since $\bar{Y}$ is finite and flat over Y by Th. 1, $\dim(\mathcal{O}_{\bar{Y},\bar{x}})=\dim(\mathcal{O}_{Y,x})$ (cf. Theorem 20 in [14]). Therefore $\dim(\mathcal{O}_{\bar{X},\bar{x}})=\dim(\mathcal{O}_{X,x})$.

In connection with the condition (1) in Theorem 1, we give the following proposition.

PROPOSITION 4. *$\bar{Y}$ is flat over Y if and only if $\pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X=\bar{\pi}_*(\mathcal{O}_{\bar{Y}})/\mathcal{O}_Y$ is a flat $\mathcal{O}_Y$-module.*

PROOF. Since the property of flatness is a local one, the assertion follows from Chap. I, § 3, n° 5, Prop. 9 in [2].

§ 3. A property for schemes to be Cohen-Macaulay

We refer the definitions of depth, local cohomology, Cohen-Macaulay ring and Cohen-Macaulay scheme to the books [8], [10] and [14].

From now to the end of § 5 in this paper, we understand that X is always normally flat along Y and Y is of pure codimension 1 in X. For the sake of simplicity, we use the following notations.

For any fixed point y of Y, we put shortly

$$A = \mathcal{O}_{X,y},\ \bar{A} = \pi_*(\mathcal{O}_{\bar{X}})_y,\ C = \mathscr{C}_y,\ \mathfrak{m} = \mathfrak{m}_y,\ \kappa = \kappa(y) \quad \text{and} \quad K = \bar{A}/\mathfrak{m}\bar{A}.$$

Then $\bar{A}$ is the normalization of A. It follows from Th. 2 that for any maximal ideal $\mathfrak{n}$ of $\bar{A}$, the dimension of $\bar{A}_{\mathfrak{n}}$ is equal to one of A. On the other hand, $H^i_{\mathfrak{m}}(\bar{A}) = H^i_{\mathfrak{m}\bar{A}}(\bar{A})$ for any $i \geqq 0$ (cf. Corollary 5.7 of Proposition 5.5 in [8]).

Since the condition that $\bar{X}$ is a Cohen-Macaulay scheme is a local property, the above facts show the following lemma.

LEMMA. *$\bar{X}$ is a Cohen-Macaulay scheme if and only if $\pi_*(\mathcal{O}_{\bar{X}})$ is a Cohen-Macaulay $\mathcal{O}_X$-module.*

THEOREM 3. *Let y be a point of Y. Then the following conditions are equivalent.*

(1) *X is Cohen-Macaulay at y.*
(2) *Y is Cohen-Macaulay at y.*
(3) *$\bar{X}$ is Cohen-Macaulay along $\pi^{-1}(y)$.*
(4) *$\bar{Y}$ is Cohen-Macaulay along $\bar{\pi}^{-1}(y)$.*

PROOF. By Cor. 2 of Th. 1, C is generated by a regular element of $\bar{A}$. Therefore the equivalence between (3) and (4) are obvious (cf. (ii) of Theorem 30 in [14]). Since $\bar{A}/C$ is a finite and flat extension of A/C by Cor. 2 of Th. 1, the equivalence between (2) and (4) follows from (21. C) in [14]. Hence we conclude that (2), (3) and (4) are equivalent.

We now show that (1) implies (2). Put $C = c\bar{A}$ and $C' = cA$. Then we have $C/C' \cong c\bar{A}/cA \cong \bar{A}/A$ because c is an $\bar{A}$-regular element. Hence C/C' is a free A/C-module by Prop. 4. Set $C/C' \cong \bar{A}/A \cong \oplus^r A/C$ for some positive integer r and consider the following exact sequence

$$0 \longrightarrow C/C' \longrightarrow A/C' \longrightarrow A/C \longrightarrow 0.$$

By our assumption, $\dim(A) = \operatorname{depth}(A)$, say d, we have $\operatorname{depth}(A/C') = \dim(A/C') = d-1$. Since $\bar{A}$ is a normal ring, it follows from the Serre's criterion for normality (cf. Theorem 39 in [14]) that $\bar{A}$ is a Cohen-Macaulay ring if $d \leqq 2$. By the equivalence between (2) and (3), we may assume that d is greater than or equal to 3. Let i be any positive integer which is less than or equal to $d-2$. Then $H^i_{\mathfrak{m}}(A/C') = 0$. By the above exact sequence, we have an exact sequence

$$H^{i-1}_{\mathfrak{m}}(A/C') \longrightarrow H^{i-1}_{\mathfrak{m}}(A/C) \longrightarrow H^i_{\mathfrak{m}}(C/C') \longrightarrow H^i_{\mathfrak{m}}(A/C').$$

Hence we have

$$H_{\mathfrak{m}}^{i-1}(A/C) \cong H_{\mathfrak{m}}^{i}(C/C') \cong \oplus^r H_{\mathfrak{m}}^{i}(A/C) \cdots\cdots\cdots (*).$$

Since $\bar{A}$ is normal and $\dim(\bar{A}) \geqq 3$, we have $\operatorname{depth}(\bar{A}_{\mathfrak{n}}) \geqq 2$ for any maximal ideal $\mathfrak{n}$ of $\bar{A}$ by the Serre's criterion for normality and Th. 2. Therefore $\operatorname{depth}(\bar{A}_{\mathfrak{n}}/C\bar{A}_{\mathfrak{n}}) = \operatorname{depth}(\bar{A}_{\mathfrak{n}}/c\bar{A}_{\mathfrak{n}}) \geqq 1$. On the other hand, $\operatorname{depth}(\bar{A}_{\mathfrak{n}}/C\bar{A}_{\mathfrak{n}}) = \operatorname{depth}(A/C) + \operatorname{depth}(\bar{A}_{\mathfrak{n}}/\mathfrak{m}\bar{A}_{\mathfrak{n}}) = \operatorname{depth}(A/C)$ because $\bar{A}/C$ is a finite and flat extension of A/C (cf. (21. C) in [14]). Hence $\operatorname{depth}(A/C) \geqq 1$, that is, $H_{\mathfrak{m}}^{0}(A/C) = 0$. Therefore we have $H_{\mathfrak{m}}^{i}(A/C) = 0$ by $(*)$. Hence we conclude that $\operatorname{depth}(A/C) \geqq d-1$. On the other hand, $\dim(A/C) = d-1$ by (iii) of Prop. 2. Therefore A/C is a Cohen-Macaulay ring. This shows (2).

Next we show that (3) implies (1). Consider the following exact sequence

$$0 \longrightarrow A \longrightarrow \bar{A} \longrightarrow \bar{A}/A \longrightarrow 0.$$

Let i be any non-negative integer which is less than or equal to $d-1$ where $d = \dim(A) = \dim(\bar{A})$. Then we have an exact sequence

$$H_{\mathfrak{m}}^{i-1}(\bar{A}/A) \longrightarrow H_{\mathfrak{m}}^{i}(A) \longrightarrow H_{\mathfrak{m}}^{i}(\bar{A}) \cdots\cdots\cdots (**)$$

where we put $H_{\mathfrak{m}}^{-1}(\bar{A}/A) = 0$. Since $\bar{A}$ is a Cohen-Macaulay ring of dimension d by our assumption, $H_{\mathfrak{m}}^{i}(\bar{A}) = 0$ by the above lemma. By $(*)$ and the equivalence between (2) and (3), we have

$$H_{\mathfrak{m}}^{i-1}(\bar{A}/A) \cong \oplus^r H_{\mathfrak{m}}^{i-1}(A/C) = 0.$$

Therefore we have $H_{\mathfrak{m}}^{i}(A) = 0$ by the above fact and $(**)$. In other words, A is a Cohen-Macaulay ring. This shows (1).

We now give easy consequences of the above theorem but we omit thier proofs.

COROLLARY 1. *$\bar{X}$ is a Cohen-Macaulay scheme if and only if so is X. And if so, Y and $\bar{Y}$ are Cohen-Macaulay schemes.*

COROLLARY 2. *$\bar{X}$ satisfies the Serre's condition (S_n)* (cf. (17. I) in [14]) *if and only if so does X.*

COROLLARY 3. *X satisfies the Serre's condition (S_2) and hence Y has no embedded component* (cf. the proof of (vi) in Theorem 2.6 in [4]). *In particular, if X is of dimension 2, then X and Y are Cohen-Macaulay schemes.*

§4. Fibres of the normal cone

In this section, we shall study some properties on the structure of the fibres $N(y)$ of the normal cone N of X along Y at any point y of Y. Under the same notations as in § 2 and § 3, we have $\bar{\pi}_*(\mathscr{G}_{\mathfrak{t}_{\mathscr{C}}}(\mathcal{O}_{\bar{X}}))_y = gr_C(\bar{A})$, $\mathscr{G}_{\mathfrak{t}_{\mathscr{C}}}(\mathcal{O}_X)_y = gr_C(A)$ and by Cor. 2 of Th. 1, $gr_C(\bar{A}) = \bar{A}/C[U]$ where U is an indeterminate, and $gr_C(A) = A/C \oplus U\bar{A}/C[U]$ because $gr_C(A)_+ = gr_C(\bar{A})_+$. Consider the following exact sequence of A/C-modules

$$0 \longrightarrow gr_C(A) \longrightarrow gr_C(\bar{A}) \longrightarrow \bar{A}/A \longrightarrow 0.$$

Then we have an exact sequence

$$0 \longrightarrow \kappa \otimes gr_C(A) \longrightarrow \kappa \otimes gr_C(\bar{A}) \longrightarrow \kappa \otimes \bar{A}/A \longrightarrow 0$$

because $\bar{A}/A$ is a flat A/C-module by Prop. 4. Therefore we have $\kappa \otimes gr_C(\bar{A}) = K[U]$ and $\kappa \otimes gr_C(A) = \kappa \oplus UK[U]$.

From now on, we put $H(y; n) = H_{Y,X}(y; n)$ and $h(y) = H(y; 1)$ for the sake of simplicity. Then we have the following proposition.

PROPOSITION 5. *$N(y)$ and $\bar{N}(y)$ are Cohen-Macaulay algebraic schemes of dimension 1 defined over the field $\kappa = \kappa(y)$. The multiplicity and the embedded dimension of $N(y)$ at the origin are same and equal to $h(y)$. In fact, $H(y; n) = h(y)$ for all $n \geqq 1$.*

PROOF. The first assertion is obvious by the above discussion. The last two assertions follow from the facts that $C^n/C^{n+1} \cong \bar{A}/C$ for any $n \geqq 1$ and $\bar{A}/C$ is a free A/C-module by Cor. 2 of Th. 1.

We now give a sufficient condition for N and $\bar{N}$ to be Cohen-Macaulay schemes.

THEOREM 4. *If X is a Cohen-Macaulay scheme, then so are N and $\bar{N}$.*

PROOF. Since τ and $\bar{\tau}$ are flat, the assertion follows from Cor. 1 of Th. 3, the above Prop. 5 and (21. C) in [14].

We refer the definition of seminormality and one of glueings to [4], [20] and [23]. Then we have the following theorem.

THEOREM 5. *For any point y of Y, $N(y)_{\text{red}}$ is a seminormal curve with an isolated singularity and its normalization is $\bar{N}(y)_{\text{red}}$.*

PROOF. By the beginning of this section, we know that $\kappa \otimes gr_C(A) \subset \kappa \otimes gr_C(\bar{A})$. Hence we have $(\kappa \otimes gr_C(A))_{\text{red}} \subset (\kappa \otimes gr_C(\bar{A}))_{\text{red}}$. On the other hand, the last ring is $\bar{K}[U]$ where $\bar{K} = K_{\text{red}} = \bar{A}/J(\bar{A})$ and $J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Therefore $(\kappa \otimes gr_C(A))_{\text{red}} = \kappa \oplus U\bar{K}[U]$. Since $\bar{K}$ is a finite product of fields, $\bar{K}[U]$ is a normal ring. Hence the last assertion is obvious by the above discussion. On the other hand, the conductor of $\kappa \oplus U\bar{K}[U]$ in $\bar{K}[U]$ is $U\bar{K}[U]$.

Therefore it is a radical ideal of $\bar{K}[U]$ and is the homogeneous maximal ideal of $\kappa\oplus U\bar{K}[U]$. Hence the first assertion follows from Corollary 2.7 of Theorem 2.6 in [4].

COROLLARY 1. *If X is an algebraic scheme defined over an algebraically closed field κ and y is any closed point of Y, then $\bar{N}(y)_{\mathrm{red}}$ is a disjoint union of affine lines and $N(y)_{\mathrm{red}}$ is the curve which is obtained by glueing the origins of the above lines.*

PROOF. Under the same notations as in the proof of the above theorem, $\bar{K}$ is the s times product of κ for some positive integer s because $\bar{K}$ is a reduced artinian ring which is finite over κ and κ is algebraically closed field by our assumption. Therefore the first assertion is trivial. Since $\kappa\oplus U\bar{K}[U]$ is isomorphic to $\kappa[U_1,\dots, U_s]/(U_iU_j \mid i\neq j)$ where $U_1,\dots, U_s$ are indeterminates (cf. Corollary 3 of Theorem 1 in [3]), we can prove the second assertion.

We now give a result on the number of branch points of X at any closed point y of Y under suitable conditions.

COROLLARY 2. *Under the same assumptions as the above corollary, if $\bar{X}$ is unramified over X, then $N(y)$ is a seminormal curve and its normalization is $\bar{N}(y)$. Moreover the number of branch points of X at y is equal to $h(y)$ and $h(y_1)=h(y_2)$ for any two points y_1, y_2 if they are contained in a same connected component of Y.*

PROOF. Under the same notations as in the beginning of this section, $K=K_{\mathrm{red}}$ because $\bar{X}$ is unramified over X. Hence we conclude that $\bar{N}(y)=\mathrm{Spec}\,(K[U])$ is reduced. Therefore $N(y)=N(y)_{\mathrm{red}}$. Hence the former assertion follows from the above corollary. The latter one follows from the fact that $\pi^{-1}(y)\cong \mathrm{Proj}\,((\bar{\pi}\bar{\tau})_*(\mathcal{O}_{\bar{N}})(y))$ by Prop. 3, Cor. 1 of Th. 1 and from (i) of Prop. 2, Prop. 5 and Cor. 1 of Th. 5.

§5. A property for schemes to be Gorenstein

For any noetherian scheme S, we define the following condition for any non-negative integer n.

(G_n): Let s be any point of S. If $\dim(\mathcal{O}_{S,s})\leqq n$, then $\mathcal{O}_{S,s}$ is a Gorenstein ring.

We shall show the following:

PROPOSITION 6. *Let y be the generic point of an irreducible component of Y. If X satisfies the condition (G_1), then the multiplicity of X at y is equal to 2 and $h(y)=2$.*

PROOF. Under the same notations as in § 3, $\bar{A}/C$ is flat over A/C by Th. 1. Let e be the multiplicity of A. Then

$$e = \dim_{\kappa}(K) = \dim_{\kappa}(\kappa \otimes \bar{A}/C) = \mathrm{rank}_{A/C}(\bar{A}/C),$$
$$\mathrm{length}_{A/C}(\bar{A}/C) = \mathrm{rank}_{A/C}(\bar{A}/C)\mathrm{length}_{A/C}(A/C).$$

Since Y is of pure codimension 1 in X, A is a Gorenstein ring of dimension 1 by our assumption. Hence $\mathrm{length}_{A/C}(\bar{A}/C)=2\,\mathrm{length}_{A/C}(A/C)$ (cf. Korollar 3.5 von Satz 3.3 in [10]). Therefore $\mathrm{rank}_{A/C}(\bar{A}/C)=2$. Hence $e=2$. On the other hand, $h(y)=H(y;\,1)=\dim_{\kappa}(\kappa \otimes C/C^2)$ and $C/C^2 \cong \bar{A}/C$ by Th. 1. Therefore $h(y)=e=2$.

PROPOSITION 7. *If X satisfies the condition (G_1), then $N(y)$ is a Gorenstein affine plane curve defined over the field $\kappa(y)$ for any point y of Y and the multiplicity at the origin is equal to* 2. *Moreover $N(y)$ is a complete intersection in the affine plane over* $\mathrm{Spec}\,(\kappa(y))$.

PROOF. Since $h(y)=2$ by (i) of Prop. 2 and the above proposition, we have $\kappa(y) \otimes gr_C(A) \cong \kappa(y)\,[U_1,\,U_2]/(f)$ where U_1 and U_2 are indeterminates and f is a form of degree 2. Therefore the assertions are obvious.

COROLLARY 1. *Let X is an algebraic scheme defined over an algebraically closed field. If X satisfies the condition (G_1) and $\bar{X}$ is unramified over X, then the number of branch points of X at any closed point of Y is equal to* 2.

The assertion is obvious and we omit the proof.

COROLLARY 2. *If X satisfies the condition (G_1), then $\pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of rank* 1.

PROOF. Since the rank of $\bar{\pi}_*(\mathcal{O}_{\bar{Y}})$ at any point y of Y is equal to $h(y)$ by Cor. 2 of Th. 1, it is equal to 2 by the proof of the above proposition. Therefore the assertion follows from Prop. 4 and from the following exact sequence of $\mathcal{O}_Y$-modules

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \bar{\pi}_*(\mathcal{O}_{\bar{Y}}) \longrightarrow \pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X \longrightarrow 0.$$

In connection with canonical modules, we shall study the $\mathcal{O}_Y$-module $\pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X$. Now for any coherent $\mathcal{O}_X$-module $\mathcal{M}$, we denote the dual module $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\,\mathcal{O}_X)$ of $\mathcal{M}$ by $\mathcal{M}^*$. Then we have the following proposition.

PROPOSITION 8. *There exist canonical isomorphisms from $\pi_*(\mathcal{O}_{\bar{X}})$ to $\mathcal{C}^*$ and from $\pi_*(\mathcal{O}_{\bar{X}})/\mathcal{O}_X$ to $\mathcal{E}xt^1_{\mathcal{O}_X}(i_*(\mathcal{O}_Y),\,\mathcal{O}_X)$ where i is the canonical injection from Y to X.*

PROOF. Consider the following exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathscr{C} \longrightarrow \mathcal{O}_X \longrightarrow i_*(\mathcal{O}_Y) \longrightarrow 0.$$

Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathscr{C}^* \longrightarrow \mathscr{E}xt^1_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X) \longrightarrow 0 \cdots (*)$$

because $i_*(\mathcal{O}_Y)^*=0$ and $\mathscr{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)=0$. Since $\mathcal{O}_X \cong \mathcal{O}_X^*$, we consider $\mathcal{O}_X$ as an $\mathcal{O}_X$-submodule of $\mathscr{C}^*$ by means of scalar multiplications. On the other hand, $\mathscr{C}$ is also a sheaf of ideals of $\pi_*(\mathcal{O}_{\bar{X}})$, we may naturally consider $\pi_*(\mathcal{O}_{\bar{X}})$ as an $\mathcal{O}_X$-submodule of $\mathscr{C}^*$ by the same method. We now show that $\pi_*(\mathcal{O}_{\bar{X}})_y = \mathscr{C}^*_y$ for any point y of X. We may assume that y is a point of Y. Under the same notations as in § 3, $\mathscr{C}^*_y = \mathrm{Hom}_A(C, A) = A :_Q C$ where Q is the full ring of quotients of A (cf. Lemma 2.1 in [10]). By Cor. 2 of Th. 1, we may put $C = c\bar{A}$ for some $\bar{A}$-regular element c of C. Then we have $A :_Q C = A :_Q c\bar{A} = 1/c(A :_Q \bar{A})$ in Q. Since $A :_Q \bar{A} \subset A$ because $\bar{A}$ has the unity, we have $A :_Q \bar{A} = A :_A \bar{A} = C$ by the definition of the conductor. Therefore $A :_Q C = 1/c(\mathrm{C}) = 1/c(c\bar{A}) = \bar{A}$. Hence we prove the first assertion. The second one follows from the first one and from the exact sequence $(*)$.

From now on we put $\Omega = \mathscr{E}xt^1_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X)$. Then we have the following corollary.

COROLLARY. *If X satisfies the condition (G_1), then Ω is a locally free $\mathcal{O}_Y$-module of rank* 1.

PROOF. The assertion follows from Cor. 2 of Prop. 7 and the above proposition.

PROPOSITION 9. *Suppose that X satisfies the condition (G_1). Then N satisfies the condition (G_n) if and only if so does Y.*

PROOF. Since τ is flat and surjective, the assertion follows from Prop. 7 and Theorem 1′ in [21].

For any coherent $\mathcal{O}_Y$-module $\mathscr{M}$, we have the natural isomorphism

$$\mathscr{H}om_{i_*(\mathcal{O}_Y)}(\mathscr{M}, i_*(\mathcal{O}_Y)^*) \cong \mathscr{M}^*.$$

Hence we have a spectral sequence of $\mathcal{O}_X$-modules

$$\mathscr{E}xt^p_{i_*(\mathcal{O}_Y)}(\mathscr{M}, \mathscr{E}xt^q_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X)) \Longrightarrow \mathscr{E}xt^{p+q}_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X) \cdots (*).$$

Then we have the following proposition.

PROPOSITION 10. *If X is a Gorenstein scheme, then we have*

$$\mathscr{E}xt^p_{i_*(\mathcal{O}_Y)}(\mathscr{M}, \Omega) \cong \mathscr{E}xt^{p+1}_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X)$$

for any coherent $\mathcal{O}_Y$-module $\mathcal{M}$.

PROOF. Since X is a Gorenstein scheme and therefore it is a Cohen-Macaulay scheme, Y is a Cohen-Macaulay scheme of dimension $\dim(X)-1$ by Cor. 1 of Th. 3. Hence we have $\mathcal{E}xt^q_{\mathcal{O}_X}(i_*(\mathcal{O}_Y), \mathcal{O}_X)=0$ if $q \neq 1$ by the duality theorem for Gorenstein schemes (cf. Theorem 6.3 in [8]). Therefore the above spectral sequence $(*)$ is degenerate. Hence we conclude the assertion.

Let y be any point of Y. Under the same notations as in § 3, put $d=\dim(A)$ and let I be an injective hull of κ as an A-module. Then $\bar{I}$ is an injective hull of κ as an A/C-module where $\bar{I}=\mathrm{Hom}_A(A/C, I)$. For any A-module M, we denote $\mathrm{Hom}_A(M, I)$ by $D(M)$. If M is an A/C-module, then we have $D(M)\cong \mathrm{Hom}_{A/C}(M, \bar{I})$. For any finitely generated A/C-module M, we know that $H^n_{\mathfrak{m}/C}(M)\cong H^n_{\mathfrak{m}}(M)$ for any non negative integer n. If X is a Gorenstein scheme, $\Omega_y \cong A/C$ by Cor. 2 of Prop. 7. Hence for any finitely generated A/C-module M, we have $\mathrm{Ext}^p_{A/C}(M, A/C)\cong \mathrm{Ext}^{p+1}_A(M, A)$ by the above proposition. On the other hand, $D(\mathrm{Ext}^{p+1}_A(M, A))\cong H^{d-1-p}_{\mathfrak{m}}(M)\cong H^{d-1-p}_{\mathfrak{m}/C}(M)$ by the duality theorem for Gorenstein rings. Therefore we have the following theorem.

THEOREM 6. *If X is a Gorenstein scheme, then so are Y and N.*

PROOF. Under the same notations as in above discussion, A/C is a Cohen-Macaulay ring of dimension $d-1$ by Cor. 1 of Th. 3. Since $\mathrm{Hom}_{A/C}(\mathrm{Ext}^p_{A/C}(M, A/C), \bar{I})\cong D(\mathrm{Ext}^{p+1}_A(M, A))\cong H^{d-1-p}_{\mathfrak{m}/C}(M)$ by the above discussion, A/C is a canonical module of A/C. Hence A/C is Gorenstein. This shows that Y is a Gorenstein scheme. Hence N is a Gorenstein scheme by Prop. 9.

COROLLARY. *If X satisfies the condition (G_n), then Y satisfies the condition (G_{n-1}) and hence so does N.*

PROOF. The assertion follows from Th. 3, Prop. 9 and the above theorem.

We now study the property for $\bar{X}$ to be Gorenstein. Since $\mathcal{C}$ is an invertible sheaf of ideals of $\pi_*(\mathcal{O}_X)$, for any point y of Y, X is Gorenstein along $\pi^{-1}(y)$ if and only if $\bar{Y}$ is so along $\bar{\pi}^{-1}(y)$ (cf. Theorem 4.1 in [1] and Theorem 206 in [12]). On the other hand, $\bar{Y}=\mathcal{P}roj_Y(\tau_*(\mathcal{O}_N))$ and $\tau_*(\mathcal{O}_N)$ is generated by $\tau_*(\mathcal{O}_N)_1$ over $\mathcal{O}_Y$, which is a subsheaf of $\tau_*(\mathcal{O}_N)$ of degree 1. Now we consider Y as the vertex of N and let λ be a canonical morphism from $N-Y$ to $\bar{Y}$. Then λ is a smooth and surjective morphism by (2.2.1) in [5]. Therefore we have the following theorem.

THEOREM 7. *If X satisfies the condition (G_n), then so does $\bar{X}$ and $\bar{Y}$ satisfies the condition (G_{n-1}).*

PROOF. The assertion follows from Cor. of Th. 6, Theorem 1′ in [21] and the above discussion.

COROLLARY. *If X is a Gorenstein scheme, then so are $\bar{X}$ and $\bar{Y}$.*

The assertion is obvious by the above theorem and we omit the proof.

§6. Examples

Let $(R, \mathfrak{m}, \kappa)$ be a reduced noetherian local ring of dimension 1 and let $\bar{R}$ be the normalization of R. Suppose that R is not normal and $\bar{R}$ is a finite extension of R. Then the conductor, say C, of R in $\bar{R}$ is an $\mathfrak{m}$-primary ideal. Since $\bar{R}$ is a principal ideal ring, Spec(R) is normally falt along Spec(R/C) if and only if $\bar{R}/C$ is a flat R/C-algebra by Cor. 2 of Th. 1. If R is a Gorenstein ring and $\bar{R}/C$ is flat over R/C, then the multiplicity of R is equal to 2 by Prop. 6.

We refer the definition of the first neighbourhood to [13]. Then we have the following proposition.

PROPOSITION 11. *If R is a Gorenstein ring and $\bar{R}/C$ is flat over R/C, then the first neighbourhood of R is $\mathfrak{m}^{-1}$.*

PROOF. The assertion follows from Theorem 12.17, Theorem 13.3 in [13] and Prop. 6.

PROPOSITION 12. *Let κ be a field and let U be an indeterminate. Put $R=\kappa[U^n, U^{n+2p-1}]_{(U^n,U^{n+2p-1})}$ with $n\geqq 2$ and $p\geqq 1$. Then $\bar{R}/C$ is flat over R/C if and only if $n=2$.*

PROOF. Since R is a Gorenstein ring, the "only if" part is obvious by Prop. 6. We now show the "if" part. Since the conductor C of $R=\kappa[U^2, U^{2p+1}]_{(U^2,U^{2p+1})}$ is $(U^{2p}, U^{2p+1})R$ and the normalization $\bar{R}$ of R is $\kappa[U]_{(U)}$, we have $C=(U^{2p})\bar{R}$. Therefore we have $R/C=\kappa[U^2]/(U^{2p})\kappa[U^2]$ and $\bar{R}/C=\kappa[U]/(U^{2p})\kappa[U]$. Hence $\bar{R}/C=R/C\oplus\bar{U}R/C$ where $\bar{U}$ is the image of U in $\bar{R}/C$. Our assertion follows from the above fact.

In case that the conductor C is the maximal ideal $\mathfrak{m}$ of R, Spec(R) is trivially normally flat along Spec(R/C). In the above proposition, this is the only one case of $p=1$. But all singularities of curves in the above proposition are cuspidal. In connection with ordinary multifold points, we give the following proposition.

PROPOSITION 13. *Under the same notations as in § 2,*

(i) *if X is a seminormal curve which is not normal, then X is normally flat along Y and Y is of pure codimension 1 in X. More generally,*

(ii) *if X is a seminormal scheme which is not normal and satisfies the Serre's* *condition* (S_2), *then* Y *is of pure codimension* 1 *in* X *and there exists an open subset* W *of* X *such that* $W \cap Y$ *is dense in* Y *and* W *is normally flat along* $W \cap Y$.

PROOF. The assertions follow from Theorem 1 in [3], Corollary 2.7 of Theorem 2.6 in [4] and Corollary of Theorem 1 (p. 189) in [11].

We now consider the following seminormal curve (cf. Corollary 3 of Theorem 1 in [2]).

$$X = \mathrm{Spec}\,(\kappa[U_1,\dots, U_n]/(U_iU_j \mid i \neq j)) \quad \text{with} \quad n \geqq 3$$

where κ is a field and U_i's are indeterminates. Under the same notations as in § 2, X is normally flat along Y by the above proposition. Although $\bar{X}$ is a regular scheme, X is not a Gorenstien scheme by Prop. 6 because the multiplicity of X at the origin is equal to $n > 2$. Therefore the converse of corollary of Th. 7 is false.

Appendix

In connection with the notion of normal flatness, L. Robbiano and G. Valla defined the concept of normal torsion-freeness in their joint work [19]. Let S be a noetherian scheme and let $\mathscr{I}$ be a sheaf of ideals of $\mathcal{O}_S$. Under the same notations as in § 1, we give the definition of normal torsion-freeness.

DEFINITION. Let T be the closed subscheme of S defined by $\mathscr{I}$. We say that S is *normally torsion-free* along T if $\mathscr{I}^n/\mathscr{I}^{n+1}$ is a torsion-free $\mathcal{O}_T$-module for any natural number n.

We now give a sufficient condition that the blowing up of a normal scheme is also normal.

PROPOSITION. *Under the same notations as in* § 1, *let* S *be normal* and $\mathscr{I}$ *be divisorial, that is to say,* $\mathscr{I}_s$ *be a divisorial ideal of* $\mathcal{O}_{S,s}$ *for any point* s *of* S. *If* S *is normally torsion-free along* T, *then we have*

(i) $\mathscr{I}^n$ *is divisorial for any* n.

(ii) $\mathscr{R}_{\mathscr{I}}(\mathcal{O}_S)$ *is a normal* $\mathcal{O}_S$*-algebra.*

In particular, the blowing up of S *along the center* T *is a normal scheme.*

PROOF. We may assume that S is a normal integral affine scheme. Put $B = \Gamma(S, \mathcal{O}_S)$ and $I = \Gamma(S, \mathscr{I})$. Then we have $\Gamma(S, \mathscr{R}_{\mathscr{I}}(\mathcal{O}_S)) = \oplus_{n \geqq 0} I^n$ where $I^0 = B$. Since I is a divisorial ideal of B, we have $\mathrm{Ass}_B(B/I) \subset \mathrm{Ht}_1(B)$ where $\mathrm{Ht}_1(B)$ is the set of prime ideals of B of height 1. On the other hand, for any element $\mathfrak{q}$ of $\mathrm{Ass}_B(I^n/I^{n+1})$ there exsits an element $\mathfrak{p}$ of $\mathrm{Ass}_B(B/I)$ such that $\mathfrak{q} \subset \mathfrak{p}$ because I^n/I^{n+1} is a torsion-free B/I-module by our assumption. Since $\mathfrak{p} \in \mathrm{Ht}_1(B)$, $\mathfrak{q} = \mathfrak{p}$.

Hence we have $\mathrm{Ass}_B(I^n/I^{n+1}) \subset \mathrm{Ass}_B(B/I) \subset \mathrm{Ht}_1(B)$. Therefore under the notations and terminologies in [16] and [17], we have $(I^n/I^{n+1})^{\sim}=0$ and hence I^n/I^{n+1} is a codivisorial B-module, that is, I^{n+1} is divisorial in I^n. Since I is divisorial, it follows from Corollary 1 of Proposition 12 in [16] that I^n is divisorial by induction on n. Therefore $\oplus_{n\geqq 0} I^n$ is a divisorial B-module by Proposition 34 in [17]. This fact implies that $\oplus_{n\geqq 0} I^n = \cap_{\mathfrak{p}} \oplus_{n\geqq 0} I^n_{\mathfrak{p}}$ by (i) of Theorem 4 in [16] and Corollary 3 of Proposition 34 in [17] where $\mathfrak{p}$ runs over the set $\mathrm{Ht}_1(B)$. Since $B_{\mathfrak{p}}$ is a principal valuation ring for any element $\mathfrak{p}$ of $\mathrm{Ht}_1(B)$, $\oplus_{n\geqq 0} I^n_{\mathfrak{p}}$ is isomorphic to the polynomial ring of one variable over $B_{\mathfrak{p}}$ and hence it is normal. Therefore $\oplus_{n\geqq 0} I^n$ is a normal ring. The last assertion is obvious.

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