HIROSHIMA MATH. J.
17 (1987), 433–446

On homomorphisms of cocommutative coalgebras and Hopf algebras

Hiroshi YANAGIHARA

(Received January 13, 1987)

Let

$$k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J \longrightarrow k$$

be an exact sequence of cocommutative Hopf algebras over a field k and let C be a cocommutative coalgebra over k. Then it is known that the induced sequence

$$\{e\} \longrightarrow \mathrm{Hom}_{coal}(C, G) \xrightarrow{j_*} \mathrm{Hom}_{coal}(C, H) \xrightarrow{\rho_*} \mathrm{Hom}_{coal}(C, J)$$

of groups is also exact, but that ρ_* is not necessarily surjective. In the paper [2] T. Shudo gave a condition for this homomorphism ρ_* to be always surjective in the case that H is a hyperalgebra. Precisely he showed that ρ_* is surjective for any connected cocommutative coalgebra C over k if and only if the Hopf algebra homomorphism j has a coalgebra retraction $\eta: H \to G$ such that $\eta \circ j$ is the identity map of G.

The main purpose of this paper is to show that the above result for hyperalgebras and connected cocommutative coalgebras is also true for any pointed cocommutative Hopf algebras and coalgebras. In §1 we shall show firstly some properties of cocommutative coalgebras over a field k and coalgebra homomorphisms between them, which are well known in connected cases. Then we shall show in Propositions 4 and 6 that the properties for coalgebra homomorphisms to have coalgebra splittings and coalgebra retractions are colocal in a sense. These results play essential roles in the proof of our main results. In §2 we shall give two theorems. Theorem 1 says that a sequence of pointed cocommutative Hopf algebras over k is exact if and only if the induced sequence of groups consisting of grouplike elements and the sequence of hyperalgebra components of the given sequence are both exact. Theorem 2 is our main result and is a generalization of Shudo's result mentioned in the above.

Throughout this paper we fix a ground field k. All coalgebras, Hopf algebras and their tensor products are defined over k, and our terminology and notations follow those in [3], [4], [5] and [6].

§ 1. Some properties of coalgebra homomorphisms

PROPOSITION 1. *Let (C, Δ, ε) be a cocommutative coalgebra over a field k and g a grouplike element of C. Denote by A the dual algebra C^* of C and by D the minimal subcoalgebra kg of C generated by g. Moreover let $\mathfrak{m}$ be the null-space $D^\perp$ of D in A and C_i the null-space $(\mathfrak{m}^{i+1})^\perp$ of the ideal $\mathfrak{m}^{i+1}$ of A in C. Then we have the following:*

(i) *$D_1 = \bigcup_{i=0}^{\infty} C_i$ is the largest connected subcoalgebra of C which contains D.*

(ii) *For any integer $i \geqslant 0$, an element x in C is contained in C_i if and only if $a \cdot x \in C_{i-1}$ for any a in $\mathfrak{m}$, where C is considered to be an A-module as defined in* §3 *in* [4] *and we understand $C_{-1} = (0)$.*

(iii) *If x is contained in C_n, then $\Delta(x) \in \sum_{i=0}^{n} C_{n-i} \otimes C_i$.*

PROOF. (i). If we put $D_x = A \cdot x$ for a non-zero element x in D_1, then D_x is a finite dimensional subcoalgebra of C by Corollary 3.9 in [4] and $A/D_x^\perp$ is isomorphic to the dual algebra D_x^* of D_x by Corollary 3.3 in [4]. Since $x \in D_1$, there is an integer i such that $x \in C_i$. By Proposition 3.8 in [4] C_i is an A-submodule of C and hence $D_x = A \cdot x$ is contained in C_i. Therefore we see

$$D_x^\perp \supset (C_i)^\perp = ((\mathfrak{m}^{i+1})^\perp)^\perp \supset \mathfrak{m}^{i+1}.$$

Now let D' be a minimal subcoalgebra of D_1. If x is a non-zero element of D', then we have $A \cdot x = D'$ and $D'^\perp = (D_x)^\perp$ is a maximal ideal of $A = C^*$. Since $D'^\perp \supset \mathfrak{m}^{i+1}$ for some i, we see that $D'^\perp$ contains $\mathfrak{m}$ and hence $D'^\perp$ must be equal to $\mathfrak{m}$. Therefore D_1 contains only one minimal subcoalgebra $D = kg$. Next let E be a subcoalgebra of C containing D and assume that E has no minimal subcoalgebra but D. Then we see that $E^\perp \subset D^\perp = \mathfrak{m}$ and $A/E^\perp$ is isomorphic to E^* by Corollary 3.3 in [4]. If we put $\mathfrak{m}' = \mathfrak{m}/E^\perp \subset E^*$ and $C_i' = (\mathfrak{m}'^{i+1})^\perp \subset E$, then we have $E = \bigcup_{i=0}^{\infty} C_i'$ by Proposition 3.11 in [4]. On the other hand we see that

$$C_i' = (\mathfrak{m}'^{i+1})^\perp = (\mathfrak{m}^{i+1} + E^\perp/E^\perp)^\perp = E \cap (\mathfrak{m}^{i+1})^\perp = E \cap C_i.$$

Therefore we see that

$$E = \bigcup_{i=0}^{\infty} C_i' = \bigcup_{i=0}^{\infty} (E \cap C_i) \subset \bigcup_{i=0}^{\infty} C_i = D_1$$

and hence that D_1 is the largest connected subcoalgebra of C containing D.

The assertions (ii) and (iii) can be shown in the exactly same way as the proof of (ii) and (iii) of Proposition 3.11 in [4] and hence we omit the detail.

COROLLARY. *Let (C, Δ, ε), g, $D = kg$, $A = C^*$ and C_i be as in Proposition 1. Let C^0 be the kernel of ε and put $C_i^0 = C^0 \cap C_i$. Then an element x in C belongs to C_n^0 if and only if*

$$\Delta(x) - x \otimes g - g \otimes x \in \textstyle\sum_{i=1}^{n-1} C_{n-i}^0 \otimes C_i^0$$

where we understand $\sum_{i=1}^{n-1} C_{n-i}^0 \otimes C_i = 0$ *for* $n=1$.

PROOF. This corollary can be shown in the same way as the proof of Proposition 3.13 in [4] and hence we omit the detail.

Let (C, Δ, ε) be a cocommutative coalgebra over k and let g be a grouplike element of C. Then an element x of C is called *a primitive element of C with respect to g*, if we have $\Delta(x) = x \otimes g + g \otimes x$. In the following we denote by $G(C)$ the set of grouplike elements of C and by $P_q(C)$ the set of primitive elements of C with respect to g. It is easy to see that $P_g(C)$ is a k-subspace of C and that we have $\varepsilon(x) = 0$ for any x in $P_g(C)$.

PROPOSITION 2. *Let g be a grouplike element of a cocommutative coalgebra C and put $D_g = kg$. If A is the dual algebra C^* of C and $\mathfrak{m}$ is the ideal of A which is the null-space $D_g^\perp$ of D_g in A, then the null-space $C_1 = (\mathfrak{m}^2)^\perp$ of $\mathfrak{m}^2$ in C is the direct sum of D_g and $P_g(C)$.*

PROOF. Let Δ and ε be the comultiplication and the coidentity of C, respectively. Since we have $\varepsilon(g) = 1$ and $\varepsilon(x) = 0$ for any x in $P_g(C)$, the sum $D_g + P_g(C)$ is a direct one. If c is an element of C_1 and we put $d = c - \varepsilon(c)g$, then d belongs to C_1 and $\varepsilon(d) = 0$. Therefore we see, by Proposition 1, (iii), that

$$\Delta(d) = d_1 \otimes g + g \otimes d_2 \quad \text{with} \quad d_i \in C_1.$$

From this equality we see that

$$0 = \varepsilon(d) = (\varepsilon \otimes \varepsilon)\Delta(d) = \varepsilon(d_1) + \varepsilon(d_2) \tag{$*$}$$

and

$$d_1 + \varepsilon(d_2)g = d = d_2 + (d_1)g \tag{$**$}$$

using $(id_C \otimes \varepsilon)\Delta(d) = d = (\varepsilon \otimes id_C)\Delta(d)$. Hence we have from $(**)$ and $(*)$

$$\begin{aligned} \Delta(d) &= d_1 \otimes g + g \otimes d_2 \\ &= (d - \varepsilon(d_2)g) \otimes g + g \otimes (d - \varepsilon(d_1)g) \\ &= d \otimes g + g \otimes d. \end{aligned}$$

This means that d is contained in $P_g(C)$ and hence that C_1 is a subspace of $D_g \oplus P_g(C)$. Conversely let x be an element of $P_g(C)$. If a and b are any elements of $\mathfrak{m} = (D_g)^\perp$, then we see by the definition of A-module structure of C given in §3 in [4] that

$$\begin{aligned}\langle a\cdot x,\, b\rangle &= \langle x,\, ab\rangle = \langle \Delta(x),\, a\otimes b\rangle\\ &= \langle x\otimes g+g\otimes x,\, a\otimes b\rangle\\ &= \langle x,\, a\rangle\langle g,\, b\rangle + \langle g,\, a\rangle\langle x,\, b\rangle = 0,\end{aligned}$$

because we have $\langle g, a\rangle=\langle g, b\rangle=0$. Therefore $a\cdot x$ belongs to $\mathfrak{m}^{\perp}=C_0$ for any $a\in\mathfrak{m}$ and hence x belongs to $C_1=(\mathfrak{m}^2)^{\perp}$ by Proposition 1, (ii). In conclusion we see that $C_1=D_g\oplus P_g(C)$.

COROLLARY. *Let C be a cocommutative coalgebra over k and let $G(C)$ be $\{g_\lambda \mid \lambda\in\Lambda\}$. Then the sum $\sum_{\lambda\in\Lambda} P_{g_\lambda}(C)$ is direct.*

PROOF. By Proposition 1, (i) there exists the largest subcoalgebra D_λ of C containing $D_{g_\lambda}=kg_\lambda$ for each λ and then, by Proposition 2, we see that D_λ contains $P_{g_\lambda}(C)$. Since the sum $\sum_{\lambda\in\Lambda} D_\lambda$ is direct by Theorem 8.0.5 in [3], our assertion follows easily.

Now let C be a cocommutative coalgebra over a field k and let $G(C)$ be the set $\{g_\lambda \mid \lambda\in\Lambda\}$ of grouplike elements of C. Then the sum $\sum_{\lambda\in\Lambda} P_{g_\lambda}(C)$ of vector subspaces $P_{g_\lambda}(C)$ is direct as seen in the above. We denote by $P(C)$ this direct sum and call it the space of primitive elements of C. The following proposition is a version of Lemma 11.0.1 of [3] in non-irreducible cases.

PROPOSITION 3. *Let C and D be cocommutative coalgebras over k and let f be a coalgebra homomorphism of C to D. Assume that C is pointed. Then f is injective if and only if the restrictions $f|_{P(C)}$ and $f|_{G(C)}$ are both injective.*

PROOF. It suffices to show the "if" part. If $G(C)=\{g_\lambda \mid \lambda\in\Lambda\}$, then we denote by C_λ the irreducible component of C containing the minimal subcoalgebra kg_λ of C. By Corollary 8.0.7 in [3] we see that C is the direct sum $\oplus_{\lambda\in\Lambda} C_\lambda$ and $f(C_\lambda)$ is an irreducible subcoalgebra of D containing $f(kg_\lambda)=kf(g_\lambda)$ as a unique minimal subcoalgebra by Theorem 8.0.8 in [3]. By our assumption we have $f(g_\lambda)\neq f(g_{\lambda'})$ for $\lambda\neq\lambda'$ and hence we see $f(C_\lambda)+f(C_{\lambda'})=f(C_\lambda)\oplus f(C_{\lambda'})$ by Theorem 8.0.5 in [3]. Therefore we see by the same theorem that

$$f(C) = f(\oplus_{\lambda\in\Lambda} C_\lambda) = \oplus_{\lambda\in\Lambda} f(C_\lambda).$$

This means that f is injective if and only if $f|_{C_\lambda}$ is injective for each λ by the injectivity of $f|_{P_{g\lambda}(C)}$ and and Lemma 11.0.1 in [3].

Let C and D be pointed cocommutative coalgebras over k and let f be a coalgebra homomorphism of C to D. If $G(D)=\{h_\mu \mid \mu\in M\}$, then we put $D_0=\sum_{\mu\in M} kh_\mu=\oplus_{\mu\in M} kh_\mu$, which is a subcoalgebra of D. In §1 of [6] we defined the h-inverse $h\text{-}f^{-1}(D_0)$ of D_0 by f, which is the largest subcoalgebra C' of C satisfying

$f(C')\subset D_0$. We call this $C'=h\text{-}f^{-1}(D_0)$ the *c-kernel* of f and denote it by *c-ker f*. If $G(C)=\{g_\lambda \mid \lambda\in\Lambda\}$, then we denote by C_λ the connected component of C containing kg_λ and we put $f_\lambda=f|_{C_\lambda}$. Since f_λ is a coalgebra homomorphism of C_λ to D and $f_\lambda(C_\lambda)$ is irreducible, there is a unique μ in M such that $f_\lambda(C_\lambda)\subset D_\mu$ where D_μ is the irreducible component of D containing kh_μ. It is clear that *c-ker* f_λ is irreducible and contained in *c-ker f*, and hence we see easily that *c-ker* $f=\bigoplus_{\lambda\in\Lambda}$ *c-ker* f_λ. In particular if C and D are connected, i.e., colocal, then *c-ker f* coincides with *h-ker f* in the sense of [5]. Therefore if we consider that f_λ is a homomorphism of C_λ to D_μ, then *c-ker f* is the direct sum $\bigoplus_{\lambda\in\Lambda}$ *h-ker* f_λ. The following lemma is well-known, but we give a proof for convenience' sake.

LEMMA 1. *Let C and D be colocal coalgebras with grouplike elements g and h, respectively, and let f be a coalgebra homomorphism of C to D. Then f is injective if and only if c-ker f=h-ker f is kg.*

PROOF.*) Since $f(g)=h$, it is easily seen that *c-ker f* is equal to kg if f is injective. Conversely assume that *c-ker* $f=kg$. If f is not injective, then there is a finite dimensional subcoalgebra C' of C such that $f'=f|_{C'}$ is not injective by Corollary 3.9 in [4]. Since *c-ker* f' is contained in *c-ker f*, we may assume that $\dim_k C$ is finite. Moreover we may assume that f is surjective. Let A and B be the dual algebras C^* and D^* of C and D, respectively, and f^* the dual algebra homomorphism of f from B to A. By our assumption A and B are both local rings and finite dimensional over k, and f^* is injective. The fact that *c-ker* $f=kg$ means by Proposition 3.2 in [4] that the ideal of A generated by the image $f^*(\mathfrak{n})$ of the maximal ideal $\mathfrak{n}$ of B coincides with the maximal ideal $\mathfrak{m}$ of A. Therefore we have $A=k+\mathfrak{m}=f^*(B)+f^*(\mathfrak{n})A$ and hence, by Nakayama's lemma (cf. Corollary 2.7 in [1]), $A=f^*(B)$. This is a contradiction, because f is not injective.

COROLLARY. *Let C and D be pointed cocommutative coalgebras over k and f a coalgebra homomorphism of C to D. Denote by C_0 the subcoalgebra $\bigoplus_{\lambda\in\Lambda} kg_\lambda$ of C where $G(C)=\{g_\lambda \mid \lambda\in\Lambda\}$. Then the following are equivalent:*

(i) *f is injective.*

(ii) *c-ker f is contained in C_0 and the restriction $f|_{G(C)}$ of f is injective.*

PROOF. It is easy to see that (i) implies (ii). Conversely assume that (ii) is true. Let C_λ be the connected component of C containing kg_λ for each λ and put $f_\lambda=f|_{C_\lambda}$. Since *c-ker* $f=\bigoplus_{\lambda\in\Lambda}$ *c-ker* f_λ is contained in $C_0=\bigoplus_{\lambda\in\Lambda} kg_\lambda$, *c-ker* f_λ is contained in kg_λ for each λ and hence f_λ is injective by Lemma 1. On the other hand since $f|_{G(C)}$ is injective, the sum $\sum_{\lambda\in\Lambda} f_\lambda(C_\lambda)$ is direct by Theorem 8.0.5 in

*) T. Shudo communicated to the author that a shorter proof of Lemma 1 can be given if we use Proposition 3. Our proof is independent of Proposition 3 and hence Lemma 11.0.1 in [3].

[3]. This means by the equality $f(C)=f(\oplus_{\lambda\in\Lambda} C_\lambda)=\oplus_{\lambda\in\Lambda} f(C_\lambda)$ that f is injective.

Let C be a pointed cocommutative coalgebra over k and let E be a subcoalgebra of C. Then we say that E has a *coalgebra retraction* in C if there exists a coalgebra homomorphism η of C to E such that $\eta|_E$ coincides with the identity map id_E of E. The property for E to have a retraction in C is colocal in a sense. To see this we need the following

LEMMA 2. *Let C and E be cocommutative coalgebras over k and assume that E has a grouplike element g. Then the mapping η of C to E given by $\eta(x)=\varepsilon(x)g$ is a coalgebra homomorphism, where ε is the coidentity of C.*

PROOF. This follows easily from the fact that $\varepsilon: C\to k$ is a coalgebra homomorphism of C to the trivial coalgebra k and the subcoalgebra kg of E is isomorphic to k as coalgebras over k.

PROPOSITION 4. *Let C be a pointed cocommutative coalgebra over k and let E be a subcoalgebra of C. Assume that $G(C)$ is equal to $\{g_\lambda \mid \lambda\in\Lambda\}$ and that $G(E)$ is the subset $\{g_\mu \mid \mu\in M\}$ of $G(C)$ with $M\subset\Lambda$. Let C_λ be the connected component of C containg kg_λ for each λ and let E_μ be the connected component of E containing kg_μ for each $\mu\in M$. Then E has a coalgebra retraction in C if and only if E_μ has a coalgebra retraction in C_μ for each $\mu\in M$.*

PROOF. First assume that there is a coalgebra homomorphism η of C to E such that $\eta|_E=id_E$. Then it is clear that $\eta|_{E_\mu}=id_{E_\mu}$ for each $\mu\in M$. Since $E_\mu\subset C_\mu$ for each $\mu\in M$ by Theorem 8.0.5 in [3], we have $\eta(C_\mu)\supset\eta(E_\mu)=E_\mu$ and hence $\eta(C_\mu)=E_\mu$. Therefore $\eta|_{C_\mu}$ is a coalgebra homomorphism of C_μ to E_μ such that $(\eta|_{C_\mu})|_{E_\mu}=id_{E_\mu}$. In other words E_μ has a coalgebra retraction in C_μ. Conversely assume that E_μ has a coalgebra retraction in C_μ for each $\mu\in M$. Let μ_0 be a fixed element of M. If λ is an element of Λ but not in M, we define a map η_λ of C_λ to E by $\eta_\lambda(x)=\varepsilon(x)g_{\mu_0}$ where ε is the coidentity of C. Then η_λ is a coalgebra homomorphism of C_λ to E by Lemma 2. If μ is an element of M, then there exists a coalgebra homomorphism η_μ of C_μ to E_μ such that $\eta_\mu|_{E_\mu}=id_{E_\mu}$. Now we define a coalgebra homomorphism η of C to E by $\eta=\oplus_{\lambda\in\Lambda}\eta_\lambda$. It is easy to see that $\eta|_E=id_E$.

Next we give another property of coalgebra homomorphisms. Let M and N be sets and let f be a map of M to N. Then a map g of N to M is called a *splitting* of f if the composite $f\circ g$ is equal to the identity id_N of N. Similarly if C and D are cocommutative coalgebras over k and if ρ is a coalgebra homomorphism of C to D, then a coalgebra homomorphism τ of D to C with $\rho\circ\tau=id_D$ is called a *coalgebra splitting* of ρ. It is clear that if $f: M\to N$ (resp. $\rho: C\to D$) has a splitting $g: N\to M$ (resp. $\tau: D\to C$), then f (resp. ρ) is a surjective. Moreover we have the following

PROPOSITION 5. *Let C and D be cocommutative coalgebras over k and let ρ be a coalgebra homomorphism of C to D. Then the following are equivalent*:
(i) *ρ has a coalgebra splitting $\tau: D \to C$.*
(ii) *For any cocommutative coalgebra F and any coalgebra homomorphism $\sigma: F \to D$ there is a coalgebra homomorphism $\omega: F \to C$ with $\sigma = \rho \circ \omega$.*

PROOF. (i)$\Rightarrow$(ii). It suffices to put $\omega = \tau \circ \sigma$. (ii)$\Rightarrow$(i). If F and σ are taken as D and id_D, respectively, then ω in (ii) is a coalgebra splitting of ρ.

The property for a coalgbra homomorphism between pointed cocommutative coalgebras to have a coalgebra splitting is also colocal in the same sense as for coalgebra retractions. Let C and D be pointed cocommutative coalgebras over k with grouplike elements $G(C) = \{g_\lambda \mid \lambda \in \Lambda\}$ and $G(D) = \{h_\mu \mid u \in M\}$, respectively. Let C_λ be the connected component of C containing kg_λ for each $\lambda \in \Lambda$ and let D_μ be that of D containing kh_μ for each $\mu \in M$. If ρ is a coalgebra homomorphism of C to D, then there is a mapping ρ' of Λ to M such that $\rho(g_\lambda) = h_{\rho'(\lambda)}$. It is easy to see that $\rho(C_\lambda)$ is contained in $D_{\rho'(\lambda)}$.

PROPOSITION 6. *Let C, D, ρ, Λ, M, C_λ, D_μ and ρ' be as above. Then ρ has a coalgebra splitting if and only if the following are satisfied*:
(i) *ρ' has a splitting $\tau': M \to \Lambda$.*
(ii) *For any μ in M the restriction $\rho_\mu: C_{\tau'(\mu)} \to D_\mu$ of ρ to the subcoalgebra $C_{\tau'(\mu)}$ of C has a coalgebra splitting.*

PROOF. Assume that ρ has a coalgebra splitting $\tau: D \to C$. Since $\tau(h_\mu)$ is a grouplike element of C, there is a unique λ in Λ such that $\tau(h_\mu) = g_\lambda$. So we define a map $\tau': M \to \Lambda$ by $\tau(h_\mu) = g_{\tau'(\mu)}$. Then we see easily from $\rho \circ \tau = id_D$ that τ' is a splitting of ρ'. Now if τ_μ is the restriction of τ to the subcoalgebra D_μ of D, then τ_μ is a coalgebra homomorphism and we see that

$$\rho_\mu \circ \tau_\mu = (\rho \circ \tau)|_{D_\mu} = (id_D)|_{D_\mu} = id_{D_\mu}.$$

Therefore ρ_μ has a coalgebra splitting τ_μ.

Conversely if our assertions (i) and (ii) are satisfied, then we see from $(\rho' \circ \tau')(\mu) = \mu$ that $\rho_\mu(C_{\tau'(\mu)}) = \rho(C_{\tau'(\mu)})$ is contained in D_μ for any μ in M. Since ρ_μ has a coalgebra splitting $\tau_\mu: D_\mu \to C_{\tau'(\mu)}$ for any $\mu \in M$, we define $\tau: D \to C$ by $\tau = \bigoplus_{\mu \in M} \tau_\mu$. Is is easy to see that τ is a coalgebra homomorphism with $\rho \circ \tau = id_D$.

§2. Strongly exact sequences of Hopf algebras

Let $(H, m, i, \Delta, \varepsilon, c)$ be a pointed cocommutative Hopf algebra over a field k, where m, i, Δ, ε and c are the multiplication, the identity, the comultiplication, the coidentity and the antipode of H, respectively. Then it is known that the set

$G(H)$ of grouplike elements of H has a group structure with unit 1_H under the composition $gg'=m(g\otimes g')$ for g and g' in $G(H)$ (cf. Proposition 4.7 in [4]). If g is an element of $G(H)$, then we denote by H_g the connected component of H containing kg. In particular we denote by H_1 the connected component of H containing $k1_H=k$. Moreover let h_g be the map of H to itself given by $h_g(x)=m(g\otimes x)=gx$ for any x in H. The following two propositions play important roles in the proof of our main results.

PROPOSITION 7. *Let $(H, m, i, \Delta, \varepsilon, c)$, H_1, H_g and h_g be as above. Then h_g is a coalgebra automorphism of H and gives a coalgebra isomorphism between H_1 and H_g.*

PROOF. By the definition of Hopf algebras we have $\Delta m=(m\otimes m)\tau(\Delta\otimes\Delta)$ where τ is the k-linear automorphism of $H\otimes H\otimes H\otimes H$ given by $\tau(x\otimes y\otimes z\otimes w)=x\otimes z\otimes y\otimes w$. Therefore if $\Delta(x)=\sum_{(x)} x_{(1)}\otimes x_{(2)}$ for any element x in H, then we have for a grouplike element g of H

$$\begin{aligned}\Delta(gx) &= (\Delta m)(g\otimes x) = (m\otimes m)\tau(\Delta\otimes\Delta)(g\otimes x)\\ &= (m\otimes m)\tau(g\otimes g\otimes(\textstyle\sum_{(x)} x_{(1)}\otimes x_{(2)}))\\ &= (m\otimes m)(\textstyle\sum_{(x)} g\otimes x_{(1)}\otimes g\otimes x_{(2)})\\ &= \textstyle\sum_{(x)} gx_{(1)}\otimes gx_{(2)} = (h_g\otimes h_g)\Delta(x)\end{aligned}$$

hence we see $\Delta h_g=(h_g\otimes h_g)\Delta$. On the other hand we have the following commutative diagram:

$$\begin{array}{ccc} H\otimes H & \xrightarrow{m} & H \\ \downarrow{\scriptstyle \varepsilon\otimes\varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ k\otimes k & \xrightarrow{i} & k \end{array}$$

Therefore we have for any x in H

$$\varepsilon m(g\otimes x) = i(\varepsilon\otimes\varepsilon)(g\otimes x) = \varepsilon(g)\varepsilon(x) = \varepsilon(x)$$

and hence $\varepsilon h_g=\varepsilon$. This means that h_g is a coalgebra endomorphism of H. Since $G(H)$ is a group and we have $h_{1_H}=id_H$ and $h_g\circ h_{g'}=h_{gg'}$ for any g and g' in $G(H)$, we see $(h_g)^{-1}=h_{c(g)}$. So h_g is a coalgebra automorphism of H. Moreover since $h_g(1_H)=g$, it is easy to see that h_g gives a coalgebra isomorphism of H_1 onto H_g.

Let H' be another pointed cocommutative Hopf algebra over k and let ρ be a Hopf algebra homomorphism of H to H'. If g is a grouplike element of H, then the image $g'=\rho(g)$ of g by ρ is that of H'. If H'_1 and $H'_{g'}$ are the connected components of H' containing $k1_{H'}$ and kg', respectively, then the restrictions ρ_1

and ρ_g of ρ to H_1 and H_g map H_1 and H_g to H'_1 and $H'_{g'}$, respectively. Moreover let $h_{g'}$ be the coalgebra automorphism of H' given by $h_{g'}(x)=g'x$ for any x in H'.

PROPOSITION 8. *Let H, H', ρ', h_g and $h_{g'}$ be as above. Then h_g gives a coalgebra isomorphism from h-ker ρ_1=c-ker ρ_1 to h-ker ρ_g=c-ker ρ_g and $h_{g'}$ gives a coalgebra isomorphism from $\rho_1(H_1)$ to $\rho_g(H_g)$.*

PROOF. Our assertions are direct consequences of the definition of c-kernels of coalgebra homomorphisms and the following commutative diagram of the coalgebras:

$$\begin{array}{ccc} H_1 & \xrightarrow{\rho_1} & H'_1 \\ {\scriptstyle h_g}\downarrow & & \downarrow{\scriptstyle h_{g'}} \\ H_g & \xrightarrow{\rho_g} & H'_{g'} \end{array}$$

where h_g and $h_{g'}$ are coalgebra isomorphisms. We omit the detail of the proof.

Now let H, H' and H'' be pointed cocommutative Hopf algebras over k and let ρ: $H \to H'$ and ρ': $H' \to H''$ be Hopf algebra homomorphisms. If g is a grouplike element of H, then $g'=\rho(g)$ and $g''=\rho'(g')=(\rho' \circ \rho)(g)$ are those of H' and H'', respectively. Let H_1(resp. H'_1, H''_1, H_g, $H'_{g'}$ or $H''_{g''}$) be the connected components of H (resp. H', H'', H, H' or H'') containing $k1_H$ (resp. $k1_{H'}$, $k1_{H''}$, kg, kg' or kg''). Then we have the following direct consequence of Proposition 8.

COROLLARY. *In the above situation $\rho(H_1)$ coincides with c-ker ρ'_1 if and only if $\rho(H_g)$ coincides with c-ker $\rho'_{g'}$, where ρ'_1 and $\rho'_{g'}$ are the restrictions of ρ' to H'_1 and $H'_{g'}$, respectively.*

Now let

$$k \xrightarrow{i_G} G \xrightarrow{j} H \xrightarrow{\rho} J \xrightarrow{\varepsilon_J} k \qquad (*)$$

be a sequence of pointed cocommutative Hopf algebras over k where j and ρ are Hopf algebra homomorphisms. We recall that the sequence $(*)$ is said to be exact in the sense of §2 in [6] if we have $i_G(k)=h\text{-}ker\ j$, $j(G)=h\text{-}ker\ \rho$ and $\rho(H)=h\text{-}ker\ \varepsilon_J$. If $G(G)$, $G(H)$ and $G(J)$ are the groups consisting of grouplike elements of G, H and J, respectively, then the above sequence $(*)$ induces the following sequence of groups:

$$1_k \xrightarrow{i'_G} G(G) \xrightarrow{j'} G(H) \xrightarrow{\rho'} G(J) \xrightarrow{\varepsilon'_J} 1_k \qquad (**)$$

Moreover if G_1, H_1 and J_1 are the connected components of G, H and J containing

$k1_G$, $k1_H$ and $k1_J$, respectively, then the sequence (*) induces also the following sequence

$$k \longrightarrow G_1 \xrightarrow{j_1} H_1 \xrightarrow{\rho_1} J_1 \longrightarrow k \tag{***}$$

of connected Hopf algebras. Then we have the following

THEOREM 1. *The sequence* (*) *of Hopf algebras is exact if and only if the sequences* (**) *of groups and* (***) *of connected Hopf algebras are both exact.*

PROOF. First assume that the sequence (*) is exact. Then it is easy to see that j' is injective, that ρ' is surjective and that the image of $\rho'\circ j'$ consists of only one element 1_J. Let g be an element of $G(H)$ such that $\rho'(g)=1_J$ and H_g the connected component of H containing kg. Then we see from Proposition 7 that $\rho(H_g)=\rho(h_g(H_1))=\rho(g)\rho(H_1)\subset J_1$. If ρ_g is the restriction of ρ to H_g, then ρ_g is a coalgebra homomorphism of H_g to J_1 and *c-ker* ρ_g is contained in *h-ker* ρ. Since g belongs to *c-ker* ρ_g and hence to *h-ker* $\rho=j(G)$, there is a unique grouplike element g' in $G(G)$ with $j'(g')=j(g')=g$. Therefore the sequence (**) of groups is exact. As seen in the above if $g\in G(H)$ is mapped into J_1 by ρ, then we have $\rho(H_g)=\rho(gH_1)=\rho(H_1)$. Since ρ is surjective, this means that $\rho_1\colon H_1\to J_1$ is also surjective. If C is *c-ker* $\rho_1=$ *h-ker* ρ_1, then C is contained in *h-ker* $\rho=j(G)$ and hence contained in $j_1(G_1)=j(G_1)=H_1\cap j(G)$ by injectivity of j. Since $(\rho\circ j)(G)$ is equal to $k1_J$, $(\rho_1\circ j_1)(G)$ is also equal to $k1_J$. Therefore C coincides with $j_1(G_1)$ and so the sequence (***) of connected Hopf algebras is exact.

Conversely assume that the sequences (**) and (***) are both exact. If g' is an element of $G(G)$, then we have $h_{j(g')}\circ j=j\circ h_{g'}$ where $h_{g'}$ and $h_{j(g')}$ are coalgebra homomorphisms given in Proposition 7. Since $j_1\colon G_1\to H_1$ is injective, we see from the above equality that the restriction $j_{g'}\colon G_{g'}\to H_{j(g')}$ of j is also injective. This means that $j\colon G\to H$ is injective, because different connected components of G is mapped to different connected components of H by the injectivity of j'. Now if g'' is any element of $G(J)$, the connected component J_g'' of J containing g'' is equal to $h_{g''}(J_1)=g''J_1$ by Proposition 7. Since $\rho'\colon G(H)\to G(J)$ is surjective, there is an element g in $G(H)$ such that $\rho'(g)=g''$. Then we have $h_{g''}\circ\rho=\rho\circ h_g$ and hence the restriction ρ_g of ρ to $H_g=gH_1$ is a surjection onto $J_{g''}$, because $\rho_1\colon H_1\to J_1$ and $h_{g''}\colon J_1\to J_{g''}$ are both surjective. Therefore ρ is surjective. Next let C be *h-ker* ρ and let $C=\bigoplus_{\lambda\in\Lambda}C_{g_\lambda}$ be the connected components decomposition of the cocommutative coalgebra C where $\{g_\lambda\mid\lambda\in\Lambda\}$ is the set of the grouplike elements of C and C_{g_λ} is the connected component of C containing kg_λ for each $\lambda\in\Lambda$. Then g_λ belongs to *ker* $\rho'=j'(G(G))$ and hence there is a unique element g'_λ in $G(G)$ such that $g_\lambda=j'(g'_\lambda)=j(g'_\lambda)$ for each $\lambda\in\Lambda$. Let H_{g_λ} and $G_{g'_\lambda}$ be the connected components of H and G containing kg_λ and kg'_λ, respectively. Then, by the corollary to Proposition 8 and the exactness of the sequence (***), we have $j(G_{g'_\lambda})=c\text{-}ker\ \rho_\lambda=C_{g_\lambda}$ where $\rho_\lambda\colon H_{g_\lambda}\to J_1$ is the restriction of ρ to H_{g_λ}. If g'_μ is any grouplike element of G and G_μ is the connected component $g'_\mu G_1$ of G containing kg'_μ, $j'(g'_\mu)$ is contained in $ker\ \rho'$ by the exactness of the sequence (**) and hence we see by the corollary to Proposition 8 and the exactness of the sequence (***) that $j(G_\mu)=c\text{-}ker\ \rho_\mu$ where ρ_μ is the restriction of ρ to the connected component H_μ of H containing $j'(g'_\mu)=j(g'_\mu)$. Moreover since $\rho(j'(g'_\mu))=(\rho'\circ j')(g'_\mu)=1_J$, we see that $j(G_\mu)=c\text{-}ker\ \rho_\mu$ is containedin $h\text{-}ker\ \rho$. Therefore we have $h\text{-}ker\ \rho=C=\sum_{\lambda\in\Lambda}C_{g_\lambda}=j(G)$.

In the paper [6] the author showed that a sequence

$$k \longrightarrow G \xrightarrow{j} H \xrightarrow{\rho} J$$

of cocommutative Hopf algebras over a field k is exact if and only if the induced sequence

$$\{e\} \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J)$$

of groups is exact for any connected cocommutative coalgebra C over k (cf. Lemma 6 in [6]). However the functor $\mathrm{Hom}_{coal}(C, *)$ is not necessarily right exact. So we give the following notion of strong exactness for pointed cocommutative Hopf algebras, which is already given in the case of hyperalgebras in [2]. Let the sequence (*) of cocommutative Hopf algebras over k be exact. Then this sequence is called *strongly exact* if the following sequence

$$\{e\} \longrightarrow \mathrm{Hom}_{coal}(C, G) \longrightarrow \mathrm{Hom}_{coal}(C, H) \longrightarrow \mathrm{Hom}_{coal}(C, J) \longrightarrow \{e\}$$

of groups is exact for any pointed cocommutative coalgebra C over k. Now we show the main result of this paper.

THEOREM 2. *Let the notation be as above, and assume that the sequence* (*) *is exact. Then the following are equivalent*:
(i) *The sequence* (*) *is strongly exact.*
(ii) *The sequence* (***) *is strongly exact.*
(iii) *ρ has a coalgebra splitting.*
(iv) *ρ_1 has a coalgebra sppltiting.*
(v) *G has a coalgebra retraction in H.*
(vi) *G_1 has a coalgebra retraction in H_1.*

PROOF. The equivalence of (i) and (iii) (resp. (ii) and (iv)) follows from Lemma 6 in [6] and Proposition 5. Moreover the equivalence of (ii) and (vi) is shown by T. Shudo in Theorem 1.8 of [2]. By our assumption and Theorem 1 the sequence (**) of groups and (***) of hyperalgebras are both exact. If (iii) is true, then ρ has a coalgebra splitting $\tau\colon J\to H$. By Proposition 6 there exists a splitting τ': $G(J)\rightarrow G(H)$ of ρ'. If $g=\tau'(1_J)$ and H_g is the connected component of H containing g, then we see that the coalgebra homomorphism $\rho_g=\rho|_{H_g}$: $H_g\rightarrow J_1$ has a coalgebra splitting τ_g: $J_1\rightarrow H_g$ by the same proposition. Since the following diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\rho_1} & J_1 \\ {\scriptstyle h_g}\downarrow & & \downarrow{\scriptstyle id_{J_1}=h_{1_J}} \\ H_g & \xrightarrow{\rho_g} & J_1 \end{array}$$

of coalgebras is commutative, we see that ρ_1 has also a coalgebra splitting. Therefore (iv) is true. Conversely assume that (iv) is true. Since the sequence (**) of groups is exact, ρ': $G(H)\rightarrow G(J)$ has a splitting τ': $G(J)\rightarrow G(H)$ such that $\tau'(1_J)=1_H$. Let J_λ be any connected component of J and let g'' be the unique grouplike element of J contained in J_λ. If we put $g=\tau'(g'')$ and H_g is the connected component of H containing g, then we have the following commutative diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\rho_1} & J_1 \\ {\scriptstyle h_g}\downarrow & & \downarrow{\scriptstyle h_{g''}} \\ H_g & \xrightarrow{\rho|_{H_g}} & J_\lambda \end{array}$$

of coalgebras with vertical isomorphisms h_g and $h_{g''}$. Since ρ_1 has a coalgebra splitting by our assumption, so does $\rho|_{H_g}$. Therefore ρ has a coalgebra splitting by Proposition 6.

Next the implication (v)⇨(vi) follows from Proposition 4. Conversely assume that (vi) is true. Let G_λ be a connected component of G and g' be the unique grouplike element of G contained in G_λ. Then the following diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{j_1} & H_1 \\ {\scriptstyle h_{g'}}\downarrow & & \downarrow{\scriptstyle h_{j'(g')}} \\ G_\lambda & \xrightarrow[j|_{G_\lambda}]{} & H_\lambda \end{array}$$

of coalgebras with vertical isomorphisms $h_{g'}$ and $h_{j'(g')}$ is commutative. Since j_1 has a coalgebra retraction in H_1 by our assumption, this commutative diagram means that G_λ ahs also a coalgebra retraction in H_λ. Therefore G has a coalgebra retraction in H by Proposition 4.

The following proposition and its corollaries are also shown in the case of hyperalgebras in [2].

PROPOSITION 9. *Let*

$$\begin{array}{ccccccccc}
k & \longrightarrow & G & \xrightarrow{j} & H & \xrightarrow{\rho} & J & \longrightarrow & k \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
k & \longrightarrow & \bar{G} & \xrightarrow{\bar{j}} & \bar{H} & \xrightarrow{\bar{\rho}} & \bar{J} & \longrightarrow & k
\end{array}$$

be a commutative diagram of pointed cocommutative Hopf algebras over a field k with exact rows. Then we have the following:

(i) *If the upper row is strongly exact and γ has a coalgebra splitting $\gamma' : \bar{J} \to J$, then the lower row is also strongly exact.*

(ii) *If the lower row is strongly exact and there is a coalgebra homomorphism $\alpha' : \bar{G} \to G$ such that $\alpha' \circ \alpha = id_G$, then the upper row is also strongly exact.*

PROOF. (i) By our assumption and Theorem 2 ρ has a coalgebra splitting $\tau : J \to H$. Therefore if we put $\bar{\tau} = \beta\tau\gamma'$, then we have $\bar{\rho}\bar{\pi} = \bar{\rho}\beta\tau\gamma' = \gamma\rho\tau\gamma' = \gamma\gamma' = id_{\bar{J}}$. This means by Theorem 2 that the lower row is strongly exact.

(ii) Similarly we have a coalgebra retraction $\bar{f} : \bar{H} \to \bar{G}$ of $\bar{j}$ by our assumption and Theorem 2. Therefore if we put $f = \alpha' \bar{f} \beta$, then we have $fj = \alpha' \bar{f} \beta j = \alpha' \bar{f} \bar{j} \alpha = \alpha' \alpha = id_G$, and hence the upper row is also strongly exact by Theorem 2.

COROLLARY 1. *Let N be normal Hopf subalgebra of a pointed cocommutative Hopf algebra H over a field k, and G be a Hopf subalgebra of H such that the join J(N, G) of N and G is equal to H. If the intersection I(N, G) of N and G has a coalgebra retraction in G, then N has also a retraction in H.*

PROOF. Our assertion follows easily from Proposition 9, (i), Theorem 2 and the following commutative diagram of Hopf algebras where the right vertical mapping is an isomorphism by Theorem 3 in [6]:

$$\begin{array}{ccccccccc}
k & \longrightarrow & I(N, G) & \longrightarrow & G & \longrightarrow & G/I(N, G) & \longrightarrow & k \\
 & & \cap\downarrow & & \cap\downarrow & & \downarrow & & \\
k & \longrightarrow & N & \longrightarrow & H & \longrightarrow & H/N & \longrightarrow & k.
\end{array}$$

COROLLARY 2. *Let N be a normal Hopf subalgebra of a pointed cocommutative Hopf algebra H over a field k and let G be a Hopf subalgebra of H containing N. If the natural surjection $\rho : H \to H/N$ has a coalgebra splitting, then the natural surjection $\bar{\rho} : G \to G/N$ has a coalgebra splitting.*

PROOF. This follows from Proposition 9, (ii), Theorem 2 and the following commutative diagram:

$$\begin{array}{ccccccccc}
k & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\bar{\rho}} & G/N & \longrightarrow & k \\
 & & \| & & \cap\downarrow & & \cap\downarrow & & \\
k & \longrightarrow & N & \longrightarrow & H & \xrightarrow{\rho} & H/N & \longrightarrow & k.
\end{array}$$

COROLLARY 3. *Let N and $\bar{N}$ be normal Hopf subalgebras of a pointed cocommutative Hopf algebra H over a field k such that $N \supset \bar{N}$. If N has a coalgebra retraction in H, then $N/\bar{N}$ has also a coalgebra retraction in $H/\bar{N}$.*

PROOF. This is a direct consequence of Proposition 9, (i), Theorem 2 and the following isomorphism

$$(H/\bar{N})/(N/\bar{N}) \cong H/N$$

of Hopf algebras obtained from the corollary to Theorem 2 in [6].

References

[1] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, London, 1969.

[2] T. Shudo, On the relatively smooth subhyperalgebras of hyperalgebras, Hiroshima Math. J. **13** (1983), 627–646.

[3] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[4] H. Yanagihara, Theory of Hopf Algebras Attached to Group Schemes, Lecture Notes in Math. 614, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[5] ———, On isomorphism theorems of formal groups, J. Algebra **55** (1978), 341–347.

[6] ———, On group theoretic properties of cocommutative Hopf algebras, Hiroshima Math. J. **9** (1979), 179–200.

Department of Mathematics,
Hyogo University of Teacher Education