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Finite element approximations of harmonic differentials on a Riemann surface

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Introduction

In the present paper we aim to establish a method of finite element approximations on a Riemann surface. Our method matches the abstract definition of Riemann surfaces, and also will offer a new technique of high practical use in numerical calculation not only for the case of Riemann surfaces but also for the case of plane domains (cf. Mizumoto and Hara [18]). It is characteristic of our method that by adopting an ordinary finite element approximation on every parametric disk, the approximations of high precision are obtained.

Let $\bar{\Omega}$ be a compact bordered or closed Riemann surface. We choose a fixed finite collection $\Phi=\{z=\varphi_j(p),\ p\in U_j;\ j=1,\cdots,m\}$ of local parameters $z=\varphi_j(p)$ and parametric disks U_j so that $\bar{\Omega}\subset\bigcup_{j=1}^m U_j$. §1 is devoted to construction of a triangulation K of $\bar{\Omega}$ with width h associated to Φ (cf. §1.2), a normal subdivision of K (cf. §1.3), and a naturalized triangulation K' associated to K (cf. §1.4). The triangulation K of $\bar{\Omega}$ is constructed as the sum of subtriangulations K_j $(j=1,\cdots,m)$ in such a way that $|K_j|\subset U_j$, each 2-simplex s of K belongs to one and only one K_j, each $s\in K_j$ is natural (see §1.2) at most except for the case when it has a common side with another $s'\in K_k$ $(k\neq j)$, and the diameter of $\varphi_j(s)$ is at most h for each $s\in K_j$ $(j=1,\cdots,m)$. Let K'_j $(j=1,\cdots,m)$ be triangulations consisting of all 2-simplices of K_j which are not minor or major, and all naturalized simplices of K_j (see §1.4). Then the triangulation K' is defined as the sum of K'_j $(j=1,\cdots,m)$.

In §2, we introduce and investigate two spaces $\Lambda=\Lambda(K)$ and $\Lambda'=\Lambda'(K')$ of differentials. The space Λ consists of locally exact differentials σ_h such that for each 2-simplex $s\in K_j$ $(j=1,\cdots,m)$ the coefficients of σ_h are constant on $\varphi_j(s)$ except that σ_h is modified on all lunes of minor or major simplices (see §1.4 and §2.1). To each $\sigma_h\in\Lambda$, we associate a differential $\sigma'_h=F(\sigma_h)$ on K' whose coefficients are constant on $\varphi_j(s)$ for each 2-simplex $s\in K'_j$ $(j=1,\cdots,m)$ and which is equal to σ_h on $\bar{\Omega}$ except for all lunes of K (cf. §2.2). The space Λ' consists of all $\sigma'_h=F(\sigma_h)$ $(\sigma_h\in\Lambda)$. We shall investigate estimates of differences of

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Dirichlet norms $\|\sigma_h\|_\Omega^2$ and $\|\sigma_h'\|_{K'}^2$ (see Lemma 2.2).

Let ω be a harmonic differential on Ω which satisfies some period conditions and some boundary conditions (see §3.1). The finite element approximations ψ_h and ω_h' of ω are defined in the spaces Λ and Λ' respectively (cf. §3.2 and §3.3 resp.). The differential ω_h' can be numerically calculated. §3 is devoted to error estimates of ψ_h and ω_h for ω, where $\omega_h = F^{-1}(\omega_h')$. We shall make use of Bramble and Zlámal's lemma (see Lemma 3.5). In Theorems 3.1 and 3.2, we obtain error estimates:

$$\|\psi_h - \omega\|^2 \leqq Ch^2 \quad \text{and} \quad \|\omega_h - \omega\|^2 \leqq C'h^2,$$

where C and C' are constants which depend only on the differential ω and the smallest value of interior angles of triangles $\varphi_j(s)$ for all $s \in K_j'$ $(j=1, \cdots, m)$. Further, in Theorem 3.2, we obtain an estimate for $\|\omega\|^2$:

$$\|\omega\|^2 \leqq \|\omega_h'\|^2 + \varepsilon(\omega_h')$$

in a special case (see §3.2), where $\varepsilon(\omega_h')$ is a quantity of $O(h^2)$ which can be numerically calculated.

Finally, in §4 we apply our results to numerical calculation of periodicity moduli of closed and compact bordered Riemann surfaces, and we shall show that calculation results for some concrete Riemann surfaces of genus one are fairly good. With respect to the problems of this type, there have been some investigations by means of finite-difference method (Gaier [11], [12], Mizumoto [13], [14], [15], Opfer [20], [21]).

Our treatment at critical points of a Riemann surface is closely related to that at boundary singularities on a plane (cf. Akin [2], Babuška [3], Babuška and Rosenzweig [4], Babuška, Szabo and Katz [5], Barnhill and Whiteman [6], Blackburn [7], Craig, Zhu and Zienkiewicz [10], Mizumoto and Hara [18], Opfer and Puri [22], Rivara [23], Schatz and Wahlbin [24], [25], Thatcher [28], Tsamasphyros [29], Whiteman and Akin [30], Yserentant [31]).

The results in the present paper (Theorems 3.1 and 3.2) may be generalized to the case of harmonic differentials on a higher dimensional Riemannian manifold.

§1. Triangulation

1. Collection Φ of local parameters Let Ω be a closed Riemann surface or a subdomain of a Riemann surface W whose closure $\bar{\Omega}$ is a compact bordered subregion of W. In the latter case, we assume that the boundary $\partial\Omega$ consists of a finite number of analytic arcs meeting at vertices p_k $(k=1, \cdots, \nu)$, and there exist parametric disks V_k $(k=1, \cdots, \nu)$ with the centers p_k and local parameters z

$=\psi_k(p)$ by which $V_k \cap \bar{\Omega}$ are mapped onto sectors $\{|z| \leqq r_k\} \cap \{0 \leqq \arg z \leqq \beta_k\}$ $(0 < \beta_k \leqq 2\pi,\ \beta_k \neq \pi)$. For conformity, if Ω is a closed Riemann surface, then we interpret that $\Omega = W$.

By $\Phi = \{z = \varphi_j(p),\ U_j;\ j = 1, \cdots, m\}$ we denote a finite collection of local parameters $z = \varphi_j(p)$ $(j = 1, \cdots, m)$ and parametric disks U_j $(j = 1, \cdots, m)$ on W which satisfies the following conditions (i)$\sim$(iv):

(i) Each U_j $(j = 1, \cdots, m)$ is a parametric disk and by the mapping $z = \varphi_j(p)$, U_j is mapped onto a disk $|z| < \rho_j$. Furthermore, each vertex p_k $(k = 1, \cdots, \nu)$ is the center of some U_j.

(ii) $\bar{\Omega}$ is covered by $\{U_j\}_{j=1}^m$.

(iii) If $U_j \cap U_k \neq \varnothing$, then there exists a constant $L(>1)$ such that for the mapping $\zeta = f(z) \equiv \varphi_k \circ \varphi_j^{-1}(z)$, $1/L < |f'(z)| < L$ on $\varphi_j(U_j \cap U_k)$.

(iv) If $U_j \cap \partial\Omega \neq \varnothing$, then $\varphi_j(U_j \cap \Omega)$ is a half disk $\{|z| < \rho_j\} \cap \{\operatorname{Im} z > 0\}$ or a sector $\{|z| < \rho_j\} \cap \{0 < \arg z < \alpha_j\}$ $(0 < \alpha_j \leqq 2\pi,\ \alpha_j \neq \pi)$.

In the latter case of (iv), by the mapping $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$, $U_j \cap \Omega$ is mapped onto a half disk $\{|\zeta| < \rho_j^{\pi/\alpha_j}\} \cap \{\operatorname{Im} \zeta > 0\}$. In this case we define anew $z = \varphi_j(p)$ and ρ_j by $\zeta = (\varphi_j(p))^{\pi/\alpha_j}$ and ρ_j^{π/α_j} respectively. Then, the local parameter $z = \varphi_j(p)$ is no longer conformal at the center of U_j.

2. Triangulation K associated to Φ For the collection Φ of local parameters and parametric disks defined in §1.1, and for a sufficiently small positive number h, we construct a triangulation $K = K^h$ of $\bar{\Omega}$ which satisfies the following conditions (i)$\sim$(v). This is called a *triangulation of $\bar{\Omega}$ with width h associated to Φ.*

(i) Each point at which $\partial\Omega$ is not analytic is a carrier of some 0-simplex of K.

(ii) K is the sum of subtriangulations $K_1, \cdots, K_m$ of K such that each 2-simplex of K belongs to one and only one K_j $(j = 1, \cdots, m)$, and the carrier $|s|$ of each 2-simplex s of K_j is contained in U_j.

If a 1-simplex $e \in K_j$ does not belong to another K_k $(k \neq j)$, or a 1-simplex e belongs to $K_j \cap K_k$ $(j \neq k)$ and the mapping $\varphi_k \circ \varphi_j^{-1}$ is an affine transformation, then e is said to be *linear*. If each edge of a 2-simplex $s \in K_j$ is linear and $\varphi_j(s)$ is an ordinary triangle, then s is called a *natural simplex*.

(iii) Each 2-simplex $s \in K_j$ which has not a common edge with any 2-simplex of another K_k $(k \neq j)$, is a natural simplex.

A 2-simplex of K_k which has a common edge with a 2-simplex $s \in K_j$ $(j \neq k)$, is said to be an *adjoint* (simplex) of s and is denoted by s'.

(iv) For each pair of a 2-simplex $s \in K_j$ and its adjoint $s' \in K_k$ with a common edge e, either one of the following three cases (a), (b), (c) occurs.

(a) Both s and s' are natural simplices.

(b) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly concave arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Fig. 1). Then s is called a *minor simplex*. The case where s' is a minor simplex and s is its adjoint may also occur.

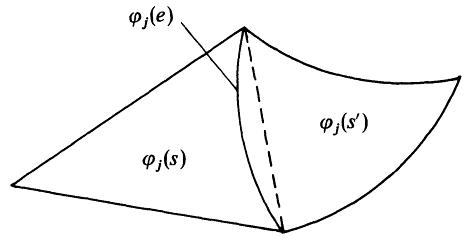


Fig. 1

(c) $\varphi_j(s)$ is a curvilinear triangle such that $\varphi_j(e)$ is a strictly convex arc w.r.t. $\varphi_j(s)$, $\varphi_k(s')$ is an ordinary triangle, and all edges of s and s' except for e are linear (cf. Fig. 2). Then s is called a *major simplex*. The case where s' is a major simplex and s is its adjoint may also occur.

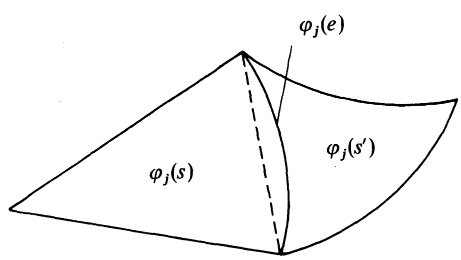


Fig. 2

If s is a minor or major simplex of K_j, then it is assumed that $|s'|\subset U_j$ for its adjoint s'.

(v) For each 2-simplex $s\in K_j$ $(j=1,\cdots, m)$, $d(\varphi_j(s))\leqq h$, where throughout the present paper we denote the diameter of a region G by $d(G)$.

Next, we assume that for the fixed Φ the class of the triangulations $K=K^h$ satisfies the following conditions (i′) and (ii′):

(i′) For each $j=1,\cdots, m$ the union of carriers of all minor and major simplices of K_j, and all their adjoints is contained in a closed subset R_j of $U_j\cap\bar{\Omega}$ which is independent of the individual triangulation K.

(ii′) The number N of minor and major simplices of K satisfies the inequality:

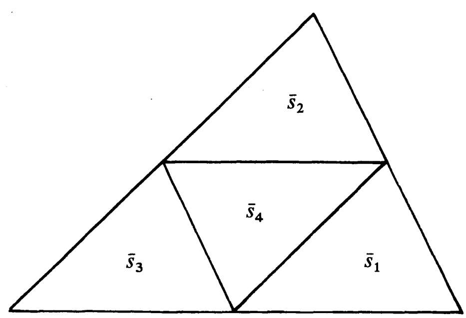
$$N\leqq M\cdot\frac{1}{h}, \tag{1.1}$$

where M is a constant which is independent of the individual triangulation K.

3. Normal subdivision of triangulation K For a triangulation $K=K^h$ of $\bar{\Omega}$ with width h associated to Φ we can construct a subdivision $K^1=K^{1,h/2}$, called the *normal subdivision of* $K=K^h$ by the following procedure:

(i) K^1 is the sum of the subtriangulations $K_1^1,\cdots,K_m^1$ which are the subdivisions of $K_1,\cdots, K_m$ respectively which are defined in the following (ii), (iii).

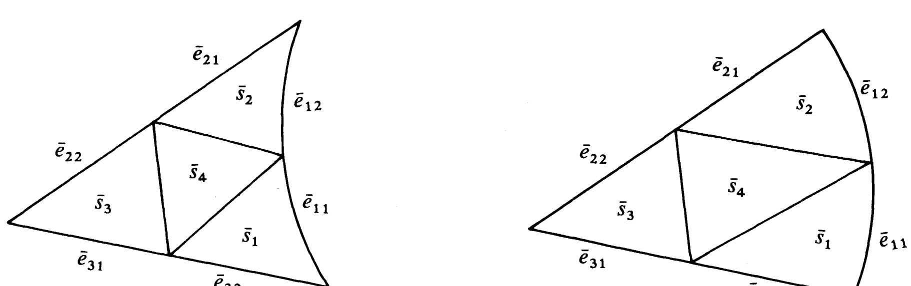
(ii) If $s\in K_j$ is a 2-simplex which is not minor or major, then s is subdivided to four 2-simplices s_1, s_2, s_3 and s_4 of K_j^1 so that $\varphi_j(s_1)$, $\varphi_j(s_2)$, $\varphi_j(s_3)$ and $\varphi_j(s_4)$ are mutually congruent ordinary triangles as in Fig. 3.



$\bar{a}=\varphi_j(a)$ (a: simplex)

Fig. 3

(iii) Let $s\in K_j$ and $s'\in K_k$ be a minor (or major) simplex and its adjoint, and let e_1, e_2 and e_3 be edges of s such that e_1 is the common edge of s and s'. We subdivide the edges e_1, e_2 and e_3 to two edges e_{11} and e_{12}, e_{21} and e_{22}, and e_{31} and e_{32} respectively so that $\varphi_k(e_{11})$ and $\varphi_k(e_{12})$, $\varphi_j(e_{21})$ and $\varphi_j(e_{22})$, and



$\bar{a}=\varphi_j(a)$ (a: simplex)

Fig. 4

$\varphi_j(e_{31})$ and $\varphi_j(e_{32})$ have the same length respectively. Then we subdivide the simplex s to two minor (or major resp.) simplices s_1 and s_2 of K_j^1 and, two natural simplices s_3 and s_4 of K_j^1 so that e_{11}, e_{12}, e_{21}, e_{22}, e_{31} and e_{32} are edges of s_1, s_2 and s_3 (cf. Fig. 4). Here we note that such a subdivision is always possible if h is sufficiently small.

We can easily see that the normal subdivision $K^1=\sum_{j=1}^m K_j^1$ is a triangulation of $\bar{\Omega}$ with width $h/2+O(h^2)$ associated to Φ (cf. (1.10)).

4. Naturalized triangulation For each minor (or major) simplex $s\in K_j$ we define the *naturalized simplex* $\natural s$ of s as the 2-simplex such that $|s|\subset|\natural s|$ ($|\natural s|\subset|s|$ resp.) and $\varphi_j(\natural s)$ is the ordinary triangle which has two common sides with $\varphi_j(s)$. Further we define a 2-simplex $\flat\ell=\flat\ell(s)$ ($\sharp\ell=\sharp\ell(s)$ resp.) with two edges whose carrier is the closed region $\overline{|\natural s|-|s|}$ ($\overline{|s|-|\natural s|}$ resp.). $\flat\ell(s)$ ($\sharp\ell(s)$ resp.) is called the *deficient* (*excessive* resp.) *lune* of s.

Each triple of a minor (or major) simplex $s\in K_j$, its adjoint $s'\in K_k$ and its deficient lune $\flat\ell$ (excessive lune $\sharp\ell$ resp.) is denoted by $(s, s', \flat\ell)$ ($(s, s', \sharp\ell)$ resp.), and is called a *triple for a minor* (*major* resp.) *simplex* s or *for a deficient* (*excessive* resp.) *lune* $\flat\ell$ ($\sharp\ell$ resp.) (cf. Fig. 5), where it is always assumed that $|\flat\ell|\subset|s'|$ for each $(s, s', \flat\ell)$.

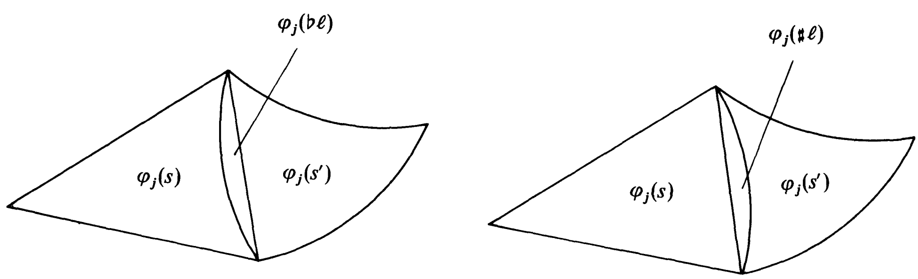


Fig. 5

For simplicity of notation, we also denote $\flat\ell=\flat\ell(s)$ or $\sharp\ell=\sharp\ell(s)$ by $\ell=\ell(s)$. If a minor or major simplex s is in K_j, then we say that $\ell=\ell(s)$ is a *lune of* K_j and write $l\in K_j$.

Now we shall define the *naturalized triangulation* K' *associated to* K.

First, K_j' $(j=1,\cdots,m)$ are defined as triangulations such that the collection of all 2-simplices of K_j' consists of all 2-simplices of K_j which are not minor or major, and of all naturalized simplices of minor and major ones of K_j. Then the triangulation K' is defined as the sum of K_j' $(j=1,\cdots,m)$. We should note that K' is no longer a triangulation of $\bar{\Omega}$, and also is not an ordinary triangulation.

5. Parametrization of lunar domains Let (s, s', ℓ) be an arbitrary triple for a deficient or excessive lune ℓ, and let e_1 and e_2 be two edges of ℓ such that $e_1 \subset \partial s$. Further, let

$$z' = (1-t)z_1 + tz_2 \qquad (0 \leqq t \leqq 1) \tag{1.2}$$

and

$$\zeta'' = (1-t)\zeta_1 + t\zeta_2 \qquad (0 \leqq t \leqq 1) \tag{1.3}$$

be parameter representations of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. The representation (1.3) induces a parameter representation of the curve $\varphi_j(e_1)$:

$$z'' = g((1-t)\zeta_1 + t\zeta_2) \qquad (0 \leqq t \leqq 1), \tag{1.4}$$

where $z = g(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$. By (1.2) and (1.4) we obtain a parameter representation of the lunar domain $\varphi_j(\ell)$:

$$\begin{aligned} z = z(t, \tau) &\equiv (1-\tau)z' + \tau z'' \\ &= (1-\tau)((1-t)z_1 + tz_2) + \tau g((1-t)\zeta_1 + t\zeta_2) \\ &\qquad\qquad (0 \leqq t \leqq 1,\ 0 \leqq \tau \leqq 1). \end{aligned} \tag{1.5}$$

6. Area of lune

LEMMA 1.1. *Let (s, s', ℓ) be a triple for an arbitrary deficient or excessive lune ℓ. Then, the estimate*

$$A(\varphi_j(\ell)) \leqq \frac{h_1{}^3}{8}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right| + O(h_1)\right) \tag{1.6}$$

holds, where throughout the present paper we denote the area of a region G by $A(G)$, $z = g(\zeta) \equiv \varphi_j \circ \varphi_k^{-1}(\zeta)$, $h_1 = d(\varphi_j(\ell))$ and ζ_1 is one of the vertices of the lunar domain $\varphi_k(\ell)$.

PROOF. Here we shall preserve the notations in §1.5. By Taylor's expansion we have

$$z'' - z_1 = g'(\zeta_1)(\zeta_2 - \zeta_1)t + \frac{1}{2}g''(\zeta_1)(\zeta_2 - \zeta_1)^2 t^2 + \cdots \tag{1.7}$$

for the point z'' of (1.4) on $\varphi_j(e_1)$, and

$$\begin{aligned} z' - z_1 &= t(z_2 - z_1) \\ &= g'(\zeta_1)(\zeta_2 - \zeta_1)t + \frac{1}{2}g''(\zeta_1)(\zeta_2 - \zeta_1)^2 t + \cdots \end{aligned} \tag{1.8}$$

for the point z' of (1.2) on $\varphi_j(-e_2)$, where we assume that the triangulation K is so chosen that $\varphi_k(e_1)$ is contained in a disk V centered at ζ_1 such that $\varphi_k^{-1}(V) \subset U_j \cap U_k$. By (1.7) and (1.8) we find that the equality

$$z''-z'=(\zeta_2-\zeta_1)^2\cdot\frac{t(t-1)}{2}\cdot g''(\zeta_1)+O((\zeta_2-\zeta_1)^3) \tag{1.9}$$

holds for the point z' of (1.2) on $\varphi_j(-e_2)$ and the point z'' of (1.4) on $\varphi_j(e_1)$ with common t.

Since $|\zeta_2-\zeta_1|\leqq h_1(1/|g'(\zeta_1)|+O(h_1))$, the equality (1.9) implies

$$|z''-z'|\leqq\frac{h_1{}^2}{8}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right|+O(h_1)\right). \tag{1.10}$$

Therefore we obtain the estimates

$$\begin{aligned} A(\varphi_j(\ell)) &\leqq |z_2-z_1|\cdot\max_{0\leqq t\leqq 1}|z'-z''| \\ &\leqq\frac{h_1{}^3}{8}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right|+O(h_1)\right). \end{aligned}$$

§2. Spaces of differentials

1. Subspace Λ of Γ_c Let $\Gamma_c^0=\Gamma_c^0(\bar{\Omega})$ be the set of all locally exact differentials σ in the class C^0 on $\bar{\Omega}$ with the finite Dirichlet norm

$$\|\sigma\|^2=\|\sigma\|_\Omega^2=\int_\Omega \sigma * \sigma<\infty,$$

where by $*\sigma$ we denote the conjugate differential of σ. Let $\Gamma_c=\Gamma_c(\bar{\Omega})$ be the completion of Γ_c^0. We should note that in Chapter 5 of Ahlfors and Sario [1], Γ_c is defined as the completion of $\Gamma_c^1\equiv\Gamma_c^0\cap C^1$.

We define a subspace $\Lambda=\Lambda(K)$ of Γ_c as the space of differentials σ_h which satisfy the following conditions (i)$\sim$(iv):

(i) $\sigma_h\in\Gamma_c$.

(ii) If $s\in K_j$ $(j=1,\cdots,m)$ is a natural simplex, then

$$\sigma_h=a_0dx+b_0dy \qquad \text{on} \quad \varphi_j(s) \quad (z=x+iy),$$

where a_0 and b_0 are constants.

(iii) Let $(s, s', \flat\ell)$ be a triple for a minor simplex s, and let e_1 and e_2 be two edges of $\flat\ell$ such that $-e_1\subset\partial s$. Then

$$\sigma_h=a_0dx+b_0dy \qquad \text{on} \quad \varphi_j(s),$$

$$\sigma_h = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on} \quad \varphi_k(s') - \varphi_k(\flat\ell),$$

and σ_h is a harmonic differential in $\flat\ell$ which satisfies the boundary conditions

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{along} \quad \varphi_j(e_1)$$

and

$$\sigma_h = \left(\alpha_0 \frac{\partial \xi}{\partial x} + \beta_0 \frac{\partial \eta}{\partial x}\right) dx + \left(\alpha_0 \frac{\partial \xi}{\partial y} + \beta_0 \frac{\partial \eta}{\partial y}\right) dy \qquad \text{along} \quad \varphi_j(e_2),$$

where a_0, b_0, α_0 and β_0 are constants, and

$$\zeta = f(z) \equiv \varphi_k \circ \varphi_j^{-1}(z) \qquad (z = x + iy,\ \zeta = \xi + i\eta).$$

(iv) Let $(s, s', \sharp\ell)$ be a triple for a major simplex s, and let e_1 and e_2 be two edges of $\sharp\ell$ such that $e_1 \subset \partial s$. Then

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{on} \quad \varphi_j(\natural s),$$
$$\sigma_h = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on} \quad \varphi_k(s'),$$

and σ_h is a harmonic differential in $\sharp\ell$ which satisfies the boundary conditions

$$\sigma_h = a_0 dx + b_0 dy \qquad \text{along} \quad \varphi_j(e_2)$$

and

$$\sigma_h = \left(\alpha_0 \frac{\partial \xi}{\partial x} + \beta_0 \frac{\partial \eta}{\partial x}\right) dx + \left(\alpha_0 \frac{\partial \xi}{\partial y} + \beta_0 \frac{\partial \eta}{\partial y}\right) dy \qquad \text{along} \quad \varphi_j(e_1),$$

where a_0, b_0, α_0 and β_0 are constants, and $\zeta = \xi + i\eta$ is as in (iii).

We note that $\sigma_h \in \Lambda$ is generally discontinuous on each edge of 2-simplices of K.

2. Space Λ' Let K' be the naturalized triangulation associated to K. For each differential $\sigma_h \in \Lambda$, we define the *differential σ_h' on K' associated to σ_h* as the differential σ_h' which satisfies the following conditions (i)~(iv):

(i) For each 2-simplex $s \in K_j'$ $(j = 1, \cdots, m)$

$$\sigma_h' = a_0 dx + b_0 dy \qquad \text{on} \quad \varphi_j(s),$$

where a_0 and b_0 are constants.

(ii) If $s \in K$ is a natural simplex, then

$$\sigma_h' = \sigma_h \qquad \text{on} \quad |s|.$$

(iii) If $(s, s', \flat\ell)$ is a triple for a minor simplex s, then

$$\sigma'_h = \sigma_h \qquad \text{on} \quad |s| \cup |s'| - |\flat\ell|.$$

(iv) If $(s, s', \sharp\ell)$ is a triple for a major simplex s, then

$$\sigma'_h = \sigma_h \qquad \text{on} \quad |\natural s| \cup |s'|.$$

We should note that the differential σ'_h is defined just twice on each deficient lune $\flat\ell$, while it is never defined on any excessive lune $\sharp\ell$. In the former case, for each triple $(s, s', \flat\ell)$ we shall denote the differential σ'_h on $\natural s \in K'_j$ and $s' \in K'_k$ by $\sigma'_{h,\natural s}$ and $\sigma'_{h,s'}$ respectively.

The space of all differentials σ'_h associated to $\sigma_h \in \Lambda$ is denoted by $\Lambda' = \Lambda'(K')$. Let σ'_h and χ'_h be two differentials of Λ'. Then the *inner product* (σ'_h, χ'_h) of σ'_h and χ'_h is defined by

$$(\sigma'_h, \chi'_h) = (\sigma'_h, \chi'_h)_{K'}$$
$$= \sum_{s \in K'} \int_{|s|} \sigma'_h * \chi'_h,$$

and the *norm* $\|\sigma'_h\|$ of σ'_h is defined by

$$\|\sigma'_h\| = \|\sigma'_h\|_{K'} = \sqrt{(\sigma'_h, \sigma'_h)_{K'}}.\text{[1)]}$$

We see that $\sigma'_h = F(\sigma_h)$ defines a one-to-one mapping of Λ onto Λ'.

3. Finite element interpolations Let σ be an element of Γ_c. We define the *finite element interpolation $\hat{\sigma}$ of σ in the space Λ* as the differential uniquely determined by the following conditions (i) and (ii):

(i) $\hat{\sigma} \in \Lambda$;

(ii) For each 1-simplex $e \in K$,

$$\int_e \hat{\sigma} = \int_e \sigma.$$

4. Harmonic differentials on a lune

LEMMA 2.1. *Let $\ell = \ell(s)$ be a deficient or excessive lune of K_j, let e_1 and e_2 be two edges of ℓ, and let σ_1 and σ_2 be exact differentials in the class C^0 on ℓ which satisfy the condition*

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1) We shall use the common notations (,) and $\| \ \|$ for both inner products and both norms of differentials of the spaces Λ and Λ'.

$$\int_{e_1} \sigma_1 = -\int_{e_2} \sigma_2 .$$

Further, let χ be the differential harmonic in ℓ and continuous on ℓ which satisfies the boundary conditions

$$\chi = \sigma_i \quad \textit{along} \quad e_i \quad (i=1, 2).$$

Then the inequalities

$$\|\chi\|_\ell^2 \leqq \iint_{\varphi_j(\ell)} \max\{(a_1{}^2+b_1{}^2), \ (a_2{}^2+b_2{}^2)\} dxdy \tag{2.1}$$
$$\leqq \|\sigma_1\|_\ell^2 + \|\sigma_2\|_\ell^2$$

hold, where

$$\|\chi\|_\ell^2 = \int_{|\ell|} \chi * \chi, \qquad \textit{etc., and}$$

$$\sigma_1 = a_1 dx + b_1 dy \ \textit{ and } \ \sigma_2 = a_2 dx + b_2 dy \qquad \textit{on} \quad \varphi_j(\ell).$$

PROOF. By making use of the parameter representation (1.5) of the lunar domain $\varphi_j(\ell)$, we define a differential σ on ℓ by

$$\sigma \circ \varphi_j^{-1}(z) = (1-\tau)\sigma_1 \circ \varphi_j^{-1}(z) + \tau \sigma_2 \circ \varphi_j^{-1}(z) \qquad (z = z(t, \tau) \in \varphi_j(\ell)).$$

We note that σ satisfies the same boundary conditions as χ on $\partial\ell$. Since χ is harmonic in ℓ, the inequality

$$\|\chi\|_\ell^2 \leqq \|\sigma\|_\ell^2 \tag{2.2}$$

holds. Further, the inequalities

$$\|\sigma\|_\ell^2 \leqq \iint_{\varphi_j(\ell)} ((1-\tau)\sqrt{a_1{}^2+b_1{}^2} + \tau\sqrt{a_2{}^2+b_2{}^2})^2 dxdy \tag{2.3}$$
$$\leqq \iint_{\varphi_j(\ell)} \max\{(a_1{}^2+b_1{}^2), \ (a_2{}^2+b_2{}^2)\} dxdy$$

hold. The inequalities (2.2) and (2.3) imply the inequality (2.1).

5. Difference of norms of σ_h and σ'_h

LEMMA 2.2. *Let σ_h be an arbitrary differential of the space Λ and let $\sigma'_h = F(\sigma_h)$.*

(i) *The inequalities*

$$\|\sigma_h\|^2 \leqq \|\sigma_h'\|^2 + \sum_{\sharp\ell\in K} \|\sigma_h\|^2_{\sharp\ell} \tag{2.4}$$

$$\leqq \|\sigma_h'\|^2 + \sum_{j=1}^{m} \sum_{\sharp\ell\in K_j} A(\varphi_j(\sharp\ell))\cdot\left(\frac{1}{\lambda}\int_{e_2}\sigma_h'\right)^2\cdot(1+\kappa h)$$

hold, where e_2 *is the edge of* $\sharp\ell$ *such that* $\varphi_j(e_2)$ *is a segment,* λ *is the length of* $\varphi_j(e_2)$ *and* κ *is a constant which depends only on the transformations* $f(z) = \varphi_k\circ\varphi_j^{-1}(z)$.

(ii)

$$\|\sigma_h'\|^2 \leqq \|\sigma_h\|^2 + \sum_{\flat\ell\in K} (\|\sigma'_{h,\natural s}\|^2_{\flat\ell} + \|\sigma'_{h,s'}\|^2_{\flat\ell}) \tag{2.5}$$

$$= \|\sigma_h\|^2 + \sum_{j=1}^{m} \sum_{\flat\ell\in K_j} \{A(\varphi_j(\flat\ell))\cdot(a_0{}^2+b_0{}^2) + A(\varphi_k(\flat\ell))\cdot(\alpha_0{}^2+\beta_0{}^2)\},$$

where for each triple $(s, s', \flat\ell)$ *the notations in* (iii) *of* §2.1 *are preserved.*

PROOF. (i) By Lemma 2.1 we see that for each triple $(s, s', \flat\ell)$

$$\|\sigma_h\|^2_{\flat\ell} \leqq \|\sigma'_{h,\natural s}\|^2_{\flat\ell} + \|\sigma'_{h,s'}\|^2_{\flat\ell}. \tag{2.6}$$

Hence the first inequality of (2.4) is obtained.

Let $(s, s', \sharp\ell)$ be a triple for an excessive lune $\sharp\ell$. We preserve the notations in (iv) of §2.1. We shall prove the inequality

$$\|\sigma_h\|^2_{\sharp\ell} \leqq A(\varphi_j(\sharp\ell))\cdot\left(\frac{1}{\lambda}\int_{e_2}\sigma_h'\right)^2\cdot(1+\kappa h), \tag{2.7}$$

from which the second inequality of (2.4) follows.

By γ and δ we denote the arguments of the oriented segments $\varphi_j(-e_2)$ and $\varphi_k(e_1)$ respectively. By making use of the parameter representation (1.5) of the lunar domain $\varphi_j(\sharp\ell)$, we define a differential σ on $\sharp\ell$ by

$$\begin{aligned} \sigma &= a\,dx + b\,dy \\ &\equiv (1-\tau)(a_0\cos\gamma + b_0\sin\gamma)\cdot((\cos\gamma)\,dx + (\sin\gamma)\,dy) \\ &\quad + \tau(\alpha_0\cos\delta + \beta_0\sin\delta)\cdot \\ &\quad \cdot\left((\cos\delta)\left(\frac{\partial\xi}{\partial x}dx + \frac{\partial\xi}{\partial y}dy\right) + (\sin\delta)\left(\frac{\partial\eta}{\partial x}dx + \frac{\partial\eta}{\partial y}dy\right)\right) \\ &\qquad\qquad (z = z(t, \tau)\in\varphi_j(\sharp\ell)). \end{aligned} \tag{2.8}$$

We note that σ satisfies the same boundary conditions as σ_h on $\partial(\sharp\ell)$. Hence

$$\|\sigma_h\|^2_{\sharp\ell} \leqq \|\sigma\|^2_{\sharp\ell} \leqq A(\varphi_j(\sharp\ell)) \max_{\varphi_j(\sharp\ell)} (a^2+b^2), \tag{2.9}$$

since σ_h is harmonic in $\sharp\ell$.

From the equation (2.8) it follows that

$$\max_{\varphi_j(\sharp\ell)} (a^2+b^2) \leqq \max\{(a_0\cos\gamma+b_0\sin\gamma)^2, \\ (\alpha_0\cos\delta+\beta_0\sin\delta)^2 \max_{\varphi_j(\sharp\ell)} |f'(z)|^2\}. \tag{2.10}$$

Further we note that

$$a_0\cos\gamma+b_0\sin\gamma=\frac{1}{\lambda}\int_{-e_2}\sigma'_h \tag{2.11}$$

and

$$\alpha_0\cos\delta+\beta_0\sin\delta=\frac{1}{\mu}\int_{e_1}\sigma'_h=\frac{1}{\mu}\int_{-e_2}\sigma'_h, \tag{2.12}$$

where

$$\lambda=\int_{\varphi_j(e_2)}|dz| \quad \text{and} \quad \mu=\int_{\varphi_j(e_1)}|f'(z)\,dz|. \tag{2.13}$$

By making use of the power series expansion of f' around a vertex z_1 of the lunar domain $\varphi_j(\sharp\ell)$, we see that

$$\max_{\varphi_j(\sharp\ell)} |f'(z)|^2 \leqq |f'(z_1)|^2(1+\kappa_1 h) \tag{2.14}$$

and

$$\mu \geqq (|f'(z_1)|-\kappa_2 h)\int_{\varphi_j(e_2)}|dz| = \lambda(|f'(z_1)|-k_2 h) \tag{2.15}$$

with constants κ_1, $\kappa_2>0$ depending only on f. Then the estimate (2.7) follows from (2.9)$\sim$(2.15).

(ii) The inequality (2.5) is obvious from the definition of σ'_h.

§3. Finite element approximations

1. Formulation of problems Let $\{C_1, C_2, C_3\}$ be a partition to three parts of the boundary $\partial\Omega$ such that each C_j ($j=1, 2, 3$) is a sum of boundary components of $\partial\Omega$, and let γ_k ($k=1,\cdots,\kappa$) be the boundary components of C_2.

Let Θ be a differential in Γ_c which satisfies the following conditions (i), (ii) and (iii):

(i) If $U_j \cap C_1 \neq \varnothing$, then $\Theta \circ \varphi_j^{-1}$ is harmonic on a neighborhood of $\varphi_j(U_j \cap C_1)$;

(ii) $\Theta = 0$ along C_2;

(iii) Θ is exact on a neighborhood of each boundary component of C_3, where the conditions (i), (ii) and (iii) may be ignored if $\partial\Omega = \varnothing$.

By Γ_Θ we denote the subspace of Γ_c consisting of all differentials σ for which there exists a function v on $\bar{\Omega}$ such that

$$\begin{aligned} dv &= \Theta - \sigma && \text{on } \bar{\Omega}, \\ v &= 0 && \text{on } C_1, \\ v &= \text{const.} && \text{on } \gamma_k \quad (k = 1, \cdots, \kappa). \end{aligned}$$

By ω we denote the harmonic differential in Γ_Θ uniquely determined by the conditions

$$\int_{\gamma_k} *\omega = 0 \qquad (k = 1, \cdots, \kappa) \tag{3.1}$$

and

$$*\omega = 0 \qquad \text{along} \quad C_3. \tag{3.2}$$

The differential ω can be constructed by the following procedure. Let χ be the harmonic component of Θ in the orthogonal decomposition of Γ_c (cf. Chapter 5 of Ahlfors and Sario [1]), and let u be the solution of the boundary value problem:

$$\begin{aligned} &u \text{ is a harmonic function on } \Omega, \\ &u = 0 \qquad \text{on } C_1, \\ &u = \text{const.} \qquad \text{on } \gamma_k, \\ &\int_{\gamma_k} *du = \int_{\gamma_k} *\chi \qquad (k = 1, \cdots, \kappa) \end{aligned}$$

and

$$*du = *\chi \qquad \text{along} \quad C_3.$$

Then, $\omega = \chi - du$. We note that the differential $\omega \circ \varphi_j^{-1}$ $(j = 1, \cdots, m)$ is harmonic on $\varphi_j(U_j \cap \bar{\Omega})$.[1)]

1) It is sufficient for our purpose that $\omega \circ \varphi_j^{-1}$ is of the class C^1 on $\varphi_j(U_j \cap \bar{\Omega})$ and hence we can weaken the assumption (i) for Θ.

LEMMA 3.1. *The harmonic differential ω satisfies the minimal property*

$$\|\omega\| = \min_{\sigma\in\Gamma_\Theta} \|\sigma\|. \tag{3.3}$$

In the equality (3.3), *the minimum of the right hand side is attained if and only if $\sigma=\omega$.*

PROOF. For each $\sigma\in\Gamma_\Theta$ there exists a function v such that

$$\begin{cases} dv = \sigma - \omega, & \\ v = 0 & \text{on} \quad C_1, \\ v = \text{const.} & \text{on} \quad \gamma_k \quad (k=1,\cdots,\kappa). \end{cases} \tag{3.4}$$

From (3.1), (3.2) and (3.4) it follows that

$$\begin{aligned} (\sigma-\omega, \omega) &= \int_{\partial\Omega} v*\omega \\ &= \int_{C_1} v*\omega + \sum_{k=1}^{\kappa}\int_{\gamma_k} v*\omega + \int_{C_3} v*\omega = 0, \end{aligned} \tag{3.5}$$

where

$$(\sigma, \tau) = (\sigma, \tau)_\Omega = \int_\Omega \sigma*\tau.$$

The equality (3.5) implies that

$$\|\sigma\|^2 = \|\omega\|^2 + \|\sigma-\omega\|^2 \geqq \|\omega\|^2.$$

In the last inequality, the equality holds if and only if $\sigma=\omega$.

The unique harmonic differential ω in Γ_Θ is called the *harmonic solution in* Γ_Θ.

Our aim is to obtain finite element approximations of ω in the spaces Λ and Λ', and error estimates between them and ω.

2. Finite element approximation ψ_h in Λ Let $\hat{\Theta}$ be the finite element interpolation of Θ in the space Λ. By Λ_Θ we denote the subspace of Λ consisting of all differentials $\sigma_h\in\Lambda$ for which there exists a function v on $\bar{\Omega}$ such that

$$\begin{aligned} &dv = \hat{\Theta} - \sigma_h, \\ &v = 0 \qquad \text{on} \quad C_1, \\ &v = \text{const.} \qquad \text{on} \quad \gamma_k \quad (k=1,\cdots,\kappa). \end{aligned}$$

By ψ_h we denote the differential of Λ_Θ such that

$$\|\psi_h\| = \min_{\sigma_h \in \Lambda_\Theta} \|\sigma_h\|. \tag{3.6}$$

We call ψ_h the *finite element approximation of* ω *in the space* Λ.

Next, we consider the special case where the differential Θ satisfies the condition:

$$\Theta = 0 \quad \text{along} \quad C_1.$$

We denote such a differential Θ by Θ_0. Since $\Lambda_{\Theta_0} \subset \Gamma_{\Theta_0}$, we see that

$$\|\omega\| \leqq \|\psi_h\|. \tag{3.7}$$

LEMMA 3.2. (i) *In the case of general* Θ, *the equality*

$$\|\psi_h - \omega\| = \min_{\sigma_h \in \Lambda_\Theta} \|\sigma_h - \omega\| \tag{3.8}$$

holds, where the minimum is attained if and only if $\sigma_h = \psi_h$.

(ii) *In the case of* $\Theta = \Theta_0$, *the equality*

$$\|\psi_h - \omega\|^2 = \|\psi_h\|^2 - \|\omega\|^2 \tag{3.9}$$

holds.

PROOF. (i) First, by a method similar to (3.5), it is shown that

$$(\omega, \sigma_h - \psi_h) = 0 \quad \text{for each} \quad \sigma_h \in \Lambda_\Theta. \tag{3.10}$$

By (3.6), standard arguments imply that

$$(\psi_h, \sigma_h - \psi_h) = 0 \quad \text{for each} \quad \sigma_h \in \Lambda_\Theta. \tag{3.11}$$

From (3.10) and (3.11), it follows that

$$\|\omega - \sigma_h\|^2 = \|\omega - \psi_h\|^2 + \|\sigma_h - \psi_h\|^2 \geqq \|\omega - \psi_h\|^2.$$

In the last inequality, the equality holds if and only if $\sigma_h = \psi_h$.

(ii) Since $\Lambda_{\Theta_0} \subset \Gamma_{\Theta_0}$, both ψ_h and ω are elements of Γ_{Θ_0}. Hence, by (3.5) $(\omega, \psi_h - \omega) = 0$ and thus

$$\|\psi_h - \omega\|^2 = \|\psi_h\|^2 - \|\omega\|^2.$$

From (3.11) the following lemma immediately follows.

LEMMA 3.3. *In the case of general* Θ, *the equality*

$$\|\sigma_h - \psi_h\|^2 = \|\sigma_h\|^2 - \|\psi_h\|^2 \tag{3.12}$$

holds for each $\sigma_h \in \Lambda_\Theta$.

3. Finite element approximation ω'_h in Λ' Let $\Lambda'_\Theta=\{\sigma'_h|\sigma'_h=F(\sigma_h),\ \sigma_h\in\Lambda_\Theta\}$. By ω'_h we denote the differential of Λ'_Θ such that

$$\|\omega'_h\| = \min_{\sigma'_h\in\Lambda'_\Theta} \|\sigma'_h\|. \tag{3.13}$$

We call ω'_h the *finite element approximation of* ω in the space Λ'.

LEMMA 3.4. *The equality*

$$\|\sigma'_h-\omega'_h\|^2 = \|\sigma'_h\|^2 - \|\omega'_h\|^2 \tag{3.14}$$

holds for each $\sigma'_h\in\Lambda'_\Theta$.

PROOF. By (3.13), standard arguments imply that

$$(\omega'_h,\ \sigma'_h-\omega'_h)=0 \quad \text{for each } \sigma'_h\in\Lambda'_\Theta. \tag{3.15}$$

This implies (3.14).

4. Lemma of Bramble and Zlámal The following lemma is due to J. H. Bramble and M. Zlámal (cf. [9]).

LEMMA 3.5. *Let Δ be a closed triangle on the z-plane ($z=x+iy$) with $d(\Delta)\leqq h$, let v be a function of the class C^2 defined on Δ such that $v=0$ at each vertex of Δ. Then, the inequality*

$$\begin{aligned}(3.16)\quad &\iint_\Delta\left(\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right)dxdy\\ &\leqq\frac{B}{\sin^2\theta}h^2\iint_\Delta\left(\left(\frac{\partial^2 v}{\partial x^2}\right)^2+2\left(\frac{\partial^2 v}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 v}{\partial y^2}\right)^2\right)dxdy\end{aligned}$$

holds, where B is an absolute constant and θ is the smallest interior angle of the triangle Δ.

5. Pointwise estimate

LEMMA 3.6. *Let Δ be a closed curvilinear triangle on the z-plane ($z=x+iy$) with $d(\Delta)\leqq h$ which is the image of some 2-simplex $s\in K_j$ ($j=1,\cdots,m$) by $z=\varphi_j(p)$, and let v be a function of the class C^2 defined on Δ such that $v=0$ at each vertex of Δ. Then,*

$$\begin{aligned}&\left|\frac{\partial v}{\partial x}\right|,\ \left|\frac{\partial v}{\partial y}\right|\\ &\leqq h\cdot\frac{4}{\sin\theta}\max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right|+2\left|\frac{\partial^2 v}{\partial x\partial y}\right|+\left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1+\kappa h)\end{aligned}$$

on Δ, where θ is the smallest interior angle of the ordinary triangle which has common vertices with Δ, and κ is a constant which depends only on $f(z) = \varphi_k \circ \varphi_j^{-1}(z)$.

PROOF. (Cf. Theorem 3.1 of Strang and Fix [27].) Let $z_0 = x_0 + iy_0$ be a fixed point and $z = x + iy$ an arbitrary point in Δ, and let $k = x - x_0$ and $l = y - y_0$. Here we choose the point z_0 so that for each $z \in \Delta$ the segment between z_0 and z is contained in Δ.

By Taylor's theorem we have that

$$v(z) = P(z) + r(z),$$

where

$$P(z) = v(z_0) + \left(k\frac{\partial}{\partial x} + l\frac{\partial}{\partial y}\right)v(z_0),$$

$$r(z) = \frac{1}{2!}\left(k\frac{\partial}{\partial x} + l\frac{\partial}{\partial y}\right)^2 v(z') \tag{3.17}$$

with some point z' on the segment between z_0 and z. First, from (3.17) the estimate

$$|r(z)| \leqq \frac{h^2}{2}\max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x \partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right|\right) \qquad (z \in \Delta) \tag{3.18}$$

immediately follows. Let z_j $(j = 1, 2, 3)$ be the vertices of Δ. Then, by the assumption of the lemma

$$v(z_j) = P(z_j) + r(z_j) = 0 \qquad (j = 1, 2, 3). \tag{3.19}$$

Since $P(z)$ is a linear function of x and y, by (3.19) we have the expression

$$P(z) = -r(z_1)\phi_1(z) - r(z_2)\phi_2(z) - r(z_3)\phi_3(z), \tag{3.20}$$

where ϕ_j $(j = 1, 2, 3)$ are linear functions of x and y such that

$$\phi_j(z_k) = \delta_{jk} \quad (j, k = 1, 2, 3)$$

with Kronecker's symbol δ_{jk}. (3.18) and (3.20) imply the estimate

$$\begin{aligned} \left|\frac{\partial P}{\partial x}\right| &\leqq |r(z_1)|\left|\frac{\partial \phi_1}{\partial x}\right| + |r(z_2)|\left|\frac{\partial \phi_2}{\partial x}\right| + |r(z_3)|\left|\frac{\partial \phi_3}{\partial x}\right| \\ &\leqq \frac{3}{2}h^2 \max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right| + 2\left|\frac{\partial^2 v}{\partial x \partial y}\right| + \left|\frac{\partial^2 v}{\partial y^2}\right|\right)\cdot \max_{1\leqq j\leqq 3}\left|\frac{\partial \phi_j}{\partial x}\right|. \end{aligned} \tag{3.21}$$

Here we can easily verify that

(3.22) $$\left|\frac{\partial\phi_j}{\partial x}\right| \leqq \frac{1}{h_1}\cdot\frac{2}{\sin\theta} \qquad (j=1,\ 2,\ 3),$$

where h_1 is the diameter of the ordinary triangle which has common vertices with Δ. From (3.21) and (3.22) it follows that

(3.23) $$\left|\frac{\partial P}{\partial x}\right| \leqq 3h\cdot\frac{1}{\sin\theta}\max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right|+2\left|\frac{\partial^2 v}{\partial x\partial y}\right|+\left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1+\kappa h).$$

By Taylor's theorem we have that

$$\frac{\partial v(z)}{\partial x}=\frac{\partial v(z_0)}{\partial x}+\left(k\frac{\partial}{\partial x}+l\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}v(z'')$$

with some point z'' on the segment between z_0 and z. Since $\partial v(z_0)/\partial x=\partial P(z_0)/\partial x$ and

$$\left|\left(k\frac{\partial}{\partial x}+l\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}v(z'')\right| \leqq h\max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right|+\left|\frac{\partial^2 v}{\partial x\partial y}\right|\right),$$

by (3.23) we obtain the estimate

$$\left|\frac{\partial v(z)}{\partial x}\right| \leqq \frac{4h}{\sin\theta}\max_{z\in\Delta}\left(\left|\frac{\partial^2 v}{\partial x^2}\right|+2\left|\frac{\partial^2 v}{\partial x\partial y}\right|+\left|\frac{\partial^2 v}{\partial y^2}\right|\right)(1+\kappa h).$$

Analogously the estimate for $|\partial v/\partial y|$ is obtained.

6. Approximation by ψ_h

THEOREM 3.1. *Let ω be the harmonic solution in Γ_Θ defined in* §3.1 *and let ψ_h be the finite element approximation of ω in the space Λ. Then,*

(3.24) $$\begin{aligned}&\|\psi_h-\omega\|^2\\ &\leqq\frac{h^2}{\sin^2\theta}\left(B\sum_{j=1}^{m}\iint_{\varphi_j(K'_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)dxdy\right.\\ &\qquad\left.+Ch^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)\right),\end{aligned}$$

where B and C are constants independent of the triangulation K and the differential Θ, θ is the smallest value of interior angles of all triangles $\varphi_j(s)$ $(s\in K'_j;\ j=1,\cdots,m)$,

$$\omega=a\,dx+b\,dy \qquad on \quad \varphi_j(U_j\cap\bar{\Omega}) \qquad (j=1,\cdots,m),$$

by $\varphi_j(K'_j)$ we denote the image set by φ_j of the carrier of K'_j, and R_j $(j=1,\cdots,m)$

are the closed subsets of $U_j \cap \bar{\Omega}$ *defined in* (i′) *of* §1.2.

PROOF. First, by (i) of Lemma 3.2,

$$\|\psi_h - \omega\| \leqq \|\hat{\omega} - \omega\|. \tag{3.25}$$

Hence it is sufficient to estimate $\|\hat{\omega} - \omega\|$.

We have

$$\|\hat{\omega} - \omega\|_\Omega^2 = \sum_{j=1}^{m} \sum_{s \in K_j} \|\hat{\omega} - \omega\|_s^2. \tag{3.26}$$

Here we note that $\omega \circ \varphi_j^{-1} (j=1, \cdots, m)$ is of the class C^1 on $\varphi_j(U_j \cap \bar{\Omega})$. Then, by Lemma 3.5,

$$\begin{aligned} &\|\hat{\omega} - \omega\|_s^2 \\ &\leqq \frac{B}{\sin^2\theta} h^2 \iint_{\varphi_j(s)} \left(\left(\frac{\partial a}{\partial x} \right)^2 + \left(\frac{\partial a}{\partial y} \right)^2 + \left(\frac{\partial b}{\partial x} \right)^2 + \left(\frac{\partial b}{\partial y} \right)^2 \right) dxdy \end{aligned} \tag{3.27}$$

for each natural simplex s of K_j. For simplicity, we denote the right hand side of (3.27) by $I[\varphi_j(s)]$.

For a triple (s, s', ℓ) for a minor simplex s, we denote the differential $\hat{\omega}'$ on $\natural s \in K'_j$ and $s' \in K'_k$ by $\hat{\omega}'_{\natural s}$ and $\hat{\omega}'_{s'}$ respectively. Then, by Lemma 2.1

$$\|\hat{\omega} - \omega\|_\ell^2 \leqq \|\hat{\omega}'_{\natural s} - \omega\|_\ell^2 + \|\hat{\omega}'_{s'} - \omega\|_\ell^2. \tag{3.28}$$

This inequality and Lemma 3.5 imply that

$$\begin{aligned} \|\hat{\omega} - \omega\|_{s+s'}^2 &\leqq \|\hat{\omega}'_{\natural s} - \omega\|_{\natural s}^2 + \|\hat{\omega}'_{s'} - \omega\|_{s'}^2 \\ &\leqq I[\varphi_j(\natural s)] + I[\varphi_k(s')]. \end{aligned} \tag{3.29}$$

Let (s, s', ℓ) be a triple for a major simplex s. Then, by Lemma 3.5

$$\|\hat{\omega} - \omega\|_s^2 \leqq I[\varphi_j(\natural s)] + \|\hat{\omega} - \omega\|_\ell^2 \tag{3.30}$$

and

$$\|\hat{\omega} - \omega\|_{s'}^2 \leqq I[\varphi_k(s')]. \tag{3.31}$$

Let

$$\hat{\omega} = a_0 dx + b_0 dy \quad \text{on} \quad \varphi_j(\natural s), \quad \text{and}$$
$$\hat{\omega} = \alpha_0 d\xi + \beta_0 d\eta \quad \text{on} \quad \varphi_k(s'),$$

where a_0, b_0, α_0 and β_0 are constants. Then we define differentials $\hat{\omega}_s$ and $\hat{\omega}_{s'+\ell}$ on s and $s'+\ell$ respectively by

$$\hat{\omega}_s = a_0 dx + b_0 dy \qquad \text{on} \quad \varphi_j(s), \quad \text{and}$$

$$\hat{\omega}_{s'+\ell} = \alpha_0 d\xi + \beta_0 d\eta \qquad \text{on} \quad \varphi_k(s'+\ell).$$

Then, by Lemma 2.1

$$\|\hat{\omega}-\omega\|_\ell^2 \leqq \|\hat{\omega}_s-\omega\|_\ell^2 + \|\hat{\omega}_{s'+\ell}-\omega\|_\ell^2. \tag{3.32}$$

Further, by Lemma 3.6

$$\begin{aligned} &\|\hat{\omega}_s-\omega\|_\ell^2 \\ &\leqq A(\varphi_j(\ell))\cdot\frac{32h^2}{\sin^2\theta}\cdot\max_{\varphi_j(s)}\left(\left|\frac{\partial a}{\partial x}\right|+\left|\frac{\partial a}{\partial y}\right|+\left|\frac{\partial b}{\partial x}\right|+\left|\frac{\partial b}{\partial y}\right|\right)^2(1+\kappa h)^2 \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\|\hat{\omega}_{s'+\ell}-\omega\|_\ell^2 \\ &\leqq A(\varphi_k(\ell))\cdot\frac{32h^2}{\sin^2\theta}\cdot\max_{\varphi_k(s'+\ell)}\left(\left|\frac{\partial \alpha}{\partial \xi}\right|+\left|\frac{\partial \alpha}{\partial \eta}\right|+\left|\frac{\partial \beta}{\partial \xi}\right|+\left|\frac{\partial \beta}{\partial \eta}\right|\right)^2(1+\kappa h)^2, \end{aligned} \tag{3.34}$$

where $\omega=a\,dx+b\,dy$ on $\varphi_j(s)$ and $\omega=\alpha\,d\xi+\beta\,d\eta$ on $\varphi_k(s'+\ell)$.

By (3.25)$\sim$(3.34), Lemma 1.1 and (1.1), the estimate (3.24) is obtained.

7. Approximation by ω_h'

THEOREM 3.2. (i) *Let ω be the harmonic solution in Γ_Θ defined in §3.1, let ω_h' be the finite element approximation of ω in the space Λ' and let $\omega_h=F^{-1}(\omega_h')$. Then*

$$\begin{aligned} &\|\omega_h-\omega\|^2 \\ &\leqq \frac{h^2}{\sin^2\theta}\left(A'\sum_{j=1}^{m}\iint_{\varphi_j(K'_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)dxdy\right. \\ &\quad \left. + B'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}\left(\left(\frac{\partial a}{\partial x}\right)^2+\left(\frac{\partial a}{\partial y}\right)^2+\left(\frac{\partial b}{\partial x}\right)^2+\left(\frac{\partial b}{\partial y}\right)^2\right)\right) \\ &\quad + C'h^2\sum_{j=1}^{m}\max_{\varphi_j(R_j)}(a^2+b^2), \end{aligned} \tag{3.35}$$

where A', B' and C' are constants independent of the triangulation K and the differential Θ, and other notations are the same as in Theorem 3.1.

(ii) *Let Θ_0 be the differential defined in §3.2, let ω be the harmonic solution in Γ_{Θ_0} and let ω_h' be the finite element approximation of ω in the space Λ'. Then the estimate*

$$\|\omega\|^2 \leqq \|\omega_h'\|^2 + \varepsilon(\omega_h') \tag{3.36}$$

holds with

$$\varepsilon(\omega_h') \equiv \sum_{j=1}^{m} \sum_{\sharp\ell\in K_j} A(\varphi_j(\sharp\ell))\cdot\left(\frac{1}{\lambda}\int_{e_2} \omega_h'\right)^2 \cdot \max\left\{1, \left(\frac{\lambda}{\mu}\right)^2 \max_{\varphi_j(\sharp\ell)} |f'(z)|^2\right\}, \tag{3.37}$$

where e_1 and e_2 are the edges of $\sharp\ell$ such that $\varphi_j(e_2)$ is a straight segment, λ and μ are the lengths of the segments $\varphi_j(e_2)$ and $\varphi_k(e_1)$ resp., and $f(z)\equiv\varphi_k\circ\varphi_j^{-1}(z)$.

PROOF. (i) First, note that

$$\|\omega_h-\omega\|^2 \leqq 2\|\psi_h-\omega\|^2 + 2\|\omega_h-\psi_h\|^2. \tag{3.38}$$

From Lemmas 2.1, 2.2 and 3.3, and (3.13), it follows that

$$\begin{aligned} \|\omega_h-\psi_h\|^2 &= \|\omega_h\|^2 - \|\psi_h\|^2 \\ &\leqq \|\omega_h'\|^2 - \|\psi_h\|^2 + \sum_{\sharp\ell\in K} \|\omega_h\|^2_{\sharp\ell} \\ &\leqq \|\psi_h'\|^2 - \|\psi_h\|^2 + \sum_{\sharp\ell\in K} \|\omega_h\|^2_{\sharp\ell} \\ \leqq \sum_{j=1}^{m} \sum_{\flat\ell\in K_j} &(A(\varphi_j(\flat\ell))\cdot(a_0'^2+b_0'^2) + A(\varphi_k(\flat\ell))\cdot(\alpha_0'^2+\beta_0'^2)) \\ + \sum_{j=1}^{m} \sum_{\sharp\ell\in K_j} &(A(\varphi_j(\sharp\ell))\cdot(a_0^2+b_0^2) + A(\varphi_k(\sharp\ell))\cdot(\alpha_0^2+\beta_0^2)), \end{aligned} \tag{3.39}$$

where for each triple $(s, s', \flat\ell)$ for $\flat\ell\in K_j$

$$\psi_h' = a_0'dx + b_0'dy \qquad \text{on} \quad \varphi_j(\natural s) \quad \text{and}$$

$$\psi_h' = \alpha_0'd\xi + \beta_0'd\eta \qquad \text{on} \quad \varphi_k(s'),$$

and for each triple $(s, s', \sharp\ell)$ for $\sharp\ell\in K_j$

$$\omega_h = a_0dx + b_0dy \qquad \text{on} \quad \varphi_j(\natural s) \quad \text{and}$$

$$\omega_h = \alpha_0d\xi + \beta_0d\eta \qquad \text{on} \quad \varphi_k(s')$$

with constants a_0', b_0', α_0', β_0', a_0, b_0, α_0 and β_0.

In the inequality (3.39), we have

$$\begin{aligned} &A(\varphi_j(\flat\ell))\cdot(a_0'^2+b_0'^2) \\ &= \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\|\psi_h\|_s^2 \\ &\leqq 2\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}(\|\psi_h-\omega\|_s^2 + \|\omega\|_s^2) \\ &\leqq 2\frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))}\|\psi_h-\omega\|_s^2 + 2A(\varphi_j(\flat\ell))\cdot\max_{\varphi_j(s)}(a^2+b^2). \end{aligned} \tag{3.40}$$

Since we can easily verify that

$$A(\varphi_j(\natural s)) > \frac{h_1{}^2}{4}\sin\theta \qquad (h_1 = d(\varphi_j(\natural s))),$$

by Lemma 1.1 we have

$$(3.41)\qquad \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(s))} = \frac{A(\varphi_j(\flat\ell))}{A(\varphi_j(\natural s)) - A(\varphi_j(\flat\ell))}$$
$$\leqq \frac{h}{2\sin\theta}\left(\left|\frac{g''(\zeta_1)}{g'(\zeta_1)^2}\right| + O(h)\right)$$

with the notations in Lemma 1.1. (3.40) and (3.14) imply

$$(3.42)\qquad \sum_{j=1}^{m}\sum_{\flat\ell\in K_j} A(\varphi_j(\flat\ell))\cdot(a_0'^2 + b_0'^2)$$
$$\leqq \frac{Ch}{\sin\theta}\sum_{j=1}^{m}\sum_{\flat\ell\in K_j}\|\psi_h - \omega\|_s^2 + 2\sum_{j=1}^{m}\sum_{\flat\ell\in K_j} A(\varphi_j(\flat\ell))\max_{\varphi_j(s)}(a^2+b^2),$$

where C is a constant depending only on the transformations of local parameters. Since similar estimates for other terms of the right hand side of (3.39) are obtained, from (3.39) it follows that

$$(3.43)\qquad \|\omega_h - \psi_h\|^2$$
$$\leqq \frac{Ch}{\sin\theta}\|\omega_h - \omega\|^2 + \frac{Ch}{\sin\theta}\|\psi_h - \omega\|^2$$
$$+ 2\sum_{j=1}^{m}\sum_{\ell\in K_j}\left(A(\varphi_j(\ell))\max_{\varphi_j(s)}(a^2+b^2) + A(\varphi_k(\ell))\max_{\varphi_k(s')}(\alpha^2+\beta^2)\right),$$

where for each triple (s, s', ℓ) for $\ell\in K_j$

$$\omega = a\,dx + b\,dy \qquad \text{on} \quad \varphi_j(s), \quad \text{and}$$
$$\omega = \alpha\,d\xi + \beta\,d\eta \qquad \text{on} \quad \varphi_k(s').$$

(3.38), (3.43), Theorem 3.1, Lemma 1.1 and (1.1) imply the estimate (3.35).

(ii) (3.7) and Lemma 3.3 and the proof of Lemma 2.2 (i) imply the inequalities

$$\|\omega\|^2 \leqq \|\psi_h\|^2 \leqq \|\omega_h\|^2$$
$$\leqq \|\omega_h'\|^2 + \sum_{j=1}^{m}\sum_{\sharp\ell\in K_j} A(\varphi_j(\sharp\ell))\left(\frac{1}{\lambda}\int_{e_2}\omega_h'\right)^2\cdot$$
$$\cdot\max\left\{1, \left(\frac{\lambda}{\mu}\right)^2 \max_{\varphi_j(\sharp\ell)} |f'(z)|^2\right\}.$$

From Lemma 2.2, (3.41), the proof of Theorem 3.1 and Theorem 3.2, the following corollary follows.

COROLLARY 3.1. *Let ω and ω'_h be the same as in Theorem 3.2, $\hat{\omega}$ be the finite element interpolation of ω in the space Λ, and $\hat{\omega}'=F(\hat{\omega})$. Then, the estimate*

$$\|\omega'_h-\hat{\omega}'\| \leqq A''h \tag{3.44}$$

holds, where A'' is a constant dependent only on ω and θ in Theorem 3.1.

§ 4. Applications

1. Periodicity moduli of Riemann surfaces Let $\bar{\Omega}$ be a closed or compact bordered Riemann surface of genus 1 with no or one boundary component. Let $\{A, B\}$ be a canonical homology basis of $\bar{\Omega}$ such that $A\times B=1$. Then there exists a unique system of harmonic differentials $\{\phi, \varrho, \chi, \tau\}$ on Ω satisfying the period and boundary conditions:

$$\int_B \phi=\int_B \chi=1. \qquad \int_A \phi=\int_A \chi=0, \tag{4.1}$$

$$\int_A \varrho=\int_A \tau=-1, \qquad \int_B \varrho=\int_B \tau=0, \tag{4.2}$$

$$\phi=\varrho=*\chi=*\tau=0 \qquad \text{along} \quad \partial\Omega \tag{4.3}$$

and

$$\int_{\partial\Omega} *\phi=\int_{\partial\Omega} *\varrho=\int_{\partial\Omega} \chi=\int_{\partial\Omega} \tau=0, \tag{4.4}$$

where the conditions (4.3) and (4.4) may be ignored if $\partial\Omega=\varnothing$. If $\partial\Omega=\varnothing$, then $\phi=\chi$ and $\varrho=\tau$.

We can easily see that

$$\begin{cases} \|\phi\|^2=\int_A *\phi, \quad \|\varrho\|^2=\int_B *\varrho, \quad \text{and} \\ (\phi, \varrho)=\int_B *\phi=\int_A *\varrho=0. \end{cases} \tag{4.5}$$

We call

$$p_1=\int_A *\phi \qquad \text{and} \qquad p_2=\int_B *\varrho$$

periodicity moduli of $\bar{\Omega}$ with respect to A and B respectively, which are the quantities determining the conformal structure of Ω. By (4.1)$\sim$(4.5) we see that

$$\tau=-\frac{*\phi}{\|\phi\|^2} \quad \text{and} \quad \chi=\frac{*\varrho}{\|\varrho\|^2}.$$

These relations imply that

$$(4.6) \qquad p_1=\|\phi\|^2=\frac{1}{\|\tau\|^2} \quad \text{and} \quad p_2=\|\varrho\|^2=\frac{1}{\|\chi\|^2}.$$

If $\partial\Omega=\varnothing$, then

$$(4.7) \qquad p_1=\|\phi\|^2=\frac{1}{\|\varrho\|^2}=\frac{1}{p_2}.$$

By making use of a relation analogous to (4.7) for the modulus of quadrilaterals on the complex plane, Gaier [11] presented a method to obtain upper and lower bounds for the modulus by the finite difference approximation.

2. Calculation of periodicity moduli Let $\{\Theta_1, \Theta_2, \Theta_3, \Theta_4\}$ be a system of differentials in $\Gamma_c(\bar{\Omega})$ satisfying the period and boundary conditions:

$$\int_B \Theta_1=\int_B \Theta_3=1, \qquad \int_A \Theta_1=\int_A \Theta_3=0,$$
$$\int_A \Theta_2=\int_A \Theta_4=-1, \qquad \int_B \Theta_2=\int_B \Theta_4=0,$$
$$\Theta_1=\Theta_2=0 \quad \text{along} \quad \partial\Omega,$$

and Θ_3 and Θ_4 are exact on a neighborhood of $\partial\Omega$. Here we interpret that $\partial\Omega=C_2$ for Θ_1 and Θ_2, and $\partial\Omega=C_3$ for Θ_3 and Θ_4 in the notations in §3.1. We note that Θ_1, Θ_2, Θ_3 and Θ_4 satisfy the conditions for the differential Θ_0 in §3.2. Then we can easily see that ϕ, ϱ, χ and τ are the harmonic solutions in Γ_{Θ_1}, Γ_{Θ_2}, Γ_{Θ_3} and Γ_{Θ_4}, respectively. Let ϕ'_h, ϱ'_h, χ'_h and τ'_h be the finite element approximations of ϕ, ϱ, χ and τ in the space Λ' respectively. Then by (ii) of Theorem 3.2 and (4.6), we obtain upper and lower bounds for p_1 and p_2:

$$(4.8) \qquad \frac{1}{\|\tau'_h\|^2+\varepsilon(\tau'_h)} \leqq p_1 \leqq \|\phi'_h\|^2+\varepsilon(\phi'_h)$$

and

$$(4.9) \qquad \frac{1}{\|\chi'_h\|^2+\varepsilon(\chi'_h)} \leqq p_2 \leqq \|\varrho'_h\|^2+\varepsilon(\varrho'_h).$$

If $\partial\Omega=\varnothing$, then $\phi=\chi$ and $\varrho=\tau$, and thus (4.8) and (4.9) imply the inequalities

$$\frac{1}{\|\varrho_h'\|^2+\varepsilon(\varrho_h')}\leqq p_1=\frac{1}{p_2}\leqq\|\phi_h'\|^2+\varepsilon(\phi_h').$$

3. Numerical example 1 (the case of a closed Riemann surface) Let $\bar{\Omega}$ be the two-sheeted covering surface with four branch points $z=-3,\ -1,\ 1,\ 3$ over the extended z-plane. Then Ω is a closed Riemann surface of genus one. A canonical homology basis $\{A,\ B\}$ of Ω is chosen as in Fig. 6. We aim to obtain good upper and lower approximate values of the periodicity moduli p_1 and p_2 of Ω with respect to A and B respectively.

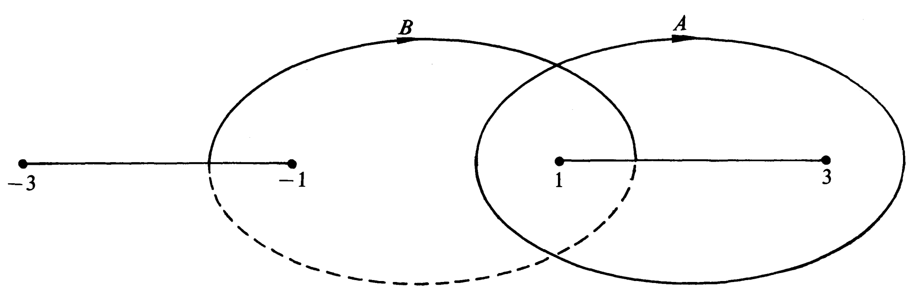


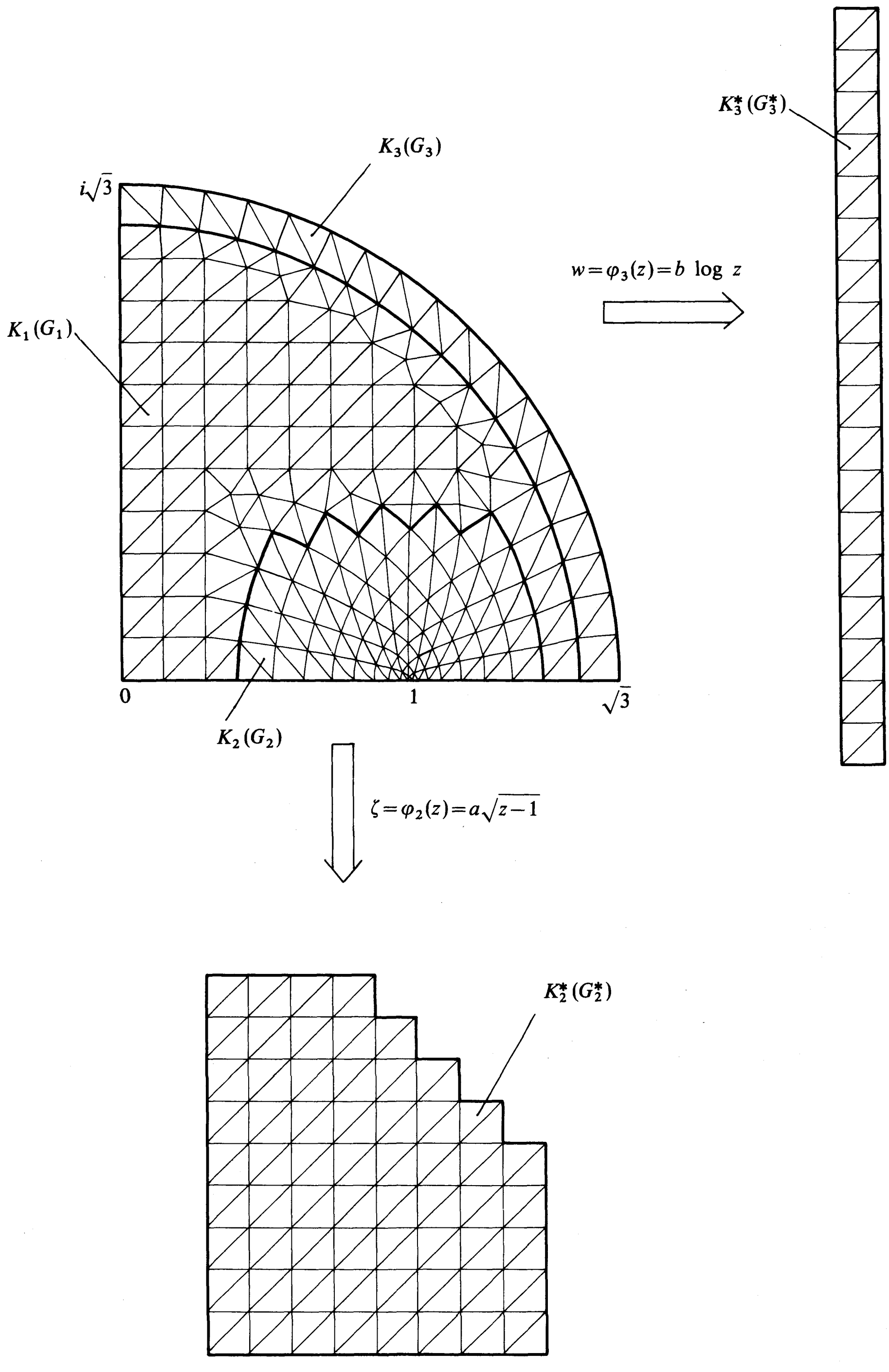
Fig. 6

First, we construct a triangulation of the closed region:

$$\bar{D}=\{z|\ |z|\leqq\sqrt{3},\ \mathrm{Re}\,z\geqq 0,\ \mathrm{Im}\,z\geqq 0\}$$

as in Fig. 7. The closed regions G_2 and G_3 are mapped onto the regions G_2^* and G_3^* resp. by the local parameters $\zeta=\varphi_2(z)=a\sqrt{z-1}$ and $w=\varphi_3(z)=b\log z$ ($a=2(\sqrt{3}-1)^{1/2}$ and $b=\sqrt{3}$) respectively, where a and b are so determined that $|d\zeta/dz|=1$ and $|dw/dz|=1$ on $|z-1|=\sqrt{3}-1$ and $|z|=\sqrt{3}$ respectively. We construct ordinary triangulations K_2^* and K_3^* of G_2^* and G_3^* as in Fig. 7 respectively. By K_2 and K_3 we denote the image triangulations of K_2^* and K_3^* by the mappings φ_2^{-1} and φ_3^{-1} respectively. The triangulation K_1 of the region $G_1=\overline{D-(G_2\cup G_3)}$ in Fig. 7 is so constructed that each 2-simplex s of K_1 is natural, minor or major according as $|s|\cap|K_2+K_3|=\varnothing$, $|s|\cap|K_2|\neq\varnothing$, or $|s|\cap|K_3|\neq\varnothing$, where if some intersection is a point then it is interpreted to be vacuous, and the local parameter $\varphi_1(z)$ of K_1 is the identity mapping $\varphi_1(z)\equiv z$.

A triangulation L_1 of the region $\bar{D}_1=\{z|\ |z|\geqq\sqrt{3},\ \mathrm{Re}\,z\geqq 0,\ \mathrm{Im}\,z\geqq 0\}$ is defined by the reflection of the triangulation $L\equiv K_1+K_2+K_3$ with respect to



$K_3^*(G_3^*)$
$K_3(G_3)$
$i\sqrt{3}$
$w=\varphi_3(z)=b\log z$
$K_1(G_1)$
0
1
$\sqrt{3}$
$K_2(G_2)$
$\zeta=\varphi_2(z)=a\sqrt{z-1}$
$K_2^*(G_2^*)$

Fig. 7

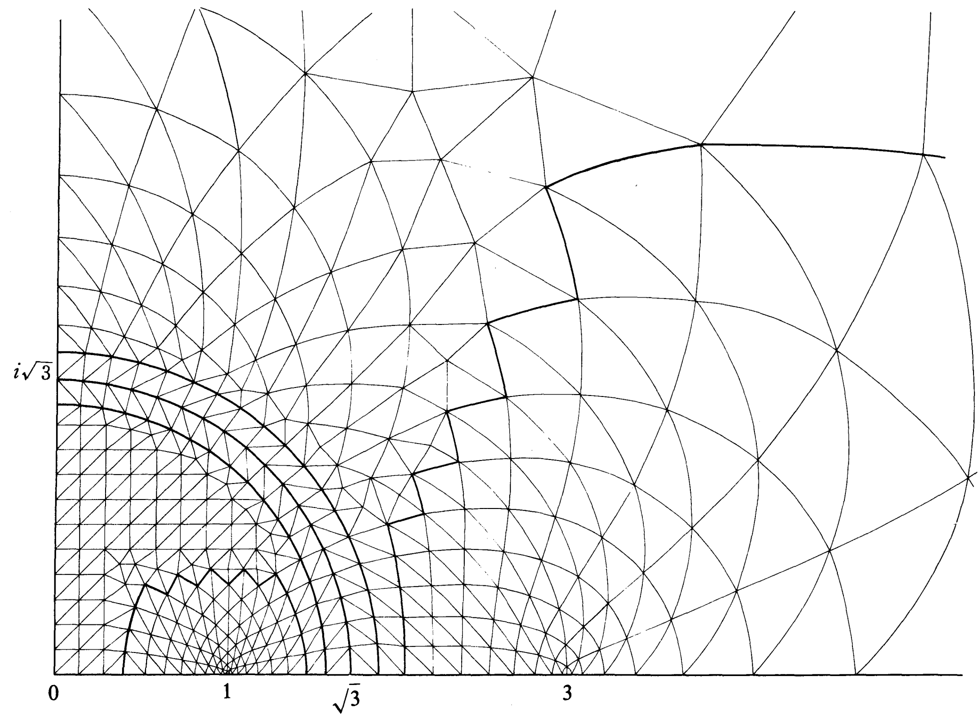


Fig. 8

the circle $|z|=\sqrt{3}$ (cf. Fig. 8). Next we define a triangulation L_2 of the fourth quadrant by the reflection of the triangulation $L+L_1$ with respect to the real axis and then a triangulation L_3 of the left half-plane by the reflection of $L+L_1+L_2$ with respect to the imaginary axis. Consequently, a triangulation L_4 of the extended z-plane is defined by $L_4=L+L_1+L_2+L_3$. Then, a triangulation K of the covering surface Ω is so constructed that the projection T of K onto the extended z-plane is the triangulation L_4. We see that the triangulation K conforms to the definition in §1.2. We denote the parts of $T^{-1}(\bar{D})$ and $T^{-1}(L)$ on the upper sheet of Ω by $\bar{D}$ and L again respectively.

Let $\phi=\chi$ and $\varrho=\tau$ be the differentials on the present Ω defined in §4.1, and let ϕ_h' and ϱ_h' be the finite element approximations of ϕ and ϱ respectively in the space $\Lambda'(K')$, where K' is the naturalized triangulation associated to the present K.

Let $\Lambda(L)$ be the space of differentials on $\bar{D}$ which are the restrictions of those in $\Lambda(K)$ to $\bar{D}$. Let $\Lambda_\phi(L)$ be the subspace of $\Lambda(L)$ which consists of the differentials σ_h in $\Lambda(L)$ satisfying the conditions:

$$\sigma_h = 0 \quad \text{along} \quad c_0 = \{z \mid 0 \leqq \operatorname{Im} z \leqq \sqrt{3},\ \operatorname{Re} z = 0\},$$

$$\sigma_h = 0 \quad \text{along} \quad c_1 = \{z \mid 1 \leqq \operatorname{Re} z \leqq \sqrt{3},\ \operatorname{Im} z = 0\}$$

and

$$\int_{B \cap \bar{D}} \sigma_h = \frac{1}{4},$$

and let $\Lambda'_\phi(L') = \{\sigma'_h = F(\sigma_h), \sigma_h \in \Lambda_\phi(L)\}$. Further, let $\Lambda_\varrho(L)$ be the subspace of $\Lambda(L)$ which consists of the differentials σ_h in $\Lambda(L)$ satisfying the conditions:

$$\sigma_h = 0 \quad \text{along} \quad c_0^* = \{z \mid 0 \leqq \operatorname{Re} z \leqq 1,\ \operatorname{Im} z = 0\},$$

$$\sigma_h = 0 \quad \text{along} \quad c_1^* = \left\{z \,\middle|\, |z| = \sqrt{3},\ 0 \leqq \arg z \leqq \frac{\pi}{2}\right\}$$

and

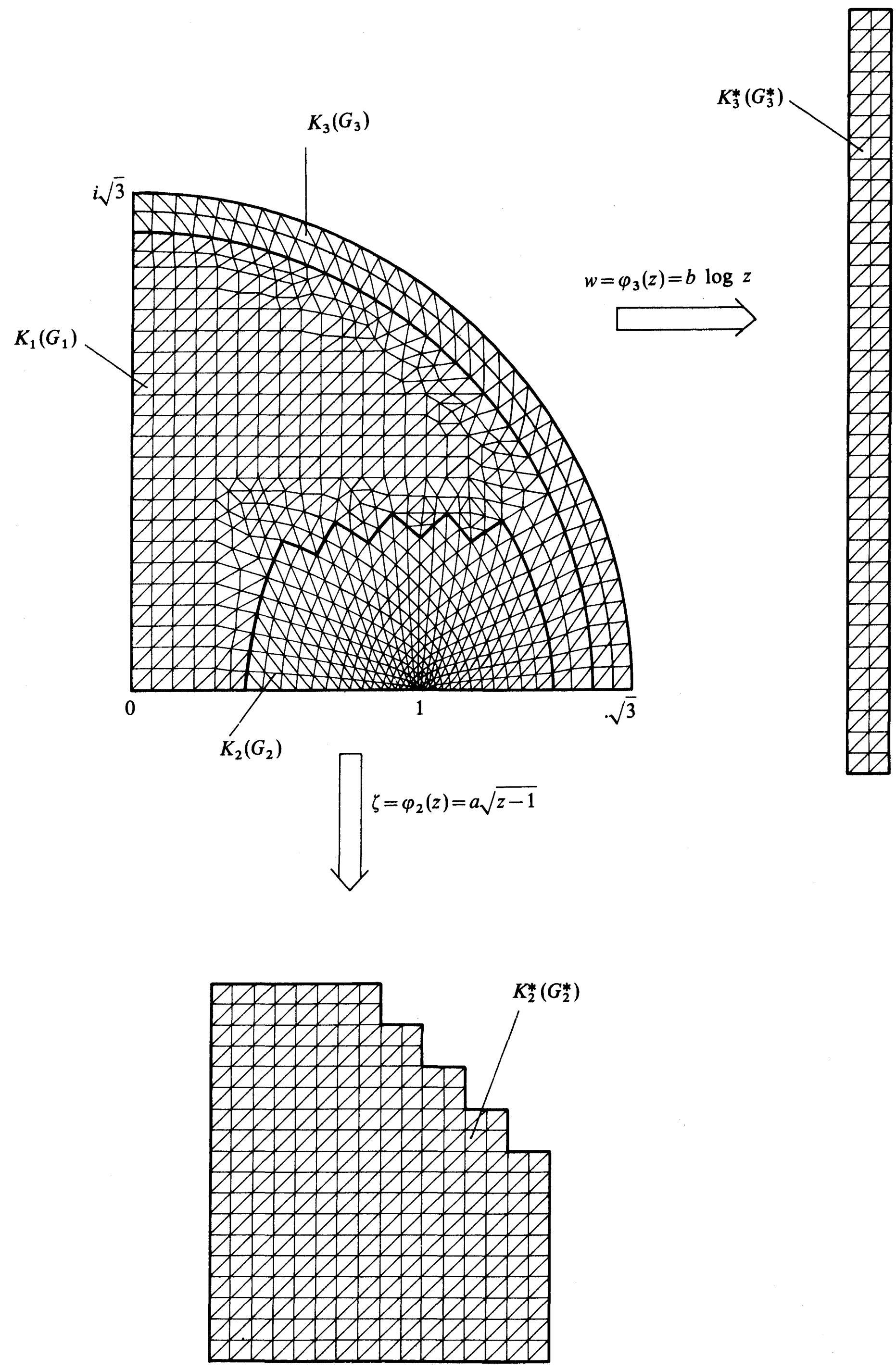
$$\int_{A \cap \bar{D}} \sigma_h = -\frac{1}{4},$$

and let $\Lambda'_\varrho(L') = \{\sigma'_h = F(\sigma_h),\ \sigma_h \in \Lambda_\varrho(L)\}$. By $\phi'_{h,L}$ and $\varrho'_{h,L}$ we denote the

Table 1. Periodicity moduli p_1 of closed Riemann surface

Exact value	$p_1 = \int_A *\phi = 0.781701$		
Finite element approximations	Original triangulation ($h=0.213758$)		
	Upper bound	$\lVert\phi'_h\rVert^2 + \varepsilon(\phi'_h)$ $= 0.782184 + 0.429347 \times 10^{-3}$ $= 0.782613$ (0.000912)	$\lVert\phi'_h - \hat{\phi}'\rVert$ $= 3.76256 \times 10^{-3}$
	Lower bound	$\dfrac{1}{\lVert\varrho'_h\rVert^2 + \varepsilon(\varrho'_h)}$ $= \dfrac{1}{1.280878 + 0.150405 \times 10^{-5}}$ $= 0.780714$ (-0.000987)	$\lVert\varrho'_h - \hat{\varrho}'\rVert$ $= 6.14254 \times 10^{-3}$
	Normal subdivision ($h=0.106879$)		
	Upper bound	$\lVert\phi'_h\rVert^2 + \varepsilon(\phi'_h)$ $= 0.781968 + 0.107413 \times 10^{-3}$ $= 0.782075$ (0.000374)	$\lVert\phi'_h - \hat{\phi}'\rVert$ $= 1.12050 \times 10^{-3}$
	Lower bound	$\dfrac{1}{\lVert\varrho'_h\rVert^2 + \varepsilon(\varrho'_h)}$ $= \dfrac{1}{1.279506 + 0.381486 \times 10^{-6}}$ $= 0.781551$ (-0.000150)	$\lVert\varrho'_h - \hat{\varrho}'\rVert$ $= 1.83821 \times 10^{-3}$

( ): Deviation from exact value.



$K_3^*(G_3^*)$
$K_3(G_3)$
$i\sqrt{3}$
$w=\varphi_3(z)=b \log z$
$K_1(G_1)$
0
1
$\sqrt{3}$
$K_2(G_2)$
$\zeta=\varphi_2(z)=a\sqrt{z-1}$
$K_2^*(G_2^*)$

Fig. 9

differentials in $\Lambda'_\phi(L')$ and $\Lambda'_\varrho(L')$ respectively which minimize norms $\|\sigma'_h\|_{L'}$ in $\Lambda'_\phi(L')$ and $\Lambda'_\varrho(L')$ respectively. Then, by making use of the symmetricity of K', the period and boundary conditions of ϕ'_h, ϱ'_h, $\phi'_{h,L}$ and $\varrho'_{h,L}$, and their minimality w.r.t. norm, we can verify that $\phi'_{h,L}$ and $\varrho'_{h,L}$ are the restrictions of ϕ'_h and ϱ'_h to L' respectively, and $\|\phi'_h\|^2_{K'}=16\|\phi'_{h,L}\|^2_{L'}$ and $\|\varrho'_h\|^2_{K'}=16\|\varrho'_{h,L}\|^2_{L'}$. Consequently, to attain our aim it is sufficient to make numerical calculations of $\phi'_{h,L}$ and $\varrho'_{h,L}$ (cf. Mizumoto and Hara [16], [17] for the calculation method).

We should note that the symmetricity of ϕ and ϱ on Ω has not been used and thus our method does not reject an application to the differentials which do not have symmetricity on Ω.

Table 1 shows the exact value of the periodicity moduli p_1 which can be calculated by making use of a complete elliptic integral, and the values of our finite element approximations. Furthermore, computation results for the normal subdivision K^1 (see Fig. 9) of the present K are shown. It can be said that the both of upper and lower bounds of p_1 are close to the exact value.

4. Numerical example 2 (the case of a compact bordered Riemann surface) Let $\bar{\Omega}$ be a two-sheeted compact bordered covering surface with three branch points $z=-1, 1, 3$ over the ellipse:

$$E=\left\{z=x+iy \,\middle|\, \frac{x^2}{16}+\frac{y}{15}\leq 1\right\}.$$

Then $\bar{\Omega}$ is a compact bordered Riemann surface of genus one with one boundary component C. A canonical homology basis $\{A, B\}$ of $\bar{\Omega}$ is chosen as in Fig. 10. We aim to obtain good upper and lower approximate values of the

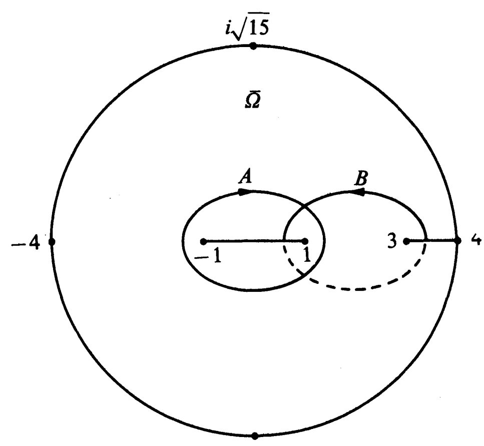
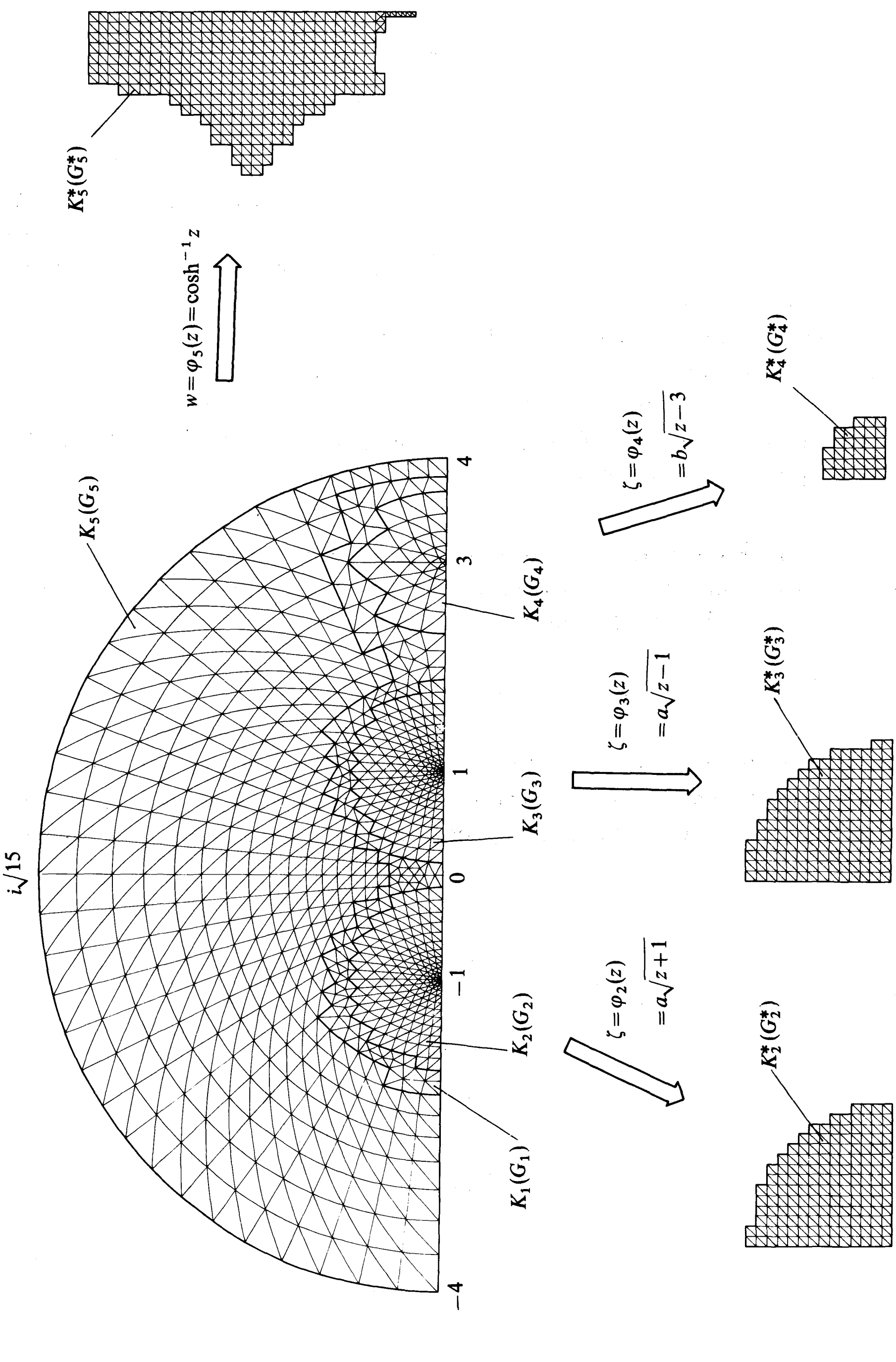


Fig. 10



$K_5^*(G_5^*)$
$w=\varphi_5(z)=\cosh^{-1}z$
$K_5(G_5)$
$i\sqrt{15}$
$K_4(G_4)$
$K_3(G_3)$
$K_2(G_2)$
$K_1(G_1)$
-4
-1
0
1
3
4
$\zeta=\varphi_4(z)$
$=b\sqrt{z-3}$
$\zeta=\varphi_3(z)$
$=a\sqrt{z-1}$
$\zeta=\varphi_2(z)$
$=a\sqrt{z+1}$
$K_4^*(G_4^*)$
$K_3^*(G_3^*)$
$K_2^*(G_2^*)$

Fig. 11

periodicity moduli p_1 and p_2 of $\bar{\Omega}$ with respect to A and B respectively.

First, we construct a triangulation of the upper half ellipse $\bar{D}=E\cap\{z|\operatorname{Im}z\geqq 0\}$ as in Fig. 11. The closed regions G_2, G_3, G_4 and G_5 are mapped onto the regions G_2^*, G_3^*, G_4^* and G_5^* resp. by the local parameters $\zeta=\varphi_2(z)=a\sqrt{z+1}$, $\zeta=\varphi_3(z)=a\sqrt{z-1}$, $\zeta=\varphi_4(z)=b\sqrt{z-3}$ and $w=\varphi_5(z)=\cosh^{-1}z$ ($a=2/5^{1/4}$ and $b=2/85^{1/4}$) respectively, where a and b are so determined that $|d\zeta/dz|$ are equal to $|dw/dz|$ at $z=z_0+i$ ($z_0=-1$, 1 or 3). We construct ordinary triangulations K_2^*, K_3^*, K_4^* and K_5^* of G_2^*, G_3^*, G_4^* and G_5^* as in Fig. 11 respectively. By K_2, K_3, K_4 and K_5 we denote the image triangulations of K_2^*, K_3^*, K_4^* and K_5^* by the mappings φ_2^{-1}, φ_3^{-1}, φ_4^{-1} and φ_5^{-1} respectively. The triangulation K_1 of the region $G_1=\overline{\Omega-(G_2\cup G_3\cup G_4\cup G_5)}$ in Fig. 11 is so constructed that each 2-simplex s of K_1 is natural, minor or major according as $|s|\cap|K_2+K_3+K_4+K_5|=\varnothing$, $|s|\cap|K_2+K_3+K_4|\neq\varnothing$, or $|s|\cap|K_5|\neq\varnothing$, with the convention as in the previous section, and the local parameter of K_1 is $\varphi_1(z)\equiv z$.

A triangulation L_1 of the lower half ellipse $\bar{D}_1=E\cap\{z|\operatorname{Im}z\leqq 0\}$ is defined by the reflection of the triangulation $L\equiv K_1+K_2+K_3+K_4+K_5$ with respect to the real axis and a triangulation L_2 of E is defined by $L_2=L+L_1$. Then, a triangulation K of the covering surface $\bar{\Omega}$ is so constructed that the projection T of K onto the z-plane is the triangulation L_2. We see that the triangulation K conforms to the definition in §1.2. We denote the parts of $T^{-1}(\bar{D})$ and $T^{-1}(L)$ on the upper sheet of $\bar{\Omega}$ by $\bar{D}$ and L again respectively.

Let ϕ, ϱ, χ and τ be the differentials on the present $\bar{\Omega}$ defined in §4.1, and let ϕ_h', ϱ_h', χ_h' and τ_h' be the finite element approximations of ϕ, ϱ, χ and τ respectively in the space $\Lambda'(K')$, where K' is the naturalized triangulation associated to the present K.

Let $\Lambda(L)$ be the space of differentials on $\bar{D}$ which are the restrictions of those in $\Lambda(K)$ to $\bar{D}$. Let $\Lambda_\phi(L)$, $\Lambda_\varrho(L)$, $\Lambda_\chi(L)$ and $\Lambda_\tau(L)$ be the subspaces of $\Lambda(L)$ which consist of the differentials σ_{h1}, σ_{h2}, σ_{h3} and σ_{h4} in $\Lambda(L)$ respectively satisfying the conditions:

$$\sigma_{h1}=\sigma_{h3}=0 \quad \text{along} \quad c_0=\{z|\ 3\leqq \operatorname{Re}z\leqq 4,\ \operatorname{Im}z=0\},$$

$$\sigma_{h1}=\sigma_{h3}=0 \quad \text{along} \quad c_1=\{z|\ -1\leqq \operatorname{Re}z\leqq 1,\ \operatorname{Im}z=0\},$$

$$\sigma_{h2}=\sigma_{h4}=0 \quad \text{along} \quad c_0^*=\{z|\ 1\leqq \operatorname{Re}z\leqq 3,\ \operatorname{Im}z=0\},$$

$$\sigma_{h2}=\sigma_{h4}=0 \quad \text{along} \quad c_1^*=\{z|\ -4\leqq \operatorname{Re}z\leqq -1,\ \operatorname{Im}z=0\},$$

$$\sigma_{h1}=\sigma_{h2}=0 \quad \text{along} \quad c=\{z=x+iy|\ x^2/16+y^2/15=1,\ y\geqq 0\},$$

$$\int_{B\cap\bar{D}}\sigma_{h1}=\int_{B\cap\bar{D}}\sigma_{h3}=\frac{1}{2}$$

and

$$\int_{A\cap\bar{D}} \sigma_{h2} = \int_{A\cap\bar{D}} \sigma_{h4} = -\frac{1}{2}.$$

Further, let $\Lambda'_\phi(L')=\{\sigma'_{h1}\}$, $\Lambda'_\varrho(L')=\{\sigma'_{h2}\}$, $\Lambda'_\chi(L')=\{\sigma'_{h3}\}$ and $\Lambda'_\tau(L')=\{\sigma'_{h4}\}$, where $\sigma'_{hj}=F(\sigma_{hj})$ $(j=1, 2, 3, 4)$. By $\phi'_{h,L}$, $\varrho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$ we denote the differentials of $\Lambda'_\phi(L')$, $\Lambda'_\varrho(L')$, $\Lambda'_\chi(L')$ and $\Lambda'_\tau(L')$ respectively which minimize norms in $\Lambda'_\phi(L')$, $\Lambda'_\varrho(L')$, $\Lambda'_\chi(L')$ and $\Lambda'_\tau(L')$ respectively. Then, by making use of the symmetricity of K', the period and boundary conditions of ϕ'_h, ϱ'_h, χ'_h, τ'_h, $\phi'_{h,L}$, $\varrho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$, and their minimality w.r.t. norm, we can verify that $\phi'_{h,L}$, $\varrho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$ are the restrictions of ϕ'_h, ϱ'_h, χ'_h and τ'_h to L' respectively, and $\|\phi'_h\|^2_{K'}=4\|\phi'_{h,L}\|^2_{L'}$, $\|\varrho'_h\|^2_{K'}=4\|\varrho'_{h,L}\|^2_{L'}$, $\|\chi'_h\|^2_{K'}=4\|\chi'_{h,L}\|^2_{L'}$ and $\|\tau'_h\|^2_{K'}=4\|\tau'_{h,L}\|^2_{L'}$. Consequently, to attain our aim it is sufficient to make numerical calculations of $\phi'_{h,L}$, $\varrho'_{h,L}$, $\chi'_{h,L}$ and $\tau'_{h,L}$.

The exact values of the periodicity moduli p_1 and p_2 can be calculated by the following procedure.

Let C_0 and C_1 be the boundary parts of the upper half ellipse domain D defined by

$$C_0=\{z|\ 3\leqq \operatorname{Re} z\leqq 4,\ \operatorname{Im} z=0\}\cup\left\{z=x+iy\ \middle|\ \frac{x^2}{16}+\frac{y^2}{15}=1,\ y\geqq 0\right\}$$

and

$$C_1=\{z|\ -1\leqq \operatorname{Re} z\leqq 1,\ \operatorname{Im} z=0\}.$$

Let Δ be the rectangular domain

$$\Delta=\{W|\ 0<\operatorname{Re} W<1,\ 0<\operatorname{Im} W<\tau\},$$

and let Γ_0 and Γ_1 be the boundary parts of Δ defined by

$$\Gamma_0=\{W|\ 0\leqq \operatorname{Im} W\leqq\tau,\ \operatorname{Re} W=0\}$$

and

$$\Gamma_1=\{W|\ 0\leqq \operatorname{Im} W\leqq\tau,\ \operatorname{Re} W=1\}.$$

If D is conformally mapped onto Δ so that C_0 and C_1 are mapped onto Γ_0 and Γ_1 respectively, then the periodicity moduli p_1 is equal to τ. The conformal map $W=f(z)$: $D\to\Delta$ is constructed by the composition of the following mappings:

(i) $w=\dfrac{2}{\cosh^{-1}4}\cdot\cosh^{-1}z-1;$

(ii) $\zeta=\operatorname{sn}(K(k)\cdot w)$, where $\dfrac{K'(k)}{K(k)}=\dfrac{2\pi}{\cosh^{-1}4};$

(iii) $\dfrac{Z-Z_1}{Z-Z_2}\cdot\dfrac{Z_3-Z_2}{Z_3-Z_1}=\dfrac{\zeta-\zeta_1}{\zeta-\zeta_2}\cdot\dfrac{\zeta_3-\zeta_2}{\zeta_3-\zeta_1}$,

where $\zeta_j=\mathrm{sn}\,(K(k)\cdot w_j)$ $(j=1, 2, 3, 4)$ with $w_1=-1+i\,(2\pi/\cosh^{-1}4)$, $w_2=-1$, $w_3=2\cosh^{-1}3/\cosh^{-1}4-1$, $w_4=1+i\,(2\pi/\cosh^{-1}4)$, and $Z_1=-1/\kappa$, $Z_2=-1$, $Z_3=1$, $Z_4=1/\kappa$ with $\kappa=(\sqrt{1/c}-\sqrt{1/c-1})^2$, $c=((\zeta_4-\zeta_1)/(\zeta_4-\zeta_2))\cdot((\zeta_3-\zeta_2)/(\zeta_3-\zeta_1))$;

(iv) $W=-\dfrac{1}{2}\left(\dfrac{1}{K(\kappa)}\displaystyle\int_0^Z\dfrac{dZ}{\sqrt{(1-Z^2)(1-\kappa^2Z^2)}}-\left(1+i\dfrac{K'(\kappa)}{K(\kappa)}\right)\right)$.

Then we see that

$$p_1=\tau=\frac{K'(\kappa)}{2K(\kappa)}.$$

Next, let C_0' and C_1' be the boundary parts of D given by

$$C_0'=\{z\mid 1\leqq \mathrm{Re}\ z\leqq 3,\ \mathrm{Im}\ z=0\}$$

and

Table 2. Periodicity moduli p_1 of compact bordered Riemann surface

Exact value	$p_1=\int_A *\phi=1.539330$		
Finite element approximations	Original triangulation ($h=0.138840$)		
	Upper bound	$\\|\phi_h'\\|^2+\varepsilon(\phi_h')$ $=1.540588+0.572262\times10^{-4}$ $=1.540645$ (0.00132)	$\\|\phi_h'-\hat{\phi}'\\|$ $=1.15335\times10^{-2}$
	Lower bound	$\dfrac{1}{\\|\tau_h'\\|^2+\varepsilon(\tau_h')}$ $=\dfrac{1}{0.649700+0.225117\times10^{-3}}$ $=1.538639$ (-0.00069)	$\\|\tau_h'-\hat{\tau}'\\|$ $=3.74131\times10^{-3}$
	Normal subdivision ($h=0.069420$)		
	Upper bound	$\\|\phi_h'\\|^2+\varepsilon(\phi_h')$ $=1.539652+0.142916\times10^{-4}$ $=1.539666$ (0.00034)	$\\|\phi_h'-\hat{\phi}'\\|$ $=5.89447\times10^{-3}$
	Lower bound	$\dfrac{1}{\\|\tau_h'\\|^2+\varepsilon(\tau_h')}$ $=\dfrac{1}{0.649652+0.558093\times10^{-4}}$ $=1.539153$ (-0.00018)	$\\|\tau_h'-\hat{\tau}'\\|$ $=1.09209\times10^{-3}$

( ): Deviation from exact value.

Table 3. Periodicity moduli p_2 of compact bordered Riemann surface

Exact value	$p_2=\int_B *\varrho=1.839350$		
Finite element approximations	Original triangulation ($h=0.138840$)		
	Upper bound	$\|\varrho'_h\|^2+\varepsilon(\varrho'_h)$ $=1.841976+0.351532\times10^{-3}$ $=1.842328$ (0.00298)	$\|\varrho'_h-\hat{\varrho}'\|$ $=7.65797\times10^{-3}$
	Lower bound	$\frac{1}{\|\chi'_h\|^2+\varepsilon(\chi'_h)}$ $=\frac{1}{0.544588+0.145580\times10^{-3}}$ $=1.835760$ (-0.00359)	$\|\chi'_h-\hat{\chi}'\|$ $=5.22574\times10^{-3}$
	Normal subdivision ($h=0.069420$)		
	Upper bound	$\|\varrho'_h\|^2+\varepsilon(\varrho'_h)$ $=1.840016+0.875764\times10^{-4}$ $=1.840104$ (0.00075)	$\|\varrho'_h-\hat{\varrho}'\|$ $=2.28613\times10^{-3}$
	Lower bound	$\frac{1}{\|\chi'_h\|^2+\varepsilon(\chi'_h)}$ $=\frac{1}{0.543904+0.361871\times10^{-4}}$ $=1.838437$ (-0.00091)	$\|\chi'_h-\hat{\chi}'\|$ $=1.73332\times10^{-3}$

( ): Deviation from exact value.

$$C'_1=\left\{z=x+iy\,\middle|\,\frac{x^2}{16}+\frac{y^2}{15}=1,\ y\geqq 0\right\}\cup\{z\mid -4\leqq \mathrm{Re}\ z\leqq -1,\ \mathrm{Im}\ z=0\}.$$

Let Δ, Γ_0 and Γ_1 be as above. If the domain D is conformally mapped onto the domain Δ so that C'_0 and C'_1 are mapped onto Γ_0 and Γ_1 respectively, then the periodicity moduli p_2 is equal to τ. The conformal map $W=f(p)$: $D\to\Delta$ is constructed similarly to the case of periodicity moduli p_1.

Tables 2 and 3 show the exact values of the periodicity moduli p_1 and p_2, and the values of our finite element approximations. Furthermore, computation results for the normal subdivision K^1 of the present K are shown. It can be said that the both of upper and lower bounds of p_1 and p_2 are close to the exact values.

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