HIROSHIMA MATH. J.
19 (1989), 203–208

On the oscillatory properties of the solutions of non-linear neutral functional differential equations of second order

D. D. BAINOV[1], A. D. MYSHKIS[2] and A. I. ZAHARIEV[1]

(Received December 14, 1987)

1. Introduction

In the present paper sufficient conditions have been obtained for oscillation or tending to zero of all bounded solutions of equations of the form

$$[A(x_t)]'' + p(t)B(x_t) = 0\,, \tag{1}$$

where $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau = \text{const} > 0$ and the functionals A, B: $C[-\tau, 0) \to \boldsymbol{R}$ are monotonic.

The oscillatory properties of linear and non-linear ordinary differential and functional differential equations have been an object of investigation by many authors [2]–[5], [8], [10]. The neutral equations of second order have numerous applications (see for instance [1], [6]) but their oscillatory and asymptotic properties are studied comparatively little. Some results in this direction for the case when the function $p(t)$ is nonnegative have been obtained in [9], [11], [12].

2. Preliminary notes and main result

DEFINITION 1. We shall say that the function $\varphi: J_\varphi \to \boldsymbol{R}$ ($J_\varphi = [t_\varphi, \infty)$, $t_\varphi \in \boldsymbol{R}$)) is oscillating if $\sup\{t|\varphi(t) = 0\} = \infty$ and $\sup\{t|\varphi(t) \neq 0\} = \infty$.

DEFINITION 2. A function $x: J_x \to \boldsymbol{R}$ will be called a solution of equation (1) if $x \in C(J_x)$, $A(x_t) \in C^2(J_x + \tau)$ and satisfies equation (1) for $t \in J_x + \tau$, where $J_x + \tau = \{t | t - \tau \in J_x\}$.

By $\Omega^{\alpha,\beta}$ $(0 < \beta \leqq \alpha)$ we shall denote the set of all continuous functionals $A: C[-\tau, 0] \to \boldsymbol{R}$ which satisfy the following conditions:

A1. For any function $\varphi \in C[-\tau, 0]$ with the property $\varphi(t) \neq 0$, $t \in [-\tau, 0]$, the following equality holds

$$\operatorname{sgn} A(\varphi) = \operatorname{sgn} \varphi(0)\,.$$

A2. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any function $\varphi \in C[-\tau, \tau]$ with the property $\min_{[-\tau,\tau]}|\varphi(t)| > 0$ the inequality $\max_{[0,\tau]}|A(\varphi_t)| < \delta$ implies the inequality $|\varphi(0)| < \varepsilon$.

A3. For all constants b_1, b_2, $0 < b_1 \leqq b_2$, and any function $\varphi \in C[-\tau, \alpha]$ with the property $\min_{[-\tau,\alpha]}|\varphi(t)| > 0$ for which the inequality $b_1 \leqq |A(\varphi_t)| \leqq b_2$, $t \in [-\tau, \alpha]$, holds, there exists a measurable set $Q \subset [-\tau, \alpha]$ and a constant $b_3 > 0$ such that $\mu(Q) \geqq \beta$ (μ is the Lebesgue measure), $|\varphi(t)| \geqq b_3$ for $t \in Q$ and the following equality holds

$$\operatorname{sgn} \varphi(t)|_Q = \operatorname{sgn} A(\varphi_t)|_{[0,\alpha]} \,.$$

EXAMPLE. It is immediately verified that for any α and correspondingly chosen β the functional A defined by the equality

$$A(\varphi) = \sum_{i=1}^{n} a_i \varphi(-\tau_i)\,,$$

$n \geqq 1$, $a_i > 0$, $0 \leqq \tau_i \leqq \tau$, $i = 1, 2, \ldots, n$, belongs to the set $\Omega^{\alpha,\beta}$.

For the function $p : J_p \to \boldsymbol{R}$ we introduce the notation

$$E_p^+ = \{t \in J_p | p(t) \geqq 0\}\,, \qquad E_p^- = \{t \in J_p | p(t) \leqq 0\}\,.$$

By P^γ, $\gamma > 0$, we shall denote the set of continuous functions $p : J_p \to \boldsymbol{R}$ satisfying the following property:

P1. There exists a number $\varepsilon > 0$ and a point $t_0 \in J_p$ such that for any $t \geqq t_0$ for which $p(t) > 0$ one can find an interval $[t', t''] \subset J_p$ with length $t'' - t' \geqq \gamma + \varepsilon$ with the property $t \in [t', t''] \subset E_p^+$ (i.e. the intervals in which the function is positive should be large enough).

By Λ we shall denote the set of continuous functionals $B : C[-\tau, 0] \to \boldsymbol{R}$ satisfying the following properties:

B1. For any element $\varphi \in C[-\tau, 0]$ with the property $\min_{[-\tau,0]}|\varphi(t)| > 0$ the following equality holds

$$\operatorname{sgn} B(\varphi) = \operatorname{sgn} \varphi(0)\,.$$

B2. For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any element $\varphi \in C[-\tau, 0]$ with the property $\min_{[-\tau,0]}|\varphi(t)| > 0$ for which the inequality $|\varphi(0)| \geqq \varepsilon$ holds, the inequality $B(\varphi) \geqq \delta$ holds as well.

B3. $B(s1(\cdot))$ is a non-decreasing function for $s \in \boldsymbol{R}$, where $1(\cdot)$ denotes the unit function $1(t) \equiv 1$, $t \in [-\tau, 0]$, and the following relation holds

$$\int_0^1 \left[\frac{1}{B(s1(\cdot))} + \frac{1}{|B(s1(\cdot))|}\right] ds < \infty\,.$$

REMARK 1. We shall note that from condition B3 it follows that no functional $B \in \Lambda$ can be linear.

LEMMA. *Let the function* $h:[a,b]\to[0,\infty)$ *be absolutely continuous,* $\varphi\in C^2[a,b]$ *and let the function* $f\in C[\min\varphi,\max\varphi]$ *be nonincreasing.*

Then the following inequality holds

$$\int_a^b h(t)\varphi''(t)f(\varphi(t))\,dt \geqq h(b)\varphi'(b)f(\varphi(b)) - h(a)\varphi'(a)f(\varphi(a)) - \int_a^b h'(t)\varphi'(t)f(\varphi(t))\,dt\,.$$

PROOF. If f is of class C^1, then the assertion of the lemma is proved by an integration by parts and in the case when f is of class C – by means of a uniform approximation of f by non-increasing functions of class C^1.

THEOREM. *Let for equation* (1) *numbers* α, β $(0<\beta\leqq\alpha)$ *exist such that the following conditions be fulfilled*:

1. $A\in\Omega^{\alpha,\beta}$.
2. $p\in P^{\alpha+\tau}$.
3. $B\in\Lambda$.
4. *For any constant* $a>0$ *the following relation holds*

$$\sup\frac{B(\varphi)}{B(A(\varphi)1(\cdot))}<\infty \qquad for \quad \varphi\in C[-\tau,\tau] \quad with \quad 0<|\varphi(t)|\leqq a\,.$$

5. *There exists a locally absolutely continuous function* $h:J_p\to(0,\infty)$ *with the properties* $\mathrm{Var}_{[t_p,t]}h=0(t)$ *for* $t\to\infty$, $\mathrm{Var}_{[t_p,\infty)}h'<\infty$, *for which the following relation holds*

$$\int_{E_p^-} h(t)|p(t)|\,dt<\infty\,. \tag{2}$$

6. *There exists a number* $\varepsilon>0$ *for which the following inequality is satisfied*

$$\limsup_{t\to\infty}\mu\{s\in[t,t+\alpha+\tau]\,|\,h(s)p(s)\leqq\varepsilon\}<\beta\,.$$

Then each bounded solution of equation (1) *either oscillates or tends to zero for* $t\to\infty$.

PROOF. Let $x:J_x\to\boldsymbol{R}$ be a bounded solution of equation (1) which is not identically equal to zero for sufficiently large values of t.

Without loss of generality we can assume that $x(t)>0$ for $t\in J_x$.

Multiplying both sides of equation (1) by the expression $h(t)/B(A(x_t)1(\cdot))$ and integrating from $t_1=t_x+\tau$ to $t>t_1$ we obtain the equality

$$\int_{t_1}^t\frac{[A(x_s)]''h(s)}{B(A(x_s)1(\cdot))}\,ds+\int_{t_1}^t h(s)p(s)\frac{B(x_s)}{B(A(x_s)1(\cdot))}\,ds=0\,.$$

Applying to the first integral the lemma and integrating once more from t_1 to $t > t_1$, we obtain the inequality

$$\text{(3)}\quad \int_{t_1}^{t} \frac{h(s)[A(x_s)]'}{B(A(x_s)1(\cdot))}\,ds - \frac{h(t_1)[A(x_t)]'|_{t=t_1}}{B(A(x_{t_1})1(\cdot))}(t-t_1)$$
$$- \int_{t_1}^{t}\left(\int_{t_1}^{s} \frac{h'(y)[A(x_y)]'}{B(A(x_y)1(\cdot))}\,dy\right)ds + \int_{t_1}^{t}\left(\int_{t_1}^{s} h(y)p(y)\frac{B(x_y)}{B(A(x_y)1(\cdot))}\,dy\right)ds \leqq 0\,.$$

Taking into account the properties of the function $h(t)$ and setting $\phi(t) = \int_0^t \frac{ds}{B(s1(\cdot))}$ we obtain for $t \to \infty$ the following relations

$$\text{(4)}\quad \begin{aligned} &\int_{t_1}^{t} \frac{h(s)[A(x_s)]'}{B(A(x_s)1(\cdot))}\,ds = \int_{t_1}^{t} h(s)\,d\phi(A(x_s)) = h(t)\phi(A(x_t)) - h(t_1)\phi(A(x_{t_1})) \\ &- \int_{t_1}^{t} \phi(A(x_s))\,dh(s) = O(t)\,, \\ &\int_{t_1}^{t} \frac{h'(s)[A(x_s)]'}{B(A(x_s)1(\cdot))}\,ds = \int_{t_1}^{t} h'(s)\,d\phi(A(x_s)) = h'(t)\phi(A(x_t)) - h'(t_1)\phi(A(x_{t_1})) \\ &- \int_{t_1}^{t} \phi(A(x_s))\,dh'(s) = O(1)\,. \end{aligned}$$

From inequality (3), in view of relations (2), (4) and condition 4 of the theorem, we obtain for $t \to \infty$ the relation

$$\text{(5)}\qquad \int_{t_1}^{t}\left(\int_{[t_1,s]\cap E_p^+} h(y)p(y)\frac{B(x_y)}{B(A(x_y)1(\cdot))}\,dy\right)ds = O(t)\,.$$

We shall prove that the following relation holds

$$\text{(6)}\qquad \int_{[t_1,\infty)\cap E_p^+} h(t)p(t)\frac{B(x_t)}{B(A(x_t)1(\cdot))}\,dt = \infty\,,$$

which obviously contradicts relation (5).

From condition A2 it follows that $\limsup_{t\to\infty} A(x_t) > 0$, so let us set $c = \limsup_{t\to\infty} A(x_t)$. On the other hand, from equation (1) it follows that the function $A(x_\cdot)$ is concave (convex) in any interval belonging to $\{J_x + \tau\} \cap E_p^+(\{J_x + \tau\} \cap E_p^-)$. In view of condition 6 of the theorem we conclude that $\sup E_p^+ = \infty$, hence there exists a sequence $\{t_i\} \subset E_p^+$ with the property $\lim_{i\to\infty}(t_{i+1} - t_i) = \infty$ such that $\lim_{i\to\infty} A(x_{t_i}) = c$. From condition P1 it follows that there exists a sequence of disjoint intervals $\{l_i\}$, $t_i \in l_i$, with length $\alpha + \tau$ such that the inequality $\inf_i \min_{l_i} A(x_t) > 0$ holds.

Then by condition A3 there exist measurable sets $Q_i \subset l_i$ with the property $\mu(Q_i) \geqq \beta$, $i = 1, 2, \ldots$, such that the inequality $\inf_i \min_{t \in Q_i} x(t) > 0$ holds. From the last inequality and condition B2 it follows that $\inf_i \inf_{t \in Q_i} B(x_t) > 0$, hence the following inequality holds

$$\inf_i \inf_{t \in Q_i} \frac{B(x_t)}{B(A(x_t)1(\cdot))} > 0 . \tag{7}$$

From condition 6 of the theorem it follows that there exist sets $Q_i' \subset Q_i$ for which $\liminf_{i \to \infty} \mu(Q_i') > 0$ and the inequality

$$\liminf_{i \to \infty} \left[\inf_{t \in Q_i'} h(t)p(t) \right] > 0 \tag{8}$$

holds. Inequalities (7) and (8) immediately imply relation (6).

REMARK 2. If, moreover, it is given that the function $p(t) \geqq 0$, then each bounded solution which for sufficiently large values of t is not identically zero oscillates. In this case, if $x(t) \geqq 0$ for $t \geqq t_x$, then the function $A(x_.)$ for $t \geqq t_x$ is concave, hence $x(t)$ may tend to zero for $t \to \infty$ only if it is identically zero for $t > t_x$.

References

[1] R. Brayton, Nonlinear oscillation in a distributed network, Quart. Appl. Math. **24** (1967), 289–301.

[2] G. J. Butler, Integral averages and the oscillation of second order ordinary differential equations, SIAM J. Math. Anal. **11** (1980), 190–200.

[3] W. J. Coles, An oscillation criterion for second order differential equations, Proc. Amer. Math. Soc. **19** (1968), 755–759.

[4] P. Hartman, On nonoscillatory linear differential equations of second order, Amer. J. Math. **74** (1952), 389–400.

[5] I. V. Kamenev, Some specific nonlinear oscillation theorems, Mat. Zametki **10** (1971), 129–134 (in Russian).

[6] V. B. Kolmanovskii and V. R. Nosov, Stability and Periodic Régimes of Controlled Systems with Aftereffect, Moscow, 1981 (in Russian).

[7] T. Kusano and H. Onose, Nonlinear oscillation of second order functional differential equations with advanced argument, J. Math. Soc. Japan **29** (1977), 541–559.

[8] A. D. Myshkis, Linear Differential Equations with Delayed Argument, Moscow, 1972 (in Russian).

[9] A. D. Myshkis, D. D. Bainov and A. I. Zahariev, Oscillatory and asymptotic properties of a class of operator-differential inequalities, Proc. Royal Soc. Edinburgh **96A** (1984), 5–13.

[10] A. Wintner, A criterion of oscillatory stability, Quart. Appl. Math. **7** (1949), 115–117.

[11] A. I. Zahariev and D. D. Bainov, Oscillating properties of a class of neutral type functional differential equations, Bull. Austral. Math. Soc. **22** (1980), 365–372.

[12] A. I. Zahariev and D. D. Bainov, On some oscillation criteria for a class of neutral type functional differential equations, J. Austral. Math. Soc. Ser. B **28** (1986), 229–239.

1) *Plovdiv University*
"Paissii Hilendrski"
Bulgaria
and
2) *Moscow Institute of*
Railway Engineering
USSR