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Homotopy coalgebras and k-fold suspensions

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ABSTRACT. We consider a weak and ordinary FG-coalgebra structure of order m on an object in a category relative to a pair of adjoint functors F and G and present some of its properties. We then specialize to the case when $F = \Sigma^k$, the k-fold suspension functor, and $G = \Omega^k$, the k-fold loop-space functor, and obtain weak and ordinary k-fold homotopy coalgebras of order m. We prove that any $(n-1)$-connected weak k-fold homotopy coalgebra of order m and of dimension $\leq (m+2)n - (m+1)k - m$ is equivalent to a k-fold suspension for any $k \geq 1$ and $n \geq 2$. We derive some consequences of this result.

Let X be a finite CW-complex which is $(n-1)$-connected, $n \geq 2$. In 1963 Berstein and Hilton proved that if X is a co-H-space of dimension $\leq 3n-3$, then X is equivalent to a suspension [2]. In 1970 Ganea proved that if X is a homotopy-associative co-H-space of dimension $\leq 4n-5$, then X is equivalent to a suspension [4]. Thus it would appear that an upper bound on the dimension of X (which is linear in n) together with restrictions on a homotopy-associative comultiplication of X would imply that X is equivalent to a suspension. The model for such results deals with the dual concept of an H-space. In this case, Stasheff's A_m-theory of H-spaces provides the necessary restriction on a multiplication for each m. For co-H-spaces, the next step was given by Saito who proved that a certain condition on a homotopy-associative comultiplication of X together with the dimensional restriction $\dim X \leq 5n-7$ implies that X is equivalent to a suspension [6]. However, the details of the argument are formidable and it is not clear what the next step should be. For a discussion of these matters, see [1, § 5].

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This paper grew out of our attempt to understand, simplify and generalize parts of Saito's work. In addition, some of our results are implicit in Ganea's paper [4]. Our approach differs from earlier work in two ways. First of all, we consider k-fold suspensions for any $k \geq 1$ and provide conditions for X to be equivalent to a k-fold suspension. Secondly, our restrictive condition on the comultiplication is given inductively in terms of the existence of a sequence of maps γ_i, $i \leq m$. For $k = 1$, $\gamma_1 : X \to \Sigma\Omega X$ is just the coretraction of X which is equivalent to the comultiplication of X. For arbitrary k, this condition is described as a weak k-fold homotopy coalgebra of order m. It would perhaps be preferable to give the appropriate condition directly in terms of the comultiplication of X rather than in terms of the existence of a sequence of maps γ_i which define a weak homotopy coalgebra. However, using the notion of a weak homotopy coalgebra, we are able to prove that any $(n-1)$-connected weak k-fold homotopy coalgebra of order m and of dimension $\leq (m+2)n - (m+1)k - m$ is equivalent to a k-fold suspension, for any $k \geq 1$ and any $n \geq 2$ (Theorem 2.4).

We next briefly summarize the contents of the paper. In Section 1 we define a weak and ordinary FG-coalgebra structure of order m on an object in a category relative to a pair of adjoint functors F and G. For $m = 1$ these notions have previously appeared in the literature. We show that a weak FG-coalgebra of order $m+1$ is an FG-coalgebra of order m. We also present a large class of examples of FG-coalgebras of order m, for all m. In Section 2 we specialize the results of Section 1 to the case $F = \Sigma^k$, the k-fold suspension functor, and $G = \Omega^k$, the k-fold loop-space functor. Then weak and ordinary FG-coalgebras of order m are just weak and ordinary k-fold homotopy coalgebras of order m. Using the results of Section 1, we prove our main result (Theorem 2.4) which is described in the previous paragraph. We then derive several consequences of this theorem.

1. Coalgebras of order m in a category

Let A be a category. We say that A has *weak pull-backs* if the following condition holds: for every pair of morphisms $f : X \to A$ and $g : Y \to A$, there is an object W and morphisms $\pi_1 : W \to X$ and $\pi_2 : W \to Y$ such that $f\pi_1 = g\pi_2$. Furthermore, if $a : Z \to X$ and $b : Z \to Y$ are any morphisms such that $fa = gb$, then there exists a morphism $h : Z \to W$ such that $\pi_1 h = a$ and $\pi_2 h = b$:

$$\begin{tikzcd} Z \arrow[dr, dashed, "h"] \arrow[drr, "b"] \arrow[ddr, "a"'] & & \\ & W \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & A. \end{tikzcd}$$

This notion differs from a pull-back in A in that the morphism h is not assumed to be unique.

Throughout this section, $\boldsymbol{A}$ and $\boldsymbol{B}$ are categories with weak pull-backs and $F:\boldsymbol{A}\to\boldsymbol{B}$ and $G:\boldsymbol{B}\to\boldsymbol{A}$ are a pair of adjoint functors. Thus the sets of morphisms $\boldsymbol{A}(A, GX)$ and $\boldsymbol{B}(FA, X)$ are naturally isomorphic. Let the natural transformations $\eta:\mathrm{id}_{\boldsymbol{A}}\to GF$ and $\varepsilon: FG\to\mathrm{id}_{\boldsymbol{B}}$ be the unit and counit of the adjunction, respectively. The following two compositions are then the identity morphisms:

$$FA \xrightarrow{F\eta_A} FGFA \xrightarrow{\varepsilon_{FA}} FA$$

and

$$GX \xrightarrow{\eta_{GX}} GFGX \xrightarrow{G\varepsilon_X} GX$$

for $A\in|\boldsymbol{A}|$ and $X\in|\boldsymbol{B}|$.

DEFINITION 1.1. (i) A *weak FG-coalgebra of order* 1 in $\boldsymbol{B}$ is a pair (X, γ_X) consisting of an object X in $\boldsymbol{B}$ and a morphism $\gamma_X: X\to FGX$ in $\boldsymbol{B}$ such that $\varepsilon_X\gamma_X = \mathrm{id}_X$:

$$\begin{tikzcd} X \arrow[r, "\gamma_X"] \arrow[dr, equal] & FGX \arrow[d, "\varepsilon_X"] \\ & X. \end{tikzcd}$$

(ii) an (ordinary) *FG-coalgebra of order* 1 in $\boldsymbol{B}$ is a weak FG-coalgebra (X, γ_X) of order 1 such that the following diagram commutes (cf. [5, p. 136]):

$$\begin{tikzcd} X \arrow[r, "\gamma_X"] \arrow[d, "\gamma_X"'] & FGX \arrow[d, "F\eta_{GX}"] \\ FGX \arrow[r, "FG\gamma_X"] & FGFGX. \end{tikzcd}$$

We next define both weak and ordinary FG-coalgebras of order m in $\boldsymbol{B}$ by induction on m.

DEFINITION 1.2. (i) *Weak FG-coalgebra structure of order m* on an object X in $\boldsymbol{B}$ consists of a sequence of objects $D_1(X), \ldots, D_m(X)$ in $\boldsymbol{A}$ such that $D_1(X) = GX$ and a sequence of morphisms $\gamma_i : X \to FD_i(X)$ in $\boldsymbol{B}$, $i = i, \ldots, m$, such that $\gamma_1 = \gamma_X : X \to FGX$. We require that the following two conditions hold:

(1) the $D_i(X)$ are inductively defined as weak pull-backs in the diagram

$$\begin{array}{ccc} D_i(X) & \xrightarrow{p_{i-1,2}} & GX \\ \Big\downarrow{\scriptstyle p_{i-1,1}} & & \Big\downarrow{\scriptstyle G\gamma_{i-1}} \\ D_{i-1}(X) & \xrightarrow{\eta_{D_{i-1}(X)}} & GFD_{i-1}(X). \end{array} \tag{1.3}$$

(2) The following composition

$$X \xrightarrow{\gamma_i} FD_i(X) \xrightarrow{\varepsilon_i} X$$

is the identity id_X, where

$$\varepsilon_i = \varepsilon_X F p_{i-1,2} : FD_i(X) \xrightarrow{Fp_{i-1,2}} FGX \xrightarrow{\varepsilon_X} X.$$

Clearly if X together with $\gamma_1, \ldots, \gamma_m$ is a weak FG-coalgebra of order m, then for $n < m$, X together with $\gamma_1, \ldots, \gamma_n$ is a weak FG-coalgebra of order n.

(ii) If X together with $\gamma_1, \ldots, \gamma_m$ is a weak FG-coalgebra of order m, then X is an (ordinary) *FG-coalgebra of order m* if the following diagram commutes for all $i = 1, \ldots, m$:

$$\begin{array}{ccc} X & \xrightarrow{\gamma_i} & FD_i(X) \\ \Big\downarrow{\scriptstyle \gamma_1} & & \Big\downarrow{\scriptstyle F\eta_{D_i(X)}} \\ FGX & \xrightarrow{FG\gamma_i} & FGFD_i(X). \end{array}$$

This condition can be regarded as a higher associative law. For a weak or ordinary FG-coalgebra of order m, the maps $\gamma_1, \ldots, \gamma_m$ are called *structure maps*. We often do not mention all but the last and thus refer to (X, γ_m) as a weak or ordinary FG-coalgebra of order m. Clearly an FG-coalgebra of order m is an FG-algebra of order n for any $n < m$.

Note that if (X, γ_m) is a weak FG-coalgebra of order m, then $D_{m+1}(X)$ and the projections $p_{m,1} : D_{m+1}(X) \to D_m(X)$ and $p_{m,2} : D_{m+1}(X) \to GX$ exist from the weak pull-back diagram (1.3) with $i = m + 1$. Furthermore, we can always define an object $W_m(X)$ in $\boldsymbol{B}$ and a morphism $\Theta_m : FD_{m+1}(X) \to W_m(X)$ via the following weak pull-back diagram:

$$(1.4)\qquad \begin{tikzcd} FD_{m+1}(X) \arrow[dr, dashed, "\theta_m"] \arrow[drr, "Fp_{m,2}"] \arrow[ddr, "Fp_{m,1}"'] & & \\ & W_m(X) \arrow[r, "\pi_{m,2}"] \arrow[d, "\pi_{m,1}"] & FGX \arrow[d, "FG\gamma_m"] \\ & FD_m(X) \arrow[r, "F\eta_m"] & FGFD_m(X), \end{tikzcd}$$

where $\eta_m : D_m(X) \to GFD_m(X)$ is $\eta_{D_m(X)}$.

The following proposition supplies examples of FG-coalgebras of order m, for all m.

PROPOSITION 1.5. *If A is an object in A and $X = FA$, then there exists $\gamma_m : X \to FD_m(X)$ such that (X, γ_m) is an FG-coalgebra of order m, for all m. Furthermore, γ_m can be taken to be $F(\alpha_m)$, for some $\alpha_m : A \to D_m(X)$.*

PROOF. We define the structure maps $\gamma_m : X \to FD_m(X)$ inductively. For $m = 1$, let $\alpha_1 = \eta_A : A \to GFA$. Then set $\gamma_1 = F(\alpha_1) : X = FA \to FGFA = FD_1(X)$. Clearly $\varepsilon_1\gamma_1 = \varepsilon_{FA}(F\eta_A) = \mathrm{id}_X$ and the diagram

$$\begin{tikzcd} X = FA \arrow[r, "F(\eta_A)"] \arrow[d, "F(\eta_A)"] & FD_1(X) = FGFA \arrow[d, "F(\eta_{GFA})"] \\ FD_1(X) = FGFA \arrow[r, "FGF(\eta_A)"] & FGFD_1(X) = FGFGFA \end{tikzcd}$$

is commutative since $\eta_{GFA}\eta_A = (GF\eta_A)\eta_A$. Thus (X, γ_1) is an FG-coalgebra of order 1.

Now suppose that (X, γ_m) is an FG-coalgebra of order m with $\gamma_i = F(\alpha_i)$ for $\alpha_i : A \to D_i(X)$, $i = 1, \ldots, m$. From (1.3) we have the diagram

$$\begin{tikzcd} A \arrow[dr, dashed, "\alpha_{m+1}"] \arrow[drr, "\eta_A"] \arrow[ddr, "\alpha_m"'] & & \\ & D_{m+1}(X) \arrow[r, "p_{m,2}"] \arrow[d, "p_{m,1}"] & GX = GFA \arrow[d, "G(\gamma_m)=GF(\alpha_m)"] \\ & D_m(X) \arrow[r, "\eta_m"] & GFD_m(X) = GFD_m(X). \end{tikzcd}$$

Now $GF(\alpha_m)\eta_A = \eta_m\alpha_m$, and so there exists a morphism $\alpha_{m+1} : A \to D_{m+1}(X)$ such that $p_{m,1}\alpha_{m+1} = \alpha_m$ and $p_{m,2}\alpha_{m+1} = \eta_A$. We set $\gamma_{m+1} = F(\alpha_{m+1}) : X \to$

$FD_{m+1}(X)$. It now easily follows that (X, γ_{m+1}) is an FG-coalgebra of order $m+1$. □

The following result is the main result of this section and is used in the applications in the next section.

THEOREM 1.6. *If (X, γ_{m+1}) is a weak FG-coalgebra of order $m+1$, then (X, γ_m) is an FG-coalgebra of order m. Moreover, if $\Theta_i : FD_{i+1}(X) \to W_i(X)$ is the morphism defined by* (1.4), $i = 1, \ldots, m$ *then*

$$\pi_{i,1}\Theta_i\gamma_{i+1} = \gamma_i$$

and

$$\pi_{i,2}\Theta_i\gamma_{i+1} = \gamma_1.$$

PROOF. For the first assertion we must show that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\gamma_i"] \arrow[d, "\gamma_1"'] & FD_i(X) \arrow[d, "F\eta_i"] \\ FGX \arrow[r, "FG\gamma_i"] & FGFD_i(X) \end{tikzcd}$$

for all $i = 1, \ldots, m$. We set $\gamma_i' = \pi_{i,1}\Theta_i\gamma_{i+1}$ and $\gamma_i'' = \pi_{i,2}\Theta_i\gamma_{i+1}$:

$$\begin{tikzcd} X \arrow[dr, "\gamma_{i+1}"] \arrow[ddrr, "\gamma_i''"] \arrow[dddrr, "\gamma_i'"'] & & & \\ & FD_{i+1}(X) \arrow[dr, "\Theta_i"] & & \\ & & W_i(X) \arrow[r, "\pi_{i,2}"] \arrow[d, "\pi_{i,1}"] & FGX \arrow[d, "FG\gamma_i"] \\ & & FD_i(X) \arrow[r, "F\eta_i"] & FGFD_i(X). \end{tikzcd}$$

Now $(F\eta_i)\gamma_i' = (F\eta_i)\pi_{i,1}\Theta_i\gamma_{i+1} = F(\eta_i p_{i,1})\gamma_{i+1}$ and $(FG\gamma_i)\gamma_i'' = (FG\gamma_i)\pi_{i,2}\Theta_i\gamma_{i+1} = F((G\gamma_i)p_{i,2})\gamma_{i+1}$. But by (1.3), $(G\gamma_i)p_{i,2} = \eta_i p_{i,1}$, and so $(F\eta_i)\gamma_i' = (FG\gamma_i)\gamma_i''$.

Thus, to establish the theorem, it suffices to prove $\gamma_i' = \gamma_i$ and $\gamma_i'' = \gamma_1$, for $i = 1, \ldots, m$. Now $\gamma_i' = \varepsilon_{FD_i(X)}(F\eta_i)\gamma_i' = \varepsilon_{FD_i(X)}(F\eta_i)\pi_{i,1}\Theta_i\gamma_{i+1} = \varepsilon_{FD_i(X)}(F\eta_i)(Fp_{i,1})\gamma_{i+1} = \varepsilon_{FD_i(X)}(FG\gamma_i)(Fp_{i,2})\gamma_{i+1} = \gamma_i\varepsilon_X(Fp_{i,2})\gamma_{i+1} = \gamma_i\varepsilon_{i+1}\gamma_{i+1}\gamma_{i+1} = \gamma_i$.

We next show $\gamma_i'' = \gamma_1$ by induction on i. For $i = 1$, we have $\gamma_1'' = (FG\varepsilon_1)(FG\gamma_1)\gamma_1'' = (FG\varepsilon_1)(FG\gamma_1)\pi_{1,2}\Theta_1\gamma_2 = (FG\varepsilon_1)F((G\gamma_1)p_{1,2})\gamma_2 = (FG\varepsilon_1)F(\eta_1 p_{1,1})\gamma_2 = (Fp_{1,1})\gamma_2 = \gamma_1' = \gamma_1$.

Now assume that $\gamma_j'' = \gamma_1$ for all $j < i$. Then $\gamma_i'' = (FG\varepsilon_i)(FG\gamma_i)\gamma_i'' = (FG\varepsilon_i)(F\eta_i)\gamma_i' = (Fp_{i-1,2})\gamma_i' = (Fp_{i-1,2})\gamma_i = \pi_{i-1,2}\Theta_{i-1}\gamma_i = \gamma_{i-1}''$. But $\gamma_{i-1}'' = \gamma_1$ by induction. Therefore $\gamma_i'' = \gamma_1$, and the induction is complete. This proves the theorem. □

2. Homotopy coalgebras of order *m*

In this section we apply the considerations of Section 1 to the homotopy category. Let Ho$\boldsymbol{Top}_*$ be the category whose objects are pointed, connected topological spaces and whose morphisms are homotopy classes of pointed mappings of such spaces. Then Ho$\boldsymbol{Top}_*$ is a category with weak pull-backs. For the weak pull-back in Ho$\boldsymbol{Top}_*$ we take the usual homotopy pull-back in $\boldsymbol{Top}_*$. We consider the k-fold suspension functor $F = \Sigma^k : \mathrm{Ho}\boldsymbol{Top}_* \to \mathrm{Ho}\boldsymbol{Top}_*$ and the k-fold loop-space functor $G = \Omega^k : \mathrm{Ho}\boldsymbol{Top}_* \to \mathrm{Ho}\boldsymbol{Top}_*$. It is well known that Σ^k and Ω^k are a pair of adjoint functors for $k \geq 1$. Thus for any space X, there are canonical mappings $\eta_X^k : X \to \Omega^k\Sigma^k X$ and $\varepsilon_X^k : \Sigma^k\Omega^k X \to X$ determined by the unit and counit of the adjunction. For $F = \Sigma^k$ and $G = \Omega^k$, we call a weak FG-coalgebra of order m and an (ordinary) FG-coalgebra of order m a *weak k-fold homotopy coalgebra of order m* and an (ordinary) *k-fold homotopy coalgebra of order m*, respectively.

Weak 1-fold homotopy coalgebras of order m, for $m = 1, 2, 3$ were considered by Saito ([6, section 6]), called homotopy coalgebras of order m and referred to as *HCAL-m*. The term weak homotopy coalgebra appears in [6], but has a meaning different from ours.

For brevity we refer to a weak or ordinary 1-fold homotopy coalgebra of order 1 as a *weak* or *ordinary homotopy coalgebra*. The structure map $\gamma : X \to \Sigma\Omega X$ is then called a *coretraction*. In [4] Ganea obtained the following characterization of homotopy coalgebras.

PROPOSITION 2.1. (i) *A space X is a co-H-space if and only if there exists a structure map $\gamma : X \to \Sigma\Omega X$ such that (X, γ) is weak homotopy coalgebra. In fact, there is a one-one correspondence between comultiplications of X and coretractions of X.*

(ii) *A 1-connected co-H-space X is homotopy associative if and only if the corresponding weak homotopy coalgebra (X, γ) is a homotopy coalgebra.*

In this section we consider weak and ordinary k-fold homotopy coalgebras of order m, for a fixed value of $k \geq 1$. To simplify our notation we write

$\varepsilon_X : \Sigma^k\Omega^k X \to X$ for ε_X^k, $\eta_X : X \to \Omega^k\Sigma^k X$ for η_X^k and $\gamma_l : X \to \Sigma^k D_l(X)$ for the structure map. We do not distinguish notationally between a map and its homotopy class, so equality of maps really signifies homotopy of maps. For a space X, $\operatorname{conn} X$ denotes the connectivity of X. Thus N is the largest integer such that $\pi_i(X) = 0$ for $i \leq N = \operatorname{conn} X$. Similarly if $f : X \to Y$ is a map of spaces, $\operatorname{conn} f$ denotes the connectivity of f. This means that $\operatorname{conn} f$ is characterized as the largest integer such that $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i < \operatorname{conn} f$ and an epimorphism for $i = \operatorname{conn} f$. This condition can be stated as $\pi_i(f) = 0$ for $i \leq \operatorname{conn} f$, where $\pi_i(f)$ is the i-th homotopy group of the map f. We assume that our spaces are 1-connected so that the above conditions on the homotopy groups for connectivity are equivalent to corresponding conditions on the homology groups.

In the following lemma we consider the diagrams of section 1 in the case $\boldsymbol{A} = \boldsymbol{B} = \mathrm{Ho}\boldsymbol{Top}_*$, $F = \Sigma^k$ and $G = \Omega^k$.

Lemma 2.2. *Suppose X is a weak k-fold homotopy coalgebra of order m and let $r \leq m$.*

(i) *Consider the diagram*

$$\begin{tikzcd} W_r(X) \arrow[r, "\pi_{r,2}"] \arrow[d, "\pi_{r,1}"] & \Sigma^k\Omega^k X \arrow[r, "\varepsilon_X"] \arrow[d, "\Sigma^k\Omega^k\gamma_r"] & X \arrow[d, "\gamma_r"] \\ \Sigma^k D_r(X) \arrow[r, "\Sigma^k\eta_r"] & \Sigma^k\Omega^k\Sigma^k D_r(X) \arrow[r, "\varepsilon_{\Sigma^k D_r(X)}"] & \Sigma^k D_r(X). \end{tikzcd}$$

If X is $(n-1)$-connected and γ_r is s-connected, $n \geq 2$ and $s \geq 1$, then $\varepsilon_X\pi_{r,2}$ is $(s+n-k)$-connected.

(ii) *Consider the diagrams*

$$\begin{tikzcd} D_{r+1}(X) \arrow[r, "p_{r,2}"] \arrow[d, "p_{r,1}"] & \Omega^k X \arrow[d, "\Omega^k\gamma_r"] \\ D_r(X) \arrow[r, "\eta_r"] & \Omega^k\Sigma^k D_r(X) \end{tikzcd}$$

and

$$\begin{tikzcd} \Sigma^k D_{r+1}(X) \arrow[dr, "\theta_r"] \arrow[drr, "\Sigma^k p_{r,2}"] \arrow[ddr, "\Sigma^k p_{r,1}"'] & & \\ & W_r(X) \arrow[r, "\pi_{r,2}"] \arrow[d, "\pi_{r,1}"] & \Sigma^k\Omega^k X \arrow[d, "\Sigma^k\Omega^k\gamma_r"] \\ & \Sigma^k D_r(X) \arrow[r, "\Sigma^k\eta_r"] & \Sigma^k\Omega^k\Sigma^k D_r(X). \end{tikzcd}$$

If η_r is p-connected and $\Omega^k\gamma_r$ is q-connected with p, $q \geq 1$ then Θ_r is $(p+q+k)$-connected.

PROOF. This result follows immediately from Ganea's Lemmas 3.1 and 3.2 ([4]) in the case $k=1$. For arbitrary k, the result is a consequence of analogues of Ganea's lemmas which are easily established using the proofs given in [4]. □

The next result provides the major step in the proof of one main result. We consider the connectivity of the maps $\gamma_l : X \to \Sigma^k D_l(X)$, $\varepsilon_X \pi_{l,2} : W_l(X) \to X$ and $\Theta_l : \Sigma^k D_{l+1}(X) \to W_l(X)$. This proposition has been proved by Saito in the case $k=1$ ([6, p. 611]).

PROPOSITION 2.3. *Let X be an $(n-1)$-connected space, $n \geq 2$, and a weak k-fold homotopy coalgebra of order m with $k \leq n-1$. Then for every $l \leq m$,*

(i) $\operatorname{conn} \gamma_l = (l+1)n - lk - l$,

(ii) $\operatorname{conn} D_l(X) = n-k-1$,

(iii) $\operatorname{conn} \varepsilon_X \pi_{l,2} = (l+2)n - (l+1)k - l$,

(iv) $\operatorname{conn} \Theta_l = (l+3)n - (l+2)k - (l+1)$.

PROOF. We prove all four statements simultaneously by induction on l. If $l=1$, then $D_1(X) = \Omega^k X$, so $D_1(X)$ is $(n-k-1)$-connected. The map $\varepsilon_1^k = \varepsilon_1 : \Sigma^k \Omega^k X \to X$ has connectivity $2n-k$ by [7, page 366]. Since $\varepsilon_1^k \gamma_1^k = \varepsilon_1 \gamma_1 = \mathrm{id}$, the map γ_1 has connectivity $2n-k-1$. This proves (i) and (ii) for $l=1$. Because γ_1 is $(2n-k-1)$-connected, $\Omega^k \gamma_1$ is $(2n-2k-1)$-connected. Also $\eta_1 : D_1(X) \to \Omega^k \Sigma^k D_1(X)$ is $(2(n-k)-1))$-connected by the Freudenthal Suspension Theorem [7, page 369]. By applying Lemma 2.2 (i) in the case $r=1$ and $s=2n-k-1$, we conclude that $\varepsilon_X \pi_{1,2}$ is $(3n-2k-1)$-connected. By applying Lemma 2.2 (ii) in the case $r=1$, $p=2n-2k-1$ and $q=2n-2k-1$, we conclude that Θ_1 is $(4n-3k-2)$-connected. This establishes the proposition for $l=1$.

Next we assume that the proposition is true for $l-1$ and prove it for l. By Theorem 1.6, $\varepsilon_X \pi_{l-1,2} \Theta_{l-1} \gamma_l = \varepsilon_X \gamma_1 = \mathrm{id}$. By induction, $\varepsilon_X \pi_{l-1,2}$ is $((l+1)n - lk - l + 1)$-connected and Θ_{l-1} is $((l+2)n - (l+1)k - l)$-connected. But $(l+1)n - lk - l + 1 \leq (l+2)n - (l+1)k - l$ since $k \leq n-1$. Thus γ_l is $((l+1)n - lk - l)$-connected. Now $\gamma_{l*} : \pi_i(X) \to \pi_i(\Sigma^k D_l(X))$ is an epimorphism for all $i \leq (l+1)n - lk - l$. However, $n-1 \leq (l+1)n - lk - l$, and so $\Sigma^k D_l(X)$ is $(n-1)$-connected. Therefore $D_l(X)$ is $(n-k-1)$-connected. This establishes (i) and (ii). By Lemma 2.2 (i) with $r=l$ and $s=(l+1)n - lk - l$, it follows that $\operatorname{conn}(\varepsilon_X \pi_{l,2})$ is $(l+1)n - lk - l + n - k = ((l+2)n - (l+1)k - l)$-connected.

Next consider $\eta_l : D_l(X) \to \Omega^k \Sigma^k D_l(X)$. Since $D_l(X)$ is $(n-k-1)$-connected, the Freudenthal Suspension Theorem [7, page 369] implies that η_l is $(2(n-k)-1)$-connected. Now apply Lemma 2.2(ii) with $r = l$, $p = 2n - 2k - 1$ and $q = (l+1)n - (l+1)k - l$. We conclude that Θ_l is $2n - 2k - 1 + (l+1)n - (l+1)k - l + k = (l+3)n - (l+2)k - l - 1$-connected. This completes the induction and establishes the proposition. □

We now state and prove our main result.

THEOREM 2.4. *Let X be a weak k-fold homotopy coalgebra of order m such that X is an $(n-1)$-connected, finite CW-complex and* $\dim X \leq (m+2)n - (m+1)k - m$, *where $n \geq 2$ and $k \leq n-1$. Then X has the homotopy type of a k-fold suspension.*

PROOF. Let $N = (m+2)n - (m+1)k - m$. Then by Proposition 2.3 the composition

$$\Sigma^k D_{m+1}(X) \xrightarrow{\Theta_m} W_m(X) \xrightarrow{\pi_{m,2}} \Sigma^k \Omega^k X \xrightarrow{\varepsilon_X} X$$

is N-connected and $\dim X \leq N$. Thus the homology group $H_N(X)$ is a free abelian group. To complete the proof we employ the variation of the homology decomposition of a space given by Berstein-Hilton [2, section 2] which we now recall.

Let Y be a CW-complex and $\rho : H_s(Y) \to A$ an epimorphism onto a free abelian group A for $s \geq 2$. Then there exists a CW-complex Y_0 and a map $\nu : Y_0 \to Y$ such that:

- $\nu_* : H_i(Y_0) \xrightarrow{\approx} H_i(Y)$ is an isomorhism for $i < s$,
- $\rho\nu_* : H_s(Y_0) \xrightarrow{\approx} A$ is an isomorphism,
- $H_i(Y_0) = 0$ for $i > s$.

We apply this with $Y = D_{m+1}(X)$ and $s = N - k$. Note that the map $\rho : H_{N-k}(D_{m+1}(X)) \to H_N(X)$ defined by the diagram

$$\begin{tikzcd} H_N(\Sigma^k D_{m+1}(X)) \arrow[r, "(\varepsilon_X \pi_{m,2} \Theta_m)_*"] & H_N(X) \\ H_{N-k}(D_{m+1}(X)) \arrow[u, "\chi", "\approx"'] \arrow[ur, "\rho"'] & \end{tikzcd}$$

is an epimorphism onto a free abelian group, where χ is the canonical isomorphism. Thus there exists a CW-complex Y_0 and a map $\nu : Y_0 \to D_{m+1}(X)$ such that

- $\nu_* : H_i(Y_0) \xrightarrow{\approx} H_i(D_{m+1}(X))$ is an isomorphism for $i < N - k$,
- $(\varepsilon_X \pi_{m,2} \Theta_m)_* \chi \nu_* : H_{N-k}(Y_0) \to H_{N-k}(X)$ is an isomorphism,
- $H_i(Y_0) = 0$ for $i > N - k$.

Since $H_i(X) = 0 = H_i(\Sigma^k Y)$ for $i > N$, it now follows that

$$\varepsilon_X \pi_{m,2} \Theta_m \Sigma^k \nu : \Sigma^k Y_0 \to X$$

is a homotopy equivalence. This completes the proof. $\square$

REMARK 2.5. Let $f = \varepsilon_X \pi_{m,2} \Theta_m \Sigma^k \nu : \Sigma^k Y_0 \to X$ be the homotopy equivalence constructed in the proof of Theorem 2.4. Then by adopting Saito's argument [6, Lemma 6.11, Corollary 6.12], it can be shown that f is compatible with the coretraction γ_1 and the canonical coretraction of $\Sigma^k Y_0$. From this it follows that f is a co-H-map, and hence a co-H-equivalence of $\Sigma^k Y_0$ with X.

In the case $m = 1$, Theorem 2.4 and Remark 2.5 yield the following result of Berstein-Ganea [3]: *If X is a weak k-fold homotopy coalgebra of order 1 and X is an $(n-1)$ connected CW-complex of dimension $\leq 3n - 2k - 1$ with $k \leq n - 1$, then X is co-H-equivalent to a k-fold suspension.*

Theorem 2.4 and Remark 2.5 yield the following result.

COROLLARY 2.6. *If X is a finite $(n-1)$-connected CW-complex and a weak k-fold homotopy coalgebra of order m for all $m \geq 1$, where $n \geq k + 2$, then X is a co-H-equivalent to a k-fold suspension.*

PROOF. We have $(m+2)n - (m+1)k - m = m(n-k-1) + 2n - k$. Since $n - k - 1 \geq 1$, we choose an integer m such that

$$\dim X \leq m(n-k-1) + 2n - k.$$

It follows that X is co-H-equivalent to a k-fold suspension. $\square$

Corollary 2.6 is a generalization of Theorem 6.4 [6] from $k = 1$ to arbitrary k, and our proof follows [6]. For $k = 1$, Corollary 2.6 yields: *A finite 2-connected CW-complex which is a weak 1-fold homotopy coalgebra of order m for all $m \geq 1$ is co-H-equivalent to a suspension.* This result is claimed in [6] for a 1-connected CW-complex, but the proof does not apply to a 1-connected space with non-trivial second homotopy group.

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