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Liouville theorems of stable F-harmonic maps for compact convex hypersurfaces

Jiancheng Liu

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Abstract. Let M^n be a compact convex hypersurface in $\mathbf{R}^{n+1}$. In this paper, we proved firstly that if the principal curvatures λ_i of M^n satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_n < \sum_{j=1}^{n-1} \lambda_j$, then there exist no nonconstant stable F-harmonic map between M and a compact Riemannian manifold when (1.2) or (1.3) holds (Theorem 1). This is a generalization or unification of the corresponding results for several varieties of harmonic map. Then, when the target manifold is δ-pinched, using a new estimate method, we obtain the Liouville-type theorem (Theorem 2) for stable F-harmonic map, which improves the results of M. Ara in [2].

1. Introduction

The instability for harmonic map (as well as p-harmonic map), from or into standard unit sphere S^n in Euclidean space $\mathbf{R}^{n+1}$, is well-known. For example, there exists no nonconstant stable harmonic (or p-harmonic) map either from S^n to any Riemannian manifold [12] (or [11]), or from any compact Riemannian manifold to S^n [6] (or [3]). In this paper, for a smooth function $F : [0, \infty) \to [0, \infty)$ such that $F'(t) > 0$ on $t \in (0, \infty)$, we concern with the instability of F-harmonic maps which is the generalization and union of the harmonic, p-harmonic or exponentially harmonic maps, introduced by M. Ara in [2].

M. Ara [1] proved that every stable F-harmonic map $u : M \to S^n$ is constant, provided that

$$(1.1) \qquad \int_M |du|^2 \left\{ |du|^2 F''\left(\frac{|du|^2}{2}\right) + (2-n)F'\left(\frac{|du|^2}{2}\right) \right\} *1 < 0.$$

In contrast with this, as far as I know there is few result when the source manifold is S^n. In this paper, however, we can prove the following instability

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property of F-harmonic maps which are from or into the compact convex hypersurfaces in Euclidean space.

THEOREM 1. *Let $M^n \subset \mathbf{R}^{n+1}$ be the compact convex hypersurface. Assuming that the principal curvatures λ_i of M^n satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_n < \sum_{j=1}^{n-1} \lambda_j$. Let $F : [0, \infty) \to [0, \infty)$ be a smooth function such that $F'(t) > 0$ on $t \in (0, \infty)$. Then every nonconstant F-harmonic map u between M^n and any compact Riemannian manifold N is unstable if there exists a constant $c_F := \inf\{c \geq 0 \mid F'(t)/t^c$ is nonincreasing$\}$ such that*

$$c_F < \frac{1}{2\lambda_n^2} \min_{1 \leq i \leq n} \left\{ \lambda_i \left(\sum_{j=1}^{n} \lambda_j - 2\lambda_i \right) \right\}, \tag{1.2}$$

or, when $F''(t) = F'(t)$ (for example $F(t) = \exp(t)$),

$$|\mathrm{d}u|^2 < \frac{1}{\lambda_n^2} \min_{1 \leq i \leq n} \left\{ \lambda_i \left(\sum_{j=1}^{n} \lambda_j - 2\lambda_i \right) \right\}. \tag{1.3}$$

REMARK 1. Theorem 1 obviously includes Ara's results [1] as a special case in a certain sense. In the case where $F(t) = (2t)^{p/2}$ ($p = 2$ or $4 \leq p < \infty$), the constant c_F equals $(p/2) - 1$, therefore, theorem 1 includes all the corresponding results in [3, 6, 11, 12] as special cases.

When $M^n = S^n$, the standard unit n-sphere, then the conditions (1.2) and (1.3) become respectively

$$c_F < \frac{n}{2} - 1, \tag{1.4}$$

$$|\mathrm{d}u|^2 < n - 2. \tag{1.5}$$

In this case, we have the following

COROLLARY 1. *With the same assumptions about the function F as in theorem 1, then every nonconstant F-harmonic map u, from S^n into any compact Riemannian manifold N or from any compact Riemannian manifold N into S^n, is unstable when (1.4) or (1.5) is true.*

REMARK 2. In the case of nonconstant harmonic maps or p-harmonic maps, the condition (1.4) implies that $n > 2$, $n > p$, respectively. Therefore, corollary 1 is an extension of [6, 12] and [3, 11] for the stability of harmonic maps and p-harmonic maps, respectively.

REMARK 3. In the case of nonconstant exponentially harmonic map, i.e. $F(t) = \exp(t)$, although the constant c_F does not exist, the condition $n - 2 > |\mathrm{d}u|^2$ is necessary. For example, taking $u : (S^n, g_0) \to (S^n, g_0)$ be the identity map, where S^n is the standard unit sphere with canonical metric g_0, it is well-known that u is stable, but in that case, $|\mathrm{d}u|^2 = n > n-2$. Hence, corollary 1 is also an extension of [5] and [7] for the stability of exponentially harmonic maps.

When the target manifolds are δ-pinched, using a new estimate method, we obtain the following Liouville theorem of stable F-harmonic map, which improves the results of M. Ara [2].

THEOREM 2. *Every stable F-harmonic map $u : M \to N$, from compact Riemannian manifold M into a compact simply-connected δ-pinched n-dimensional Riemannian manifold N, is constant, provided that there exists a constant $c_F := \inf\{c \geq 0 \mid F'(t)/t^c$ is non-increasing$\}$ such that n and δ satisfy* (1.4) *(or equivalently $2c_F + 1 < n-1$) and*

$$\tilde{\Phi}_{n,F}(\delta) := (2c_F + 1)\left\{\frac{n}{4}k_3^2(\delta) + k_3(\delta) + 1\right\} - \frac{2\delta}{1+\delta}(n-1) < 0. \tag{1.6}$$

REMARK 4. M. Ara [2] proved that: every stable F-harmonic map, from compact Riemannian manifold into n-dimensional δ-pinching manifold, is constant, provided that n and δ satisfy $n > 2(c_F + 1)$ and

$$\Phi_{n,F}(\delta) := (2c_F + 1)\left\{\frac{n+1}{4}k_3^2(\delta) + \sqrt{n+1}k_3(\delta) + 1\right\} - \frac{2\delta}{1+\delta}(n-1) < 0. \tag{1.7}$$

Obviously, $\tilde{\Phi}_{n,F}(\delta) < \Phi_{n,F}(\delta)$, so theorem 2 is much better than that of M. Ara in [2]. In the case where $F(t) = (2t)^{p/2}$ ($p = 2$ or $4 \leq p < \infty$), the constant c_F equals $(p/2) - 1$. So Theorem 2 is a unification and generalization of the well-known results for harmonic maps and p-harmonic maps obtained by T. Okayasu in [8] and H. Takeuchi in [11].

2. Preliminaries

Let $F : [0,\infty) \to [0,\infty)$ be a C^2-function such that $F'(t) > 0$ on $t \in (0,\infty)$. For a smooth map $u : (M,g) \to (N,h)$ between compact Riemannian manifolds (M,g) and (N,h) with Riemannian metric g and h, respectively. Following M. Ara [1], u is F-harmonic if it represents a critical point of the F-energy integral

$$E_F(u) = \int_M F\left(\frac{|\mathrm{d}u|^2}{2}\right) *1, \tag{2.1}$$

where $|\mathrm{d}u|^2$ is the energy density defined as $\sum_{i=1}^{m} |\mathrm{d}u(e_i)|^2$, $m = \dim M$, for a local orthonormal frame field $\{e_i\}$ on M, and $*1$ is the volume element of (M,g).

For example, when $F(t) = t$, $(2t)^{p/2}/p$, $(1+2t)^{\alpha}(\alpha > 1, m = 2)$ and $\exp(t)$, F-energy is the energy, the p-energy, the α-energy of Sacks-Uhlenbeck [10] and the exponential energy respectively. So F-harmonic map can be viewed as one of the unified theory for several varieties of harmonic map. On the other hand, we can see results for harmonic maps, p-harmonic maps or exponentially harmonic maps in a different viewpoint.

Denoted by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and N respectively. Let $u^{-1}TN$ be the induced vector bundle by u over M, and $\Gamma(u^{-1}TN)$ the space of all sections of $u^{-1}TN$. $\tilde{\nabla}$ denotes the induced connection on the induced bundle $u^{-1}TN$ from $\overline{\nabla}$ and u defined by $\tilde{\nabla}_X W = \overline{\nabla}_{u_* X} W$, here X is a tangent vector of M and W is a section of $u^{-1}TN$. With these symbols, then, the Euler-Lagrange equation of the F-energy functional E_F can be written

$$
\begin{aligned}
(2.2)\qquad \tau_F(u) &= \sum_{i=1}^{m}\left\{\tilde{\nabla}_{e_i}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)u_* e_i\right) - F'\left(\frac{|\mathrm{d}u|^2}{2}\right)u_*\nabla_{e_i}e_i\right\} \\
&= F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\tau(u) + u_*\left\{\operatorname{grad}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right)\right\},
\end{aligned}
$$

where $\tau(u)$ is the tension field along u. From now on we use the summation convention.

We need the following second variation formula for F-harmonic maps (cf. [1]). Let $u: M \to N$ be an F-harmonic map. Let $u_{s,t}: M \to N(-\varepsilon < s, t < \varepsilon)$ be a compactly supported two-parameters variation such that $u_{0,0} = u$, and set $V = \frac{\partial u_{s,t}}{\partial t}\Big|_{s,t=0}$, $W = \frac{\partial u_{s,t}}{\partial s}\Big|_{s,t=0}$. Then

$$
\begin{aligned}
(2.3)\qquad & \frac{\partial^2}{\partial s \partial t} E_F(u_{s,t})|_{s,t=0} \\
&= \int_M F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle\tilde{\nabla}V, \mathrm{d}u\rangle\langle\tilde{\nabla}W, \mathrm{d}u\rangle * 1 \\
&\quad + \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\cdot\left\{\langle\tilde{\nabla}V, \tilde{\nabla}W\rangle - \sum_{i=1}^{m} h(R^N(V, u_* e_i)u_* e_i, W)\right\} * 1,
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $T^*M \otimes u^{-1}TN$ and R^N is the curvature tensor of the manifold N, i.e. $R^N(X, Y) = [\overline{\nabla}_X, \overline{\nabla}_Y] - \overline{\nabla}_{[X,Y]}$ for vector fields X, Y on N.

We put

$$
(2.4)\qquad I(V, W) = \frac{\partial^2}{\partial s \partial t} E_F(u_{s,t})\Big|_{s,t=0}.
$$

An F-harmonic map u is called F-stable or stable if $I(V, V) \geq 0$ for any compactly supported vector field V along u.

3. F-harmonic maps from compact convex hypersurfaces

In the following two sections, we will discuss separately when the source manifolds or the target manifolds are compact convex hypersurfaces in Euclidean space, and obtain Proposition 1 and Proposition 2 respectively. Then, combining two Propositions, we had proved Theorem 1 at the end of section 4.

In this section, we study the instability for F-harmonic map from compact convex hypersurfaces into a compact Riemannian manifold, and obtain the following

Proposition 1. *With the same assumptions on M^n and F as in Theorem* 1. *Let $u : M^n \to N$ be a nonconstant F-harmonic map into compact Riemannian manifold N, if* (1.2) *or* (1.3) *holds, then u is unstable.*

Proof. In order to prove the instability of $u : M^n \to N$, we need to consider some special variational vector fields along u. To do this, choosing an orthogonal frame field $\{e_i, e_{n+1}\}$, $i = 1, \dots, n$, of $\mathbf{R}^{n+1}$, such that $\{e_i\}$ are tangent to $M^n \subset \mathbf{R}^{n+1}$, e_{n+1} is normal to M^n and $\nabla_{e_i} e_j|_P = 0$. Meanwhile, taking a fixed orthonormal basis E_A, $A = 1, \dots, n+1$, of $\mathbf{R}^{n+1}$ and setting

$$V_A = \sum_{i=1}^{n} v_A^i e_i, \qquad v_A^i = \langle E_A, e_i \rangle, \qquad v_A^{n+1} = \langle E_A, e_{n+1} \rangle, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical Euclidean inner product. Then $u_* V_A \in \Gamma(u^{-1}TN)$ and

$$\sum_A v_A^i v_A^j = \sum_A \langle E_A, e_i \rangle \langle E_A, e_j \rangle = \delta_{ij}, \tag{3.2}$$

$$\nabla_{e_i} V_A = v_A^{n+1} h_{ij} e_j, \tag{3.3}$$

$$\nabla_{e_i}(\nabla_{e_i} V_A) = -v_A^k h_{ik} h_{ij} e_j + v_A^{n+1} (\nabla_{e_i} h_{ij}) e_j, \tag{3.4}$$

$$\begin{aligned} \tilde{\nabla}_{e_i}(\mathrm{d}u(\nabla_{e_i} V_A)) = {} & -v_A^k h_{ik} h_{ij} (\mathrm{d}u(e_j)) + v_A^{n+1} (\tilde{\nabla}_{e_i} h_{ij})(\mathrm{d}u(e_j)) \\ & + v_A^{n+1} h_{ij} \tilde{\nabla}_{u_* e_i} u_* e_j, \end{aligned} \tag{3.5}$$

where, h_{ij} denotes the components of the second fundamental form of M^n in $\mathbf{R}^{n+1}$.

From now on, suppose that $u : M^n \to N$ is a nonconstant F-harmonic map, we shall use the variational vector fields $u_* V_A$ to prove the instability of

u. Firstly, by using F-harmonicity condition $\mathrm{d}^*\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\mathrm{d}u\right) = 0$ and (3.2), we have

$$\sum_A \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \triangle\, \mathrm{d}u(V_A), \mathrm{d}u(V_A)\rangle *1 \tag{3.6}$$

$$= \sum_A \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) v_A^i v_A^j \langle \triangle\, \mathrm{d}u(e_i), \mathrm{d}u(e_j)\rangle *1$$

$$= \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \triangle\, \mathrm{d}u(e_i), \mathrm{d}u(e_i)\rangle *1$$

$$= \int_{M^n} \left\langle \mathrm{d}^*\, \mathrm{d}u, \mathrm{d}^*\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\mathrm{d}u\right)\right\rangle *1 = 0.$$

It follows from Weitzenböck formula that

$$-R^N(u_* V_A, \mathrm{d}u(e_i))\mathrm{d}u(e_i) + u_*\, \mathrm{Ric}^{M^n}(V_A) = \triangle\, \mathrm{d}u(V_A) + \tilde{\nabla}^2\, \mathrm{d}u(V_A). \tag{3.7}$$

With respect to the variational vector fields $u_* V_A$ along u, it follows from (3.6) and (3.7) that

$$\sum_A I(u_* V_A, u_* V_A) \tag{3.8}$$

$$= \sum_A \int_{M^n} F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \tilde{\nabla} u_* V_A, \mathrm{d}u\rangle^2 *1$$

$$+ \sum_A \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{|\tilde{\nabla} u_* V_A|^2 + \langle \tilde{\nabla}^2\, \mathrm{d}u(V_A), u_* V_A\rangle$$

$$- \langle u_*\, \mathrm{Ric}^{M^n}(V_A), u_* V_A\rangle\} *1.$$

Notice that $\tilde{\nabla}^2 = \tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i}e_i}$, for any fixed point $P \in M$, choose $\{e_i\}$ such that $\nabla_{e_i}e_j|_P = 0$. Then

$$\tilde{\nabla}^2\, \mathrm{d}u(V_A) = \tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)) - 2\tilde{\nabla}_{e_i}(\mathrm{d}u(\nabla_{e_i} V_A)) + \mathrm{d}u(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} V_A), \tag{3.9}$$

and

$$\int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)), \mathrm{d}u(V_A)\rangle *1 \tag{3.10}$$

$$= -\int_{M^n} \left\langle \tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)), \tilde{\nabla}_{e_i}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\mathrm{d}u(V_A)\right)\right\rangle *1$$

$$= -\int_{M^n} \left\langle \tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)), \tilde{\nabla}_{e_i}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right)\mathrm{d}u(V_A)\right\rangle *1$$

$$- \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\tilde{\nabla}(\mathrm{d}u(V_A))|^2 *1.$$

Substituting (3.9) and (3.10) into (3.8), we get therefore

$$\begin{aligned}
(3.11)\qquad \sum_A I(u_* V_A, u_* V_A) &= \sum_A \int_{M^n} \left\{ F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle \tilde{\nabla} u_* V_A, \mathrm{d}u\rangle^2 \right. \\
&\qquad \left. - \left\langle \tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)), \tilde{\nabla}_{e_i}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right)\mathrm{d}u(V_A)\right\rangle\right\} *1 \\
&\qquad + \sum_A \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\{\langle -2\tilde{\nabla}_{e_i}(\mathrm{d}u(\nabla_{e_i} V_A)) \\
&\qquad + \mathrm{d}u(\nabla_{e_i}\nabla_{e_i} V_A) - u_* \operatorname{Ric}^{M^n}(V_A), u_* V_A\rangle\} *1 \\
&:= \int_{M^n} \{(\mathrm{I}) + (\mathrm{II})\} *1.
\end{aligned}$$

In the following, we shall estimate the two parts (I) and (II) in (3.11) separately. Because trace is independent of the choice of orthonormal basis, we can take pointwisely $\{e_i, e_{n+1}\}$ such that $h_{ij} = \lambda_i \delta_{ij}$. From Gauss formula it follows that

$$(3.12)\qquad \operatorname{Ric}^{M^n} = v_A^i (h_{kk} h_{ij} - h_{il} h_{jl}) e_j.$$

Using (3.3), (3.4), (3.5) and (3.12), we can easily obtain

$$\begin{aligned}
(3.13)\qquad \int_{M^n} (\mathrm{II}) *1 &= \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum_A \{\langle 2 v_A^k h_{ik} h_{ij} u_* e_j - v_A^i h_{kk} h_{ij} u_* e_j, v_A^l u_* e_l\rangle\} *1 \\
&\quad - \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \sum_A \{\langle v_A^{n+1} (\nabla_{e_i} h_{ij}) u_* e_j \\
&\qquad + 2 v_A^{n+1} h_{ij} \tilde{\nabla}_{u_* e_i} u_* e_j, v_A^l u_* e_l\rangle\} *1 \\
&= \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \langle 2 h_{il} h_{ij} u_* e_j - h_{kk} h_{lj} u_* e_j, u_* e_l\rangle *1 \\
&\le \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \max_{1 \le i \le n}\left(2\lambda_i - \sum_{k=1}^n \lambda_k\right)\lambda_i |\mathrm{d}u|^2 *1.
\end{aligned}$$

In order to estimate part (I) in (3.11), a straightforward computation then shows

$$\sum_A \left\langle \tilde{\nabla}_{e_i}(\mathrm{d}u(V_A)), \tilde{\nabla}_{e_i}\left(F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right)\mathrm{d}u(V_A)\right\rangle \tag{3.14}$$

$$= \sum_A F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\tilde{\nabla}_{e_i}\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle v_A^{n+1}h_{ik}u_*e_k + v_A^k\tilde{\nabla}_{u_*e_i}u_*e_k, v_A^j u_*e_j\rangle$$

$$= F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle\tilde{\nabla}_{e_i}\,\mathrm{d}u, \mathrm{d}u\rangle^2,$$

and

$$\sum_A F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle\tilde{\nabla}u_*V_A, \mathrm{d}u\rangle^2 \tag{3.15}$$

$$= \sum_A F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\langle v_A^{n+1}h_{ik}u_*e_k + v_A^k\tilde{\nabla}_{u_*e_i}u_*e_k, u_*e_i\rangle^2$$

$$= F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\{h_{ik}h_{jl}\langle u_*e_k, u_*e_i\rangle\langle u_*e_l, u_*e_j\rangle$$

$$+ 2\langle\overline{\nabla}_{u_*e_i}u_*e_k, u_*e_i\rangle\langle\overline{\nabla}_{u_*e_j}u_*e_k, u_*e_j\rangle\}$$

$$= F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\{\lambda_i\lambda_j\langle u_*e_i, u_*e_i\rangle\langle u_*e_j, u_*e_j\rangle + \langle\tilde{\nabla}_{e_i}\,\mathrm{d}u, \mathrm{d}u\rangle^2\}.$$

Then, it follows from (3.14) and (3.15) that

$$\int_{M^n}(\mathrm{I})*1 = \int_{M^n} F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\lambda_i\lambda_j\langle u_*e_i, u_*e_i\rangle\langle u_*e_j, u_*e_j\rangle*1. \tag{3.16}$$

By the assumption in Theorem 1, if $F''(t) = F'(t)$, then (3.16) leads to the inequality

$$\int_{M^n}(\mathrm{I})*1 \le \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\lambda_n^2|\mathrm{d}u|^4*1; \tag{3.17}$$

if there exists a constant c_F such that $F'(t)/t^{c_F}$ is non-increasing, it follows that $F''(t)t \le c_F F'(t)$ on $t \in (0, \infty)$, thus (3.16) implies

$$\int_{M^n}(\mathrm{I})*1 \le \int_{M^n} 2c_F F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\lambda_n^2|\mathrm{d}u|^2*1. \tag{3.18}$$

Finally, substituting (3.13), (3.17) or (3.18) into (3.11), we obtain

$$(3.19)\qquad \sum_A I(u_*V_A, u_*V_A)$$
$$\le \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2\left\{\lambda_n^2|\mathrm{d}u|^2 + \max_{1\le i\le n}\left(2\lambda_i - \sum_{k=1}^n \lambda_k\right)\lambda_i\right\}*1,$$

or

$$(3.20)\qquad \sum_A I(u_*V_A, u_*V_A)$$
$$\le \int_{M^n} F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2\left\{2c_F\lambda_n^2 + \max_{1\le i\le n}\left(2\lambda_i - \sum_{k=1}^n \lambda_k\right)\lambda_i\right\}*1,$$

either of which implies that $\sum_A I(u_*V_A, u_*V_A) < 0$ if u is nonconstant and (1.2) or (1.3) holds. Therefore, there exists at least one $V_i \in \{V_1, \dots, V_{n+1}\}$ such that

$$I(u_*V_i, u_*V_i) < 0.$$

That is, F-harmonic map u is not stable. This completes the proof of the Proposition 1.

4. F-harmonic maps into compact convex hypersurfaces

In this section, we study the instability for F-harmonic map from any compact Riemannian manifold N into compact convex hypersurface $M^n \subset \mathbf{R}^{n+1}$, and obtain the following

Proposition 2. *With the same assumptions on M^n and F as in Theorem* 1. *Let $u : N \to M^n$ be a nonconstant F-harmonic map from m-dimensional compact Riemannian manifold N, if* (1.2) *or* (1.3) *holds, then u is unstable.*

Proof. Denoted by $\tilde{\nabla}$ the induced connection on $u^{-1}TM^n$ (notice that, we use the same symbol $\tilde{\nabla}$ as in proposition 1, but different meaning). Taking the same vector fields V_A, $A = 1, \dots, n+1$, as in proposition 1. A straightforward computation similar to Theorem 5 in [9], we obtain

$$(4.1)\qquad \sum_A I(V_A, V_A) = \int_N \left\{F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\left(\sum_{\alpha,i}\lambda_i u_{\alpha i}^2\right)^2\right.$$
$$\left. + F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\sum_{\alpha,i}\lambda_i\left(2\lambda_i - \sum_{j=1}^n \lambda_j\right)u_{\alpha i}^2\right\}*1,$$

where $\sum_{\alpha,i} u_{\alpha i}^2 = |\mathrm{d}u|^2$. According to the assumptions on the function F, if $F''(t) = F'(t)$, then we get from (4.1)

$$\sum_A I(V_A, V_A) \tag{4.2}$$
$$\leq \int_N F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2\left\{\lambda_n^2|\mathrm{d}u|^2 - \min_{1\leq i\leq n}\left(\sum_{j=1}^n \lambda_j - 2\lambda_i\right)\lambda_i\right\}*1,$$

or, if there exists a constant c_F such that $F'(t)/t^{c_F}$ is non-increasing, then we obtain from (4.1)

$$\sum_A I(V_A, V_A) \tag{4.3}$$
$$\leq \int_N F'\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2\left\{2\lambda_n^2 c_F - \min_{1\leq i\leq n}\left(\sum_{j=1}^n \lambda_j - 2\lambda_i\right)\lambda_i\right\}*1.$$

Now, if $u : N \to M^n$ is a nonconstant F-harmonic map and (1.2) or (1.3) holds, then we have from (4.2) or (4.3),

$$\sum_A I(V_A, V_A) < 0,$$

which implies that there exists at least one $V_i \in \{V_1, \dots, V_{n+1}\}$ such that

$$I(V_i, V_i) < 0,$$

so, F-harmonic map $u : N \to M^n$ is unstable. This completes the proof of Proposition 2.

Proof of Theorem 1. Combining Proposition 1 (in section 3) and Proposition 2 (in section 4), Theorem 1 had been proved immediately.

5. F-harmonic maps into δ-pinched manifolds

Proof of Theorem 2. From now on, we assume that (N, h) is a compact simply-connected δ-pinched Riemannian manifold (i.e. its sectional curvature k_N satisfy $\delta < k_N \leq 1$). Deform the Riemannian metric h of N conformally to $\frac{1+\delta}{2} \cdot h$ (also denoted by h). We can set the sectional curvature equal to $\frac{2\delta}{1+\delta}$. Let E denote the Whitney sum $E = TN \oplus \epsilon(N)$ of the tangent bundle TN and the trivial line bundle $\epsilon(N) = N \times \mathbf{R}$ with the canonical metric. Let e be a cross-section of unit length in $\epsilon(N)$. We define a metric connection ∇'' on E as follows:

$$\nabla''_X Y = {}^N\nabla_X Y - h(X, Y)e;$$
$$\nabla''_X e = X,$$

where X and Y are vector fields on N. According to the results in [4], there exists a flat connection ∇' such that

$$\|\nabla' - \nabla''\| \le \frac{1}{2}k_3(\delta), \tag{5.1}$$

where the distance of two connections ∇', ∇'' defined by

$$\|\nabla' - \nabla''\| = \max\{\|\nabla'_X Y - \nabla''_X Y\|; X \in TN, \|X\| = 1, Y \in \Gamma(E), \|Y\| = 1\},$$

and

$$k_1(\delta) = \frac{4(1-\delta)}{3\delta}\left[1 + \left(\sqrt{\delta}\sin\frac{1}{2}\pi\sqrt{\delta}\right)^{-1}\right],$$

$$k_2(\delta) = \left[\frac{1}{2}(1+\delta)\right]^{-1} \cdot k_1(\delta),$$

$$k_3(\delta) = k_2(\delta)\sqrt{1 + \left(1 - \frac{1}{24}\pi^2(k_1(\delta))^2\right)^{-2}}.$$

Taking a cross-section W of E and denoting W^T the TN-component of W. Then, with the orthonormal frame field $\{e_i\}_{i=1}^m$, $m = \dim M$, we obtain

$$\begin{aligned}
I(W^T, W^T) &\\
= \int_M & F''\left(\frac{|\mathrm{d}u|^2}{2}\right)\sum_{i=1}^m \langle\tilde{\nabla}_{e_i} W^T, u_*e_i\rangle^2 *1 \\
&+ \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \cdot \sum_{i=1}^m \{|\tilde{\nabla}_{e_i} W^T|^2 - h(R^N(W^T, u_*e_i)u_*e_i, W^T)\} *1 \\
\le \int_M & \left\{F''\left(\frac{|\mathrm{d}u|^2}{2}\right)|\mathrm{d}u|^2 + F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\right\}\sum_{i=1}^m |\tilde{\nabla}_{e_i} W^T|^2 *1 \\
&- \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\sum_{i=1}^m h(R^N(W^T, u_*e_i)u_*e_i, W^T) *1 \\
\le \int_M & F'\left(\frac{|\mathrm{d}u|^2}{2}\right)\sum_{i=1}^m \{(2c_F + 1)|\tilde{\nabla}_{e_i} W^T|^2 \\
&\qquad - h(R^N(W^T, u_*e_i)u_*e_i, W^T)\} *1,
\end{aligned} \tag{5.2}$$

where we have used in the last inequality the assumption which says that $F'(t)/t^{c_F}$ is non-increasing, i.e. $F''(t)t \le c_F F'(t)$ on $t \in (0, \infty)$.

Meanwhile, we observe that

$$\begin{aligned}\tilde{\nabla}_{e_i} W^T &= {}^N\nabla_{u_* e_i} W^T \\ &= \nabla''_{u_* e_i} W^T + \langle W^T, u_* e_i \rangle e \\ &= \nabla''_{u_* e_i}(W - \langle W, e \rangle e) + \langle W^T, u_* e_i \rangle e \\ &= (\nabla''_{u_* e_i} W)^T - \langle W, e \rangle u_* e_i.\end{aligned}$$

Then, we have

$$\sum_{i=1}^m |\tilde{\nabla}_{e_i} W^T|^2 = \sum_{i=1}^m |(\nabla''_{u_* e_i} W)^T|^2 + \langle W, e \rangle^2 |\mathrm{d}u|^2 - 2\sum_{i=1}^m \langle W, e \rangle \langle \nabla''_{u_* e_i} W^T, u_* e_i \rangle. \tag{5.3}$$

Since N is δ-pinched, so

$$h(R^N(W^T, u_* e_i) u_* e_i, W^T) \ge \frac{2\delta}{1+\delta}\{|W^T|^2 |u_* e_i|^2 - \langle W^T, u_* e_i \rangle^2\}. \tag{5.4}$$

Substituting (5.3) and (5.4) into (5.2), we obtain

$$I(W^T, W^T) \le \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \cdot q(W) {*1}, \tag{5.5}$$

where

$$\begin{aligned}q(W) = (2c_F + 1)\Bigg\{ &\sum_{i=1}^m |(\nabla''_{u_* e_i} W)^T|^2 + \langle W, e \rangle^2 |\mathrm{d}u|^2 \\ &- 2\sum_{i=1}^m \langle W, e \rangle \langle \nabla''_{u_* e_i} W^T, u_* e_i \rangle \Bigg\} \\ &- \frac{2\delta}{1+\delta}\sum_{i=1}^m \{|W^T|^2 |u_* e_i|^2 - \langle W^T, u_* e_i \rangle^2\}.\end{aligned} \tag{5.6}$$

Let $\mathscr{W} := \{W \in \Gamma(E); \nabla' W = 0\}$, then $\mathscr{W}$ with natural inner product is isomorphic to $\mathbf{R}^{n+1}$. Define a quadratic form Q on $\mathscr{W}$ by $Q(W) := \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \cdot q(W){*1}$. Taking an orthonormal basis $\{W_1, W_2, \dots, W_n, W_{n+1}\}$ of $\mathscr{W}$ with respect to its natural inner product, from (5.5), we obtain

$$\sum_{j=1}^{n+1} I(W_j^T, W_j^T) \le \sum_{j=1}^{n+1} Q(W_j) = \text{trace } Q = \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right)(\text{trace}(q))*1.$$

Since trace(q) is independent of the choice of orthonormal basis on each fibre of E, at each point $x \in M$, we can choose an orthonormal basis $\{W_1, W_2, \dots, W_n, W_{n+1}\}$ of $\mathscr{W}$ with respect to a natural inner product, such that $W_1, W_2, \dots, W_n$ tangent to N. Then, at point $x \in M$, we have from (5.1) and (5.6)

$$\begin{aligned}
\text{trace}(q) = (2c_F + 1)\Bigg\{ & \sum_{j=1}^{n+1}\sum_{i=1}^{m} |(\nabla''_{u_*e_i} W_j)^T|^2 \\
& + \sum_{j=1}^{n+1} \langle W_j, e\rangle^2 |\mathrm{d}u|^2 - 2\sum_{j=1}^{n+1}\sum_{i=1}^{m} \langle W_j, e\rangle \langle \nabla''_{u_*e_i} W_j^T, u_*e_i\rangle \Bigg\} \\
& - \frac{2\delta}{1+\delta}\sum_{j=1}^{n+1}\sum_{i=1}^{m}\{|W_j^T|^2|u_*e_i|^2 - \langle W_j^T, u_*e_i\rangle^2\} \\
= (2c_F + 1)\Bigg\{ & \sum_{j=1}^{n+1}\sum_{i=1}^{m}\sum_{k=1}^{n} \langle \nabla''_{u_*e_i} W_k, W_j\rangle^2 + |\mathrm{d}u|^2 \\
& - 2\sum_{i=1}^{m} \langle \nabla''_{u_*e_i} W_{n+1}, u_*e_i\rangle \Bigg\} - \frac{2\delta}{1+\delta}(n-1)|\mathrm{d}u|^2 \\
\le (2c_F + 1)\Bigg\{ & \sum_{i=1}^{m}\sum_{k=1}^{n} |\nabla''_{u_*e_i} W_k|^2 + |\mathrm{d}u|^2 + k_3(\delta)|\mathrm{d}u|^2 \Bigg\} \\
& - \frac{2\delta}{1+\delta}(n-1)|\mathrm{d}u|^2 \\
\le \Bigg\{ & (2c_F + 1)\left[\frac{n}{4}k_3^2(\delta) + k_3(\delta) + 1\right] - \frac{2\delta}{1+\delta}(n-1)\Bigg\}|\mathrm{d}u|^2 \\
= \tilde{\Phi}_{n,F}&(\delta)|\mathrm{d}u|^2.
\end{aligned}$$

Hence,

$$\text{trace } I = \sum_{j=1}^{n+1} I(W_j^T, W_j^T) \le \int_M F'\left(\frac{|\mathrm{d}u|^2}{2}\right) \cdot \tilde{\Phi}_{n,F}(\delta) \cdot |\mathrm{d}u|^2 * 1. \tag{5.7}$$

If u is F-stable, then trace $I \ge 0$, when $\tilde{\Phi}_{n,F}(\delta) < 0$, from (5.7), we must have $|\mathrm{d}u| = 0$, i.e. u is constant, which completes the proof of theorem 2.

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Jiancheng Liu
Department of Mathematics
Northwest Normal University
Lanzhou 730070, *China*
E-mail: liujc@nwnu.edu.cn