Hiroshima Math. J.
31 (2001), 213–262

Asymptotic expansions of the null distributions of three test statistics in a nonnormal GMANOVA model

Hirokazu Yanagihara

(Received August 22, 2000)
(Revised November 11, 2000)

Abstract. This paper deals with three test statistics for testing a linear hypothesis and estimators of regression coefficients in the GMANOVA model which was proposed by Potthof and Roy (1964), without assuming normal error. The test statistics considered include the likelihood ratio statistic, the Lawley-Hotelling trace criterion and the Bartlett-Nanda-Pillai trace criterion, which have been proposed under normality. We obtain asymptotic expansions of the null distributions of three test statistics up to the order n^{-1}, where n is the sample size. The results are generalizations of the formulas in Wakaki, Yanagihara and Fujikoshi (2000). In addition, asymptotic expansions of the distribution functions of several standardized statistics on regression coefficients are derived.

1. Introduction

The GMANOVA model considered is defined by

$$Y = A\Xi X' + \mathscr{E}, \tag{1.1}$$

where $Y = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)'$ is an $n \times p$ observation matrix of response variables, $A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)'$ is an $n \times k$ between-individuals design matrix of explanatory variables with full rank k, X is a $p \times q$ within-individuals design matrix of explanatory variables with full rank q $(\le p)$, Ξ is a $k \times q$ unknown parameter matrix and $\mathscr{E} = (\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n)'$ is an $n \times p$ error matrix. It is assumed that each vector $\boldsymbol{\varepsilon}_j$ is *i.i.d.*, i.e., independently and identically distributed with $\mathrm{E}(\boldsymbol{\varepsilon}_j) = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{\varepsilon}_j) = \Sigma$. This model can be applied to analysis of growth curve data, and hence it is also called the growth curve model.

We consider to test for a general linear hypothesis

$$H_0 : C\Xi D = O, \tag{1.2}$$

where C is a known $c \times k$ matrix with rank c $(\le k)$, D is a known $q \times d$ matrix with rank d $(\le q)$ and O is a $c \times d$ matrix all of whose elements are 0.

2000 *Mathematics Subject Classification*. primary 62H10, secondly 62E20.

Key words and phrases. Confidence interval, Conservativeness, General multivariate linear hypothesis, Linear combination, MLE, Nonnormality, Robustness.

The GMANOVA model (1.1) with normal error was introduced by Potthoff and Roy (1964) and have been extensively studied by many authors. The maximum likelihood estimators $\hat{\Xi}$ and $\hat{\Sigma}$ of Ξ and Σ, and the likelihood ratio test statistic were obtained by Khatri (1966) and Gleser and Olkin (1970). Fujikoshi (1974) studied properties of some test statistics, including the LR test statistic, and gave asymptotic expansions of their non-null distributions. Gleser and Olkin (1970) were the first to derive the exact density of MLE $\hat{\Xi}$. Asymptotic expansions of the distributions of $\hat{\Xi}$ and its linear transform have been studied by Fujikoshi (1985, 1993a) and von Rosen (1997). Various aspects of statistical inference under normality have been also discussed in literature. For these results, see. e.g., Kariya (1985), von Rosen (1991), Fujikoshi (1993b), Kshirsagar and Smith (1995), and Srivastava and von Rosen (1999).

The above results are based on the assumption that the error vectors $\varepsilon_1, \dots, \varepsilon_n$ are independently and identically distributed as a multivariate normal distribution with means $\mathbf{0}$ and covariance matrix Σ. Khatri (1988) discussed robustness for test statistic under elliptical distribution. However, the nonnormal case has not been investigated so much, except for the case $X = I_p$, i.e., MANOVA case. For MANOVA case, Ito (1969, 1980), Chase and Bulgren (1971) and Everitt (1979) studied robustness of certain test statistics by simulation. Wakaki, Yanagihara and Fujikoshi (2000) obtained asymptotic expansions of the null distributions of three test statistics in nonnormal multivariate linear model. These results include several expansions obtained by Kano (1995), Fujikoshi (1997b, 2001), Fujikoshi, Ohmae and Yanagihara (1999) and Yanagihara (1999), as special cases. Our main purpose is to extend the asymptotic expansion formulas in a multivariate linear model to the ones in the GMANOVA model.

The present paper is organized in the following way. In § 2, we describe three test statistics. It is shown that our test statistics can be expresses in terms of a random matrix U, which is a kind of Studentized version of $\hat{\Xi}$. Using this expression, we derive perturbation expansions of our test statistics. In § 3, we give an asymptotic expansion of the distribution function of U. Further, asymptotic expansions of other standardized statistics of $\hat{\Xi}$ are obtained in § 4. In § 5, we obtain asymptotic expansions of the null distributions of three test statistics, by expanding their characteristic function. Moreover, in § 6, we discuss robustness of testing under nonnormality and derive a result on conservativeness based on the asymptotic expansion formulas. Some applications of the asymptotic expansions of test statistics are given in § 7. In § 8, numerical accuracies are studied for some confidence interval of Ξ and asymptotic expansions of the null distributions for some test statistics under nonnormality.

2. Test statistics and perturbation expansion

First, we summarize typical three test criteria that have been proposed under normality. Let S_h and S_e be the variation matrices due to the hypothesis and the error, respectively, i.e.,

$$S_h = (C\hat{\Xi}D)'(CRC')^{-1}(C\hat{\Xi}D), \qquad S_e = D'(X'S^{-1}X)^{-1}D,$$

where

$$\hat{\Xi} = (A'A)^{-1}A'YS^{-1}X(X'S^{-1}X)^{-1},$$

$$R = (A'A)^{-1} + (A'A)^{-1}A'Y\{S^{-1} - S^{-1}X(X'S^{-1}X)X'S^{-1}\}Y'A(A'A)^{-1},$$

and $S = Y'(I_n - P_A)Y$. Here P_A is the projection matrix to the linear space $\mathfrak{R}(A)$ generated by the column vectors of A. Then the following three criteria have been proposed, in particular, under normality.

(i) the likelihood ratio statistic:

$$T_{LR} = -\{n - k - (p - q) + s_1\} \log(|S_e|/|S_e + S_h|),$$

(ii) the Lawley-Hotelling trace criterion:

$$T_{HL} = \{n - k - (p - q) + s_2\} \operatorname{tr}(S_h S_e^{-1}),$$

(iii) the Bartlett-Nanda-Pillai trace criterion:

$$T_{BNP} = \{n - k - (p - q) + s_3\} \operatorname{tr}\{S_h(S_h + S_e)^{-1}\},$$

where the constants s_j's are the Bartlett corrections in the normal case, and they are given as follows: $s_1 = -(d - c + 1)/2$, $s_2 = -(d + 1)$ and $s_3 = c$. For the special case $q = p$, note that three criteria are reduced to the ones in the usual MANOVA model. Therefore, as in the MANOVA model, it may be suggested to use the criteria for nonnormal models.

Under normality, the distributions of these statistics have been extensively studied. Fujikoshi (1974) obtained asymptotic expansions of the non-null distributions for three test statistics. Under nonnormality it is easily seen that the null distributions of these statistics converge to χ^2_{cd} as the sample size n tends to infinity under an appropriate regularity condition on the design matrix (see Huber (1973)). Our main purpose is to obtain asymptotic expansions of the null distributions of these statistics up to the order n^{-1} under a general condition.

Note that the three test statistics are invariant under the transformations from $[Y', X]$ to $\Sigma^{-1/2}[Y', X]$. Therefore, without loss of generality we may assume $\Sigma = I_p$ by replacing X with $\Sigma^{-1/2}X$. In the following, we shall do that, and we regard X as $\Sigma^{-1/2}X$. We consider expressing the test statistics in terms of

$$Z = (A'A)^{-1/2}A'\mathscr{E}, \qquad V = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}(\varepsilon_j\varepsilon_j' - I_p). \tag{2.1}$$

Note that $(n^{-1}S)^{-1}$ can be expanded as

$$\left(\frac{1}{n}S\right)^{-1} = I_p - \frac{1}{n}V + \frac{1}{n}(V^2 + Z'Z) + \mathrm{O}_p(n^{-3/2}).$$

Therefore,

$$\begin{aligned}
&\frac{1}{n}(X'S^{-1}X)^{-1} \\
&\quad = (X'X)^{-1/2}\Bigg[I_q + \frac{1}{\sqrt{n}}(X'X)^{-1/2}X'VX(X'X)^{-1/2} \\
&\qquad\qquad - \frac{1}{n}(X'X)^{-1/2}X'\{V(I_p - P_X)V + Z'Z\}X(X'X)^{-1/2}\Bigg] \\
&\qquad \times (X'X)^{-1/2} + \mathrm{O}_p(n^{-3/2}).
\end{aligned}$$

By using these results, we define modified matrices $\tilde{S}_e$, $\tilde{\Xi}$ and $\tilde{R}$ by the following relations, respectively.

$$\begin{aligned}
\left(\frac{1}{n}S_e\right)^{-1} &= \{D'(X'X)^{-1}D\}^{-1/2}\tilde{S}_e^2\{D'(X'X)^{-1}D\}^{-1/2}, \\
\hat{\Xi} &= (A'A)^{-1/2}Z\tilde{\Xi}X(X'X)^{-1/2}, \\
(A'A)^{-1/2}C'(CRC')^{-1}C(A'A)^{-1/2} &= \tilde{R}\Omega\tilde{R},
\end{aligned}$$

where

$$\Omega = (A'A)^{-1/2}C'\{C(A'A)^{-1}C'\}^{-1}C'(A'A)^{-1/2}.$$

Then, we obtain $\Omega^2 = \Omega$ and get $\mathrm{rank}(\Omega) = \mathrm{tr}(\Omega) = c$. Further, the random matrices $\tilde{S}_e$, $\tilde{\Xi}$ and $\tilde{R}$ can be expanded as

$$\begin{aligned}
\tilde{S}_e &= I_d - \frac{1}{2\sqrt{n}}L'VL \\
&\quad + \frac{1}{2n}L'\left\{V\left(I_p - P_X + \frac{3}{4}Q\right)V + Z'Z\right\}L + \mathrm{O}_p(n^{-3/2}), \\
\tilde{\Xi} &= I_p - \frac{1}{\sqrt{n}}(I_p - P_X)V \\
&\quad + \frac{1}{n}(I_p - P_X)\{V(I_p - P_X)V + Z'Z\} + \mathrm{O}_p(n^{-3/2}), \\
\tilde{R} &= I_k - \frac{1}{2n}\Omega Z(I_p - P_X)Z'\Omega + \mathrm{O}_p(n^{-3/2}),
\end{aligned} \tag{2.2}$$

where

$$L = X(X'X)^{-1}D\{D'(X'X)^{-1}D\}^{-1/2},$$

$$Q = LL' = X(X'X)^{-1}D\{D'(X'X)^{-1}D\}^{-1}D'(X'X)^{-1}X'.$$

Using these expressions, the three test statistics can be expanded as

$$T_G = \mathrm{tr}(U'\Omega U) + \frac{1}{n}[\{r_1 - k - (p-q)\}\ \mathrm{tr}(U'\Omega U) + r_2\ \mathrm{tr}\{(U'\Omega U)^2\}] + \mathrm{O}_p(n^{-3/2}), \tag{2.3}$$

where

$$U = \tilde{R}Z\tilde{\Xi}L\tilde{S}_e. \tag{2.4}$$

Here the constants r_1 and r_2 are defined as follows;

(i) $T_{LR}\ : r_1 = s_1,\ \ r_2 = -1/2,$

(ii) $T_{HL}\ : r_1 = s_2,\ \ r_2 = 0,$

(iii) $T_{BNP} : r_1 = s_3,\ \ r_2 = -1.$

In our derivation, first we derive an asymptotic expansion of the distribution of U. Then, using the result, we obtain an asymptotic expansion of the null distribution of T_G.

3. Edgeworth expansion of U

In this section, we obtain an asymptotic expansion of the distribution function of U up to the order n^{-1}. Without loss of generality, we assume that $\Sigma = I_p$ as in a previous section. So, we regard X as $\Sigma^{-1/2}X$ in the following expressions. Let $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_n$ be a sequence of *i.i.d.* random vectors with $\mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{\varepsilon}) = I_p$. We write a moment of $\boldsymbol{\varepsilon}$ as

$$\mu_{i_1\dots i_m} = \mathrm{E}[\varepsilon_{i_1}\dots\varepsilon_{i_m}],$$

where ε_j denotes the jth element of $\boldsymbol{\varepsilon}$. Similarly, the corresponding cumulant of $\boldsymbol{\varepsilon}$ is expressed as $\kappa_{i_1\dots i_m}$. Further, we use the following real matrix notation for arguments of some characteristic functions.

$$T = [t_{ab}] : k \times d\ \text{ matrix},$$

$$T_1 = [t_{ab}^{(1)}] : k \times p\ \text{ matrix},$$

$$T_2 = [(1+\delta_{ab})t_{ab}^{(2)}/2] : p \times p\ \text{ matrix},$$

where δ_{ab} is the Kronecker delta, i.e., $\delta_{aa} = 1$ and $\delta_{ab} = 0$ for $a \neq b$.

In order to get a valid expansion for the distribution function of U up the order n^{-1}, we make some assumptions for the between-individuals design matrix A and the distribution of $\boldsymbol{\varepsilon}$. Let λ_n be the smallest eigenvalue of $A'A$, and $M_n = \max\{\|\boldsymbol{a}_j\| : j = 1, \dots n\}$, where $\|\cdot\|$ denotes the Euclidean norm. We make the following assumptions.

B1. $\limsup_{n\to\infty} \dfrac{1}{n}\sum_{j=1}^{n} \|\boldsymbol{a}_j\|^4 < \infty$,

B2. $\liminf_{n\to\infty} \dfrac{\lambda_n}{n} > 0$,

B3. For some constant $\delta > 0$, $M_n = \mathrm{O}(n^{1/2-\delta})$,

B4. $\mathrm{E}(\|\boldsymbol{\varepsilon}\|^8) < \infty$,

B5. The Cramér's condition for $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon\varepsilon}'$;

$$\limsup_{\|\boldsymbol{t}\|+\|T_2\|\to\infty} |\mathrm{E}[\exp\{i\boldsymbol{t}'\boldsymbol{\varepsilon} + i\ \mathrm{tr}(T_2\boldsymbol{\varepsilon\varepsilon}')\}]| < 1,$$

where $\boldsymbol{t}$ is a $p \times 1$ real vector. Here, we define the norm of a matrix T_2 as $\|T_2\| = [\sum_{a=1}^{p}\sum_{b=1}^{p}\{(1+\delta_{ab})t_{ab}^{(2)}\}^2/4]^{1/2}$.

From (2.2) and (2.4), the random matrix U can be expanded as

$$U = U_0 + \frac{1}{\sqrt{n}}U_1 + \frac{1}{n}U_2 + \mathrm{O}_p(n^{-3/2}), \tag{3.1}$$

where

$$U_0 = ZL,$$

$$U_1 = -\frac{1}{2}Z\{Q + 2(I_p - P_X)\}VL,$$

$$U_2 = \frac{1}{8}Z[2\{Q + 2(I_p - P_X)\}V\{Q + 2(I_p - P_X)\} + QVQ]VL$$
$$+ \frac{1}{2}Z\{Q + 2(I_p - P_X)\}Z'ZL - \frac{1}{2}\Omega Z(I_p - P_X)Z'\Omega ZL.$$

Using (3.1), the characteristic function $C_U(T)$ of U can be expanded under the assumptions B1, B2, B3 and B4 as

$$\begin{aligned}
C_U(T) &= \mathrm{E}[\exp\{i\ \mathrm{tr}(T'U)\}]\\
&= \mathrm{E}[\exp\{i\ \mathrm{tr}(T'U_0)\}] + \frac{1}{\sqrt{n}}\mathrm{E}[i\ \mathrm{tr}(T'U_1)\exp\{i\ \mathrm{tr}(T'U_0)\}]\\
&\quad + \frac{1}{n}\mathrm{E}\left[\left\{i\ \mathrm{tr}(T'U_2) + \frac{i^2}{2}\{\mathrm{tr}(T'U_1)\}^2\right\}\exp\{i\ \mathrm{tr}(T'U_0)\}\right] + \mathrm{o}(n^{-1})\\
&= C_U^{(0)}(T) + \frac{1}{\sqrt{n}}C_U^{(1)}(T) + \frac{1}{n}C_U^{(2)}(T) + \mathrm{o}(n^{-1}).
\end{aligned}$$

Now we need to evaluate each term in the expansion of $C_U(T)$. Here we note that, $\mathrm{rank}(L) = d$, which can be essentially done in the same way as in Wakaki, Yanagihara and Fujikoshi (2000). The method is based on the use of differentials for $\Psi(T_1, T_2)$, which is defined by

$$\Psi(T_1, T_2) = \mathrm{E}[\exp\{i\ \mathrm{tr}(T_1'Z + n^{-1/2}T_2V)\}].$$

Therefore, letting $T_1 = TL'$ we have

$$\begin{aligned}
C_U^{(0)}(T) &= \Psi(T_1, O) \\
&= \exp\left\{ \frac{i^2}{2}\ \mathrm{tr}(T_1'T_1) + \frac{i^3}{6\sqrt{n}} \sum_{a'b'c'}^{k} \sum_{abc}^{p} t_{a'a}^{(1)} t_{b'b}^{(1)} t_{c'c}^{(1)} \bar{\chi}_{a'b'c'} \kappa_{abc} \right. \\
&\qquad \left. + \frac{i^4}{24n} \sum_{a'b'c'd'}^{k} \sum_{abcd}^{p} t_{a'a}^{(1)} t_{b'b}^{(1)} t_{c'c}^{(1)} t_{d'd}^{(1)} \bar{\chi}_{a'b'c'd'} \kappa_{abcd} + \mathrm{o}(n^{-1}) \right\},
\end{aligned}$$

where

$$\bar{\chi}_{a_1\ldots a_j} = \frac{1}{n} \sum_{i=1}^{n} \prod_{l=1}^{j} \chi_{ia_l}, \qquad \sqrt{n}(A'A)^{-1/2}\boldsymbol{a}_i = (\chi_{i1}, \ldots, \chi_{ik})'. \tag{3.2}$$

Further,

$$\begin{aligned}
C_U^{(1)}(T) &= -\frac{i}{2} \mathrm{E}[\mathrm{tr}\{T'Z(Q + 2(I_p - P_X)VL\}\ \exp\{i\ \mathrm{tr}(T'ZL)]\}] \\
&= -\frac{i}{2} \mathrm{E}[\mathrm{tr}\{T_1'Z(Q + 2(I_p - P_X))V\}\ \exp\{i\ \mathrm{tr}(T_1'Z)\}] \\
&= -\frac{i}{2}(i)^{-2} \sum_{a'}^{k} \sum_{abc}^{p} t_{a'c}^{(1)}(q_{ab} + 2\rho_{ab}) \frac{\partial^2}{\partial t_{a'a}^{(1)} \partial t_{bc}^{(2)}} \Psi(T_1, T_2)|_{T_2=O},
\end{aligned} \tag{3.3}$$

where q_{ab} and ρ_{ab} the (a,b)th elements of Q and $I_p - P_X$ resrpectively, and $\sum_{a_1\ldots a_j}^{k} = \sum_{a_1=1}^{k} \cdots \sum_{a_j=1}^{k}$. Note that

$$\begin{aligned}
&\frac{\partial^2}{\partial t_{a'a}^{(1)} \partial t_{bc}^{(2)}} \Psi(T_1, T_2)|_{T_2=O} \\
&= \left[i^2 \bar{\chi}_{a'} \kappa_{abc} + i^4 t_{a'a}^{(1)} \sum_{b'}^{k} \sum_{d}^{p} t_{b'd}^{(1)} \bar{\chi}_{b'} \kappa_{bcd} \right. \\
&\qquad + \frac{1}{\sqrt{n}} \left\{ i^3 \sum_{d}^{p} t_{a'd}^{(1)} (\mu_{abcd} - \delta_{ad}\delta_{bc}) + \frac{i^5}{2} t_{a'a}^{(1)} \sum_{b'}^{k} \sum_{de}^{p} t_{b'd}^{(1)} t_{b'e}^{(1)} (\mu_{bcde} - \delta_{bc}\delta_{de}) \right. \\
&\qquad \left. \left. + \frac{i^5}{2} \sum_{b'c'd'}^{k} \sum_{def}^{p} t_{b'e}^{(1)} t_{c'f}^{(1)} t_{d'd}^{(1)} \bar{\chi}_{a'b'c'} \bar{\chi}_{d'} \kappa_{aef} \kappa_{bcd} \right\} \right] \Psi(T_1, O) + \mathrm{o}(n^{-1/2}).
\end{aligned}$$

Moreover, it holds that $(I_p - P_X)L = 0$, $\mathrm{tr}(I_p - P_X) = p - q$, $(I_p - P_X)^2 = I_p - P_X$, $L'L = I_d$ and $Q^2 = Q$, in other words

$$\sum_{b}^{p} = \rho_{ab} l_{bj} = 0, \qquad \sum_{a}^{p} \rho_{aa} = p - q, \qquad \sum_{c}^{p} \rho_{ac}\rho_{bc} = \rho_{ab},$$

$$\sum_{a}^{p} l_{ai} l_{aj} = \delta_{ij}, \qquad \sum_{c}^{p} q_{ac} q_{bc} = q_{ab},$$

where l_{ab} is the (a,b)th element of L. By substituting these equations into (3.3) and replacing $t_{a'a}^{(1)}$ with $\sum_{j=1}^{d} t_{a'j} l_{aj}$, we can evaluate $C_U^{(0)}(T)$ and $C_U^{(1)}(T)$. Similarly, we can evaluate $C_U^{(2)}(T)$. Therefore, we can obtain an expansion of $C_U(T)$, whose formal inversion yields a valid expansion of the distribution function of U as in the following Theorem 3.1.

Some additional notations on cumulants need to be defined before describing Theorem 3.1. The quantity $\mathscr{K}_{(l_{*a}q_{*1}q_{*1})}$, which depends on the third order cumulants and the elements of L and Q is defined as

$$\mathscr{K}_{(l_{*a}q_{*1}q_{*1})} = \sum_{a'b'c'}^{p} l_{a'a} q_{b'c'} \kappa_{a'b'c'}. \tag{3.4}$$

In this expression, the order of indices $*$ in $\mathscr{K}$ corresponds to the one of indices in $\kappa_{a'b'c'}$. So, the l accompanying with index a' appears to the first order of indices in $\mathscr{K}$. Similarly, the second and third order of indices in $\mathscr{K}$ are the q with indices b' and c', respectively. Further, the same number in indices expresses as the same element of symmetric matrix. Along the same line as (3.4), we define

$$\mathscr{K}_{(l_{*a}\rho_{*1}\rho_{*2})(l_{*b}\rho_{*1}\rho_{*2})} = \sum_{a'b'c'd'e'f'}^{p} l_{a'a} l_{d'b} \rho_{b'e'} \rho_{c'f'} \kappa_{a'b'c'} \kappa_{d'e'f'}.$$

Other constants are defined similarly.

THEOREM 3.1. *Suppose that the design matrix A and the error matrix $\mathscr{E}$ in* (1.1) *satisfy the assumptions* B1, B2, B3, B4 *and* B5. *Let $\boldsymbol{u} = \mathrm{vec}(U)$, then the distribution function of U can be expanded as*

$$\begin{aligned} &\mathrm{P}(\mathrm{vec}(U) \le \boldsymbol{x}) \\ &\quad = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{kd}} \phi_{kd}(\boldsymbol{u}) \left[1 + \frac{1}{\sqrt{n}} R_1(\boldsymbol{u}) + \frac{1}{n} R_2(\boldsymbol{u})\right] d\boldsymbol{u} + \mathrm{o}(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} R_1(\boldsymbol{u}) = &-\frac{1}{2}\sum_{a'}^{k}\sum_{ab}^{d} \bar{\chi}_{a'}(\mathscr{K}_{(l_{*a}q_{*1}q_{*1})} + 2\mathscr{K}_{(l_{*a}\rho_{*1}\rho_{*1})}) H_{a'a}(\boldsymbol{u}) \\ &+ \frac{1}{6}\sum_{a'b'c'}^{k}\sum_{abc}^{d} (\bar{\chi}_{a'b'c'} - 3\bar{\chi}_{a'}\delta_{b'c'}) \mathscr{K}_{(l_{*a}l_{*b}l_{*c})} H_{a'a,b'b,c'c,d'd}(\boldsymbol{u}), \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
R_2(\boldsymbol{u}) = \frac{1}{8}\sum_{a'b'}^{k}\sum_{abcd}^{d}\bar{\chi}_{a'}\bar{\chi}_{b'}(&\mathscr{K}_{(l_{*a}q_{*1}q_{*1})(l_{*b}q_{*2}q_{*2})} + 3\mathscr{K}_{(l_{*a}l_{*b}q_{*1})(q_{*1}q_{*2}q_{*2})} \\
&+ 4\mathscr{K}_{(l_{*a}q_{*1}q_{*2})(l_{*b}q_{*1}q_{*2})} + 4\mathscr{K}_{(l_{*a}\rho_{*1}\rho_{*1})(l_{*b}\rho_{*2}\rho_{*2})} + 8\mathscr{K}_{(l_{*a}l_{*b}\rho_{*1})(\rho_{*1}\rho_{*2}\rho_{*2})} \\
&+ 12\mathscr{K}_{(l_{*a}\rho_{*1}\rho_{*2})(l_{*b}\rho_{*1}\rho_{*2})} + 4\mathscr{K}_{(l_{*a}l_{*b}\rho_{*1})(q_{*1}q_{*1}\rho_{*1})} + 12\mathscr{K}_{(l_{*a}q_{*1}\rho_{*1})(l_{*b}q_{*1}\rho_{*1})} \\
&+ 4\mathscr{K}_{(l_{*a}l_{*b}q_{*1})(q_{*1}\rho_{*1}\rho_{*1})} + 4\mathscr{K}_{(l_{*a}q_{*1}q_{*1})(l_{*b}\rho_{*1}\rho_{*1})})H_{a'a,b'b}(\boldsymbol{u}) \\
+ \frac{1}{2}\sum_{a'}^{k}\sum_{a}^{d}&\{(k+3p-2q+1) + \omega_{a'a'}(p-q)\}H_{a'a,a'a}(\boldsymbol{u}) \\
+ \frac{1}{24}\sum_{a'b'c'd'}^{k}\sum_{abcd}^{d}&[(\bar{\chi}_{a'b'c'd'} - 3\delta_{a'b'}\delta_{c'd'})\mathscr{K}_{(l_{*a}l_{*b}l_{*c}l_{*d})} \\
&- 2\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\{\mathscr{K}_{(l_{*a}l_{*b}l_{*c})(l_{*d}q_{*1}q_{*1})} + 3\mathscr{K}_{(l_{*a}l_{*d}q_{*1})(l_{*b}l_{*c}q_{*1})} \\
&+ 2\mathscr{K}_{(l_{*a}l_{*b}l_{*c})(l_{*d}\rho_{*1}\rho_{*1})} + 6\mathscr{K}_{(l_{*a}l_{*d}\rho_{*1})(l_{*b}l_{*c}\rho_{*1})}\} + 3\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'}\{\mathscr{K}_{(l_{*a}q_{*1}q_{*1})(l_{*b}l_{*c}l_{*d})} \\
&+ \mathscr{K}_{(l_{*a}l_{*b}q_{*1})(l_{*c}l_{*d}q_{*1})} + 2\mathscr{K}_{(l_{*a}l_{*c}q_{*1})(l_{*b}l_{*d}q_{*1})} + 4\mathscr{K}_{(l_{*a}\rho_{*1}\rho_{*1})(l_{*b}l_{*c}l_{*d})} \\
&+ 4\mathscr{K}_{(l_{*a}l_{*b}\rho_{*1})(l_{*c}l_{*d}\rho_{*1})} + 8\mathscr{K}_{(l_{*a}l_{*c}\rho_{*1})(l_{*b}l_{*d}\rho_{*1})}\} \\
&+ 6\delta_{a'b'}\delta_{c'd'}\delta_{ac}\delta_{bc}]H_{a'a,a'b,c'c,d'd}(\boldsymbol{u}) \\
+ \frac{1}{72}\sum_{a'b'c'd'e'f'}^{k}\sum_{abcbdef}^{d}&(\bar{\chi}_{a'b'c'}\bar{\chi}_{d'e'f'} - 6\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\delta_{e'f'} + 9\bar{\chi}_{a'}\delta_{b'c'}\bar{\chi}_{d'}\delta_{e'f'}) \\
&\times \mathscr{K}_{(l_{*a}l_{*b}l_{*c})(l_{*d}l_{*e}l_{*f})}H_{a'a,b'b,c'c,d'd,e'e,f'f}(\boldsymbol{u}). \qquad (3.6)
\end{aligned}
$$

Here $\phi_{kd}(\boldsymbol{u})$ is the probability density function of $\mathrm{N}_{kd}(\boldsymbol{0}, I_{kd})$ *given by* $\phi_{kd}(\boldsymbol{u}) = (2\pi)^{-kd/2}\exp(-\boldsymbol{u}'\boldsymbol{u}/2)$, *and* $H_{a_1'a_1,\dots,a_j'a_j}(\boldsymbol{u})$ *is the multivariate Hermite polynomial.*

In Theorem 3.1, the multivariate Hermite polynomial is defined by

$$H_{a_1'a_1,\dots,a_j'a_j}(\boldsymbol{u}) = (-1)^j\frac{\partial^j}{\partial u_{a_1'a_1}\dots\partial u_{a_j'a_j}}\phi_{kd}(\boldsymbol{u}),$$

where $u_{a'a}$ is the (a', a)th element of U. For example

$$H_{a'a}(\boldsymbol{u}) = u_{a'a},$$

$$H_{a'a,b'b}(\boldsymbol{u}) = u_{a'a}u_{b'b} - \delta_{ab}\delta_{a'b'},$$

$$H_{a'a,b'b,c'c}(\boldsymbol{u}) = u_{a'a}u_{b'b}u_{c'c} - \sum_{[3]} u_{a'a}\delta_{bc}\delta_{b'c'},$$

$$H_{a'a,b'b,c'c,d'd}(\boldsymbol{u}) = u_{a'a}u_{b'b}u_{c'c}u_{d'd} - \sum_{[6]} u_{a'a}u_{b'b}\delta_{cd}\delta_{c'd'} + \sum_{[3]} \delta_{ab}\delta_{cd}\delta_{a'b'}\delta_{c'd'},$$

$$H_{a'a,b'b,c'c,d'd,e'e,f'f}(\boldsymbol{u}) = u_{a'a}u_{b'b}u_{c'c}u_{d'd}u_{e'e}u_{f'f} - \sum_{[15]} u_{a'a}u_{b'b}u_{c'c}u_{d'd}\delta_{ef}\delta_{e'f'}$$
$$+ \sum_{[45]} u_{a'a}u_{b'b}\delta_{cd}\delta_{ef}\delta_{c'd'}\delta_{e'f'} + \sum_{[45]} \delta_{ab}\delta_{cd}\delta_{ef}\delta_{a'b'}\delta_{c'd'}\delta_{e'f'}.$$

Here $\sum_{[j]}$ means the sum of all j possible combinations of the sets a_i' and a_i, for example

$$\sum_{[3]} \delta_{a'b'}\delta_{c'd'}\delta_{ab}\delta_{cd} = \delta_{a'b'}\delta_{c'd'}\delta_{ab}\delta_{cd} + \delta_{a'c'}\delta_{b'd'}\delta_{ac}\delta_{bd} + \delta_{a'd'}\delta_{b'c'}\delta_{ad}\delta_{bc}.$$

It may be noted that we can demonstrate the validity of the expansion by the argument similar to the one as in Wakaki, Yanagihara and Fujikoshi (2000), which is based on the same manner as in Bhattacharya and Ghosh (1978). Moreover, the moment condition B4 will be replaced with $\mathrm{E}(\|\boldsymbol{\varepsilon}\|^4) < \infty$ as in Hall (1987).

4. Asymptotic expansions of the distribution functions of $\hat{\Xi}$ and its linear combination

4.1. Two types of standardizations

In this section we consider asymptotic expansions of the distribution functions for $\hat{\Xi}$ and its linear combination, where $\hat{\Xi}$ is the maximum likelihood estimator of Ξ under normality. Related to the construction of confidence intervals of Ξ and its linear combination, we consider following two types of standardizations.

(1) standardized $\hat{\Xi}$:

$$U_S = (A'A)^{1/2}(\hat{\Xi} - \Xi)(X'\Sigma^{-1}X)^{1/2},$$

(2) Studentized $\hat{\Xi}$:

$$U_T = \sqrt{n}(A'A)^{1/2}(\hat{\Xi} - \Xi)(X'S^{-1}X)^{1/2},$$

(3) standardized linear combination of $\hat{\Xi}$:

$$U_{SL} = \boldsymbol{a}'(\hat{\Xi} - \Xi)\boldsymbol{b}/\tau,$$

(4) Studentized linear combination of $\hat{\Xi}$:

$$U_{TL} = \boldsymbol{a}'(\hat{\Xi} - \Xi)\boldsymbol{b}/\hat{\tau},$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are $k \times 1$ and $q \times 1$ fixed vectors, respectively, and positive values τ and $\hat{\tau}$ are defined by

$$\tau^2 = \boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\boldsymbol{b}'(X'\Sigma^{-1}X)^{-1}\boldsymbol{b}, \qquad \hat{\tau}^2 = \frac{1}{n}\boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\boldsymbol{b}'(X'S^{-1}X)^{-1}\boldsymbol{b}.$$

Note that these standardizations have been proposed under normality. However, we shall see that such standardizations do work asymptotically under nonnormality. Under normality, Fujikoshi (1987, 1993a) and von Rosen (1997) derived asymptotic expansions of the distributions of these statistics. Further, its error bounds were discussed in Fujikoshi (1987, 1993a).

In this section, without loss of generality, we assume that $\Sigma = I_p$ as in previous sections. So, X is regarded as $\Sigma^{-1/2}X$. Therefore, τ^2 is rewritten as

$$\tau^2 = \boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\boldsymbol{b}'(X'X)^{-1}\boldsymbol{b}.$$

4.2. Asymptotic expansion of U_S

Let $M = X(X'X)^{-1/2}$ whose (a,b)th elements of M is denoted as m_{ab}. From (2.1), U_S can be expanded as

$$\begin{aligned} U_S = {} & ZM - \frac{1}{\sqrt{n}}Z(I_p - P_X)VM \\ & + \frac{1}{n}Z(I_p - P_X)\{V(I_p - P_X) + Z'Z\}M + \mathrm{O}_p(n^{-3/2}). \end{aligned} \tag{4.1}$$

From (4.1) we can expand the characteristic function $C_{U_S}(T_3)$ of U_S as

$$\begin{aligned} C_{U_S}(T_3) = {} & \mathrm{E}[\exp\{i\,\mathrm{tr}(T_3'ZM)\}] \\ & - \frac{i}{\sqrt{n}}\mathrm{E}[\mathrm{tr}\{T_3'Z(I_p - P_X)VM\}\exp\{i\,\mathrm{tr}(T_3'ZM)\}] \\ & + \frac{i}{n}\mathrm{E}[\mathrm{tr}\{T_3'Z(I_p - P_X)\{V(I_p - P_X) + Z'Z\}M\}\exp\{i\,\mathrm{tr}(T_3'ZM)\}] \\ & + \frac{i^2}{2n}\mathrm{E}[\{\mathrm{tr}(T_3'Z(I_p - P_X)VM)\}^2\exp\{i\,\mathrm{tr}(T_3'ZM)\}] + \mathrm{o}(n^{-1}), \end{aligned}$$

where T_3 is a $k \times q$ real matrix. Letting $T_1 = T_3M'$, we can see that the characteristic function can be evaluated by the same method as in Section 3. In this case, using the relations $M'M = I_q$, $MM' = P_X$ and $(I_p - P_X)M = 0$, we can obtain an expansion of $C_{U_S}(T_3)$, whose inversion yields an asymptotic expansion of the distribution function of U_S as in Theorem 4.1.

THEOREM 4.1. *Suppose that the design matrix A and the error matrix $\mathscr{E}$ in* (1.1) *satisfy the assumptions* B1, B2, B3, B4 *and* B5. *Let $\boldsymbol{u} = \mathrm{vec}(U_S)$, then the distribution function of U_S can be expanded as*

$$
\begin{aligned}
&\mathrm{P}(\mathrm{vec}(U_S) \le \boldsymbol{x})\\
&\quad = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{kq}} \phi_{kq}(\boldsymbol{u}) \left[1 + \frac{1}{\sqrt{n}} R_{S,1}(\boldsymbol{u}) + \frac{1}{n} R_{S,2}(\boldsymbol{u})\right] d\boldsymbol{u} + \mathrm{o}(n^{-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
R_{S,1}(\boldsymbol{u}) = &- \sum_{a'}^{k} \sum_{a}^{q} \bar{\chi}_{a'} \mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})} H_{a'a}(\boldsymbol{u})\\
&+ \frac{1}{6} \sum_{a'b'c'}^{k} \sum_{abc}^{q} \bar{\chi}_{a'b'c'} \mathscr{K}_{(m_{*a}m_{*b}m_{*c})} H_{a'a,b'b,c'c}(\boldsymbol{u}), \qquad (4.2)
\end{aligned}
$$

$$
\begin{aligned}
R_{S,2}(\boldsymbol{u}) = &\frac{1}{2} \sum_{a'b'}^{k} \sum_{ab}^{q} [(p-q)\delta_{a'b'}\delta_{ab} - \delta_{a'b'} \mathscr{K}_{(m_{*a}m_{*b}\rho_{*1}\rho_{*1})}\\
&\quad + \bar{\chi}_{a'}\bar{\chi}_{b'} \{\mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})(m_{*b}\rho_{*2}\rho_{*2})} + 2\mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(\rho_{*1}\rho_{*2}\rho_{*2})}\\
&\quad + 3\mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*2})(m_{*b}\rho_{*1}\rho_{*2})}\}] H_{a'a,b'b}(\boldsymbol{u})\\
&+ \frac{1}{24} \sum_{a'b'c'd'}^{k} \sum_{abcd}^{q} [12\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'} \mathscr{K}_{(m_{*a}m_{*c}\rho_{*1})(m_{*b}m_{*d}\rho_{*1})}\\
&\quad - 4\bar{\chi}_{a'b'c'}\bar{\chi}_{d'} \{\mathscr{K}_{(m_{*a}m_{*b}m_{*c})(m_{*d}\rho_{*1}\rho_{*1})} + 3\mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(m_{*c}m_{*d}\rho_{*1})}\}\\
&\quad + \bar{\chi}_{a'b'c'd'} \mathscr{K}_{(m_{*a}m_{*b}m_{*c}m_{*d})}] H_{a'a,b'b,c'c,d'd}(\boldsymbol{u})\\
&+ \frac{1}{72} \sum_{a'b'c'd'e'f'}^{k} \sum_{abcdef}^{q} \bar{\chi}_{a'b'c'}\bar{\chi}_{d'e'f'}\\
&\quad \times \mathscr{K}_{(m_{*a}m_{*b}m_{*c})(m_{*d}m_{*e}m_{*f})} H_{a'a,b'b,c'c,d'd,e'e,f'f}(\boldsymbol{u}). \qquad (4.3)
\end{aligned}
$$

Specially, when $\boldsymbol{\varepsilon}$ is distributed as $\mathrm{N}_p(\boldsymbol{0}, \Sigma)$,

$$
R_{S,1}(\boldsymbol{u}) = 0, \qquad R_{S,2}(\boldsymbol{u}) = \frac{1}{2}(p-q) \sum_{a'}^{k} \sum_{a}^{q} H_{a'a,a'a}(\boldsymbol{u}).
$$

Therefore,

$$
\begin{aligned}
&\mathrm{P}(\mathrm{vec}(U_S) \le \boldsymbol{x})\\
&\quad = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{kq}} \phi_{kq}(\boldsymbol{u}) \left[1 + \frac{1}{2n}(p-q) \sum_{a'}^{k} \sum_{a}^{q} H_{a'a,a'a}(\boldsymbol{u})\right] d\boldsymbol{u} + \mathrm{o}(n^{-1}).
\end{aligned}
$$

This result coincides with the formula in Fujikoshi (1987).

4.3. Asymptotic expansion of U_T

Let λ_i be the eigenvalues of $X'X$ and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_q)$. Further, let H be an orthogonal matrix of order q such that $(X'X) = H\Lambda H'$. Then, using a perturbation formula (see, Okamoto and Fujikoshi (1976)), we have

$$\sqrt{n}(X'S^{-1}X)^{1/2} = (X'X)^{1/2} + \frac{1}{\sqrt{n}}H\mathscr{S}^{(1)}H' + \frac{1}{n}H\mathscr{S}^{(2)}H' + \mathrm{O}_p(n^{-3/2}), \quad (4.4)$$

where the (i,j)th elements of $\mathscr{S}^{(1)}$ and $\mathscr{S}^{(2)}$ are defined by

$$(\mathscr{S}^{(1)})_{ij} = \frac{(B)_{ij}}{\sqrt{\lambda_i} + \sqrt{\lambda_j}}, \qquad (\mathscr{S}^{(2)})_{ij} = -\frac{(\mathscr{S}^{(1)^2})_{ij}}{\sqrt{\lambda_i} + \sqrt{\lambda_j}}.$$

Here $(\cdot)_{ij}$ is denoted as the (i,j)th element of the matrix in the parenthesis and the matrix B is defined by

$$B = -H'X'\left[V - \frac{1}{\sqrt{n}}(V^2 + Z'Z)\right]XH.$$

Substituting (4.4) into U_T, we can represent as

$$\begin{aligned} U_T = U_S &+ \frac{1}{\sqrt{n}}U_S(X'X)^{-1/2}H\mathscr{S}^{(1)}H' \\ &+ \frac{1}{n}U_S(X'X)^{-1/2}H\mathscr{S}^{(2)}H' + \mathrm{O}_p(n^{-3/2}). \end{aligned} \quad (4.5)$$

From (4.5), the characteristic function $C_{U_T}(T_3)$ of U_T can be expanded as

$$\begin{aligned} C_{U_T}(T_3) &= C_{U_S}(T_3) \\ &\quad + \frac{i}{\sqrt{n}}\mathrm{E}[\mathrm{tr}\{T_3'(X'X)^{-1/2}H\mathscr{S}^{(1)}H'\}\exp\{i\,\mathrm{tr}(T_3'U_S)\}] \\ &\quad + \frac{i}{n}\mathrm{E}[\mathrm{tr}\{T_3'(X'X)^{-1/2}H\mathscr{S}^{(2)}H'\}\exp\{i\,\mathrm{tr}(T_3'U_S)\}] \\ &\quad + \frac{i^2}{2n}\mathrm{E}[\{\mathrm{tr}\{T_3'(X'X)^{-1/2}H\mathscr{S}^{(1)}H'\}\}^2\exp\{i\,\mathrm{tr}(T_3'U_S)\}] + \mathrm{o}(n^{-1}). \end{aligned}$$

By computing $C_{U_T}(T_3)$ and inverting the resultant expansion, we can obtain an asymptotic expansion of U_T as in Theorem 4.2.

Theorem 4.2. *Suppose that the design matrix A and the error matrix $\mathscr{E}$ in* (1.1) *satisfy the assumptions* B1, B2, B3, B4 *and* B5. *Let $\boldsymbol{u} = \mathrm{vec}(U_T)$,*

$$\eta_{ij} = \frac{1}{\sqrt{\lambda_i} + \sqrt{\lambda_j}}, \qquad \nu_{ij} = \frac{\eta_{ij}}{\lambda_i},$$

and h_{ab}, $h_{ab}^{(1)}$ and $h_{ab}^{(2)}$ denote the (a,b)th elements of H, XH and $(X'X)^{1/2}H$ respectively. Then the distribution function of U_T can be expanded as

$$\begin{aligned}
&\mathrm{P}(\mathrm{vec}(U_T) \le \boldsymbol{x}) \\
&\quad = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{kq}} \phi_{kq}(\boldsymbol{u}) \left[1 + \frac{1}{\sqrt{n}} R_{T,1}(\boldsymbol{u}) + \frac{1}{n} R_{T,2}(\boldsymbol{u}) \right] d\boldsymbol{u} + \mathrm{o}(n^{-1}),
\end{aligned}$$

where

$$\begin{aligned}
R_{T,1}(\boldsymbol{u}) &= R_{S,1}(\boldsymbol{u}) + \sum_{a'}^{k} \sum_{abc}^{q} H_{a'a}(\boldsymbol{u}) \bar{\chi}_{a'} v_{bc} h_{ac} \mathscr{K}_{(h_{*b}^{(1)} h_{*b}^{(1)} h_{*c}^{(1)})} \\
&\quad - \sum_{a'b'c'}^{k} \sum_{abcde}^{q} H_{a'a,b'b,c'c}(\boldsymbol{u}) \bar{\chi}_{a'} \delta_{b'c'} h_{cd}^{(2)} v_{de} h_{be} \mathscr{K}_{(m_{*a} h_{*d}^{(1)} h_{*e}^{(1)})},
\end{aligned}$$

$$\begin{aligned}
&R_{T,2}(\boldsymbol{u}) \\
&\quad = R_{S,2}(\boldsymbol{u}) - \frac{1}{2} \sum_{a'}^{k} \sum_{abcde}^{q} H_{a'a,a'b}(\boldsymbol{u}) v_{cd} h_{ad} \\
&\qquad \times [2\delta_{de} \{ \mathscr{K}_{(m_{*b} h_{*c}^{(1)} h_{*c}^{(1)} h_{*d}^{(1)})} - h_{bc}^{(2)} \mathscr{K}_{(h_{*c}^{(1)} h_{*d}^{(1)} \delta_{*1} \delta_{*1})} \} \\
&\qquad + 2h_{bc}^2 \eta_{ce} \eta_{de} \mathscr{K}_{(h_{*c}^{(1)} h_{*d}^{(1)} h_{*e}^{(1)} h_{*e}^{(1)})} - \eta_{ce} h_{be} \mathscr{K}_{(h_{*c}^{(1)} h_{*c}^{(1)} h_{*d}^{(1)} h_{*e}^{(1)})} \\
&\qquad + \delta_{de} \{ 2h_{bd}^{(2)} (1 + \delta_{cd}) \lambda_c - 2(p+1) \lambda_c \delta_{cd} - 2(k+1) h_{bc}^{(2)} \delta_{cd} \\
&\qquad - 2h_{cd}^{(2)} - \eta_{ce} \lambda_c \lambda_d h_{bd} (1 + \delta_{cd}) + 2h_{bc}^{(2)} \eta_{ce}^2 \delta_{cd} (1 + \delta_{ce}) \}] \\
&\quad + \frac{1}{2} \sum_{a'b'}^{k} \sum_{abcdef} H_{a'a,b'b}(\boldsymbol{u}) \bar{\chi}_{a'} \bar{\chi}_{b'} v_{cd} h_{ad} [2\delta_{de} \delta_{df} \{ \mathscr{K}_{(m_{*b} h_{*c}^{(1)} h_{*d}^{(1)})(h_{*c}^{(1)} \rho_{*1} \rho_{*1})} \\
&\qquad + 2\mathscr{K}_{(m_{*b} h_{*c}^{(1)} \rho_{*1})(h_{*c}^{(1)} h_{*d}^{(1)} \rho_{*1})} + \mathscr{K}_{(m_{*b} \rho_{*1} \rho_{*1})(h_{*c}^{(1)} (h_{*c}^{(1)} h_{*d})} \\
&\qquad + \mathscr{K}_{(m_{*b} h_{*d}^{(1)} \delta_{*1})(h_{*c}^{(1)} h_{*c}^{(1)} \delta_{*1})} + \mathscr{K}_{(m_{*b} h_{*c}^{(1)} \delta_{*1})(h_{*c}^{(1)} h_{*d}^{(1)} \delta_{*1})} \} \\
&\qquad - 2\eta_{ce} \eta_{de} \delta_{ef} \{ \mathscr{K}_{(m_{*b} h_{*d}^{(1)} h_{*e}^{(1)})(h_{*c}^{(1)} h_{*c}^{(1)} h_{*e}^{(1)})} + \mathscr{K}_{(m_{*b} h_{*c}^{(1)} h_{*e}^{(1)})(h_{*c}^{(1)} h_{*d}^{(1)} h_{*e}^{(1)})} \} \\
&\qquad + v_{ef} b_{bf} \{ \mathscr{K}_{(h_{*c}^{(1)} h_{*c}^{(1)} h_{*d}^{(1)})(h_{*e}^{(1)} h_{*e}^{(1)} h_{*f}^{(1)})} + \mathscr{K}_{(h_{*c}^{(1)} h_{*d}^{(1)} h_{*e}^{(1)})(h_{*c}^{(1)} h_{*e}^{(1)} h_{*f}^{(1)})} \} \\
&\quad - \frac{1}{2} \sum_{a'b'} \sum_{abcdefgh} H_{a'a,a'b,b'c,b'd}(\boldsymbol{u}) h_{be}^{(2)} v_{ef} h_{af} \\
&\qquad \times [\delta_{fg} \delta_{fh} \mathscr{K}_{(m_{*c} m_{*d} h_{*e}^{(1)} h_{*f}^{(1)})} + h_{dg}^{(2)} v_{gh} h_{ch} \mathscr{K}_{(h_{*e}^{(1)} h_{*+f}^{(1)} h_{*g}^{(1)} h_{*h}^{(1)})} \\
&\qquad + \delta_{fg} \delta_{fh} \{ 2h_{ce}^{(2)} h_{df}^{(2)} + h_{de}^{(2)} h_{cf}^{(2)} + \lambda_e \lambda_f (h_{de}^{(2)} h_{cf} + h_{df}^{(2)} h_{ce}) \}]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{6}\sum_{a'b'c'd'}^{k}\sum_{abcdefgh}^{q} H_{a'a,b'b,c'c,d'd}(\boldsymbol{u})v_{ef} \\
&\quad\times[\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\delta_{fg}\delta_{fh}\{h_{df}\mathscr{K}_{(m_{*a}m_{*b}m_{*c})(h^{(1)}_{*e}h^{(1)}_{*e}h^{(1)}_{*f})} + 3h_{af}\mathscr{K}_{(m_{*b}m_{*c}h^{(1)}_{*e})(m_{*d}h^{(1)}_{*e}h^{(1)}_{*f})}\} \\
&\quad-3\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'}\{2h^{(2)}_{de}h_{cf}\delta_{fg}\delta_{fh}(\mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(h^{(1)}_{*e}h^{(1)}_{*f}\rho_{*1})} \\
&\quad+\mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})(m_{*b}h^{(1)}_{*e}h^{(1)}_{*f})} + \mathscr{K}_{(m_{*a}h^{(1)}_{*e}\delta_{*1})(m_{*b}h^{(1)}_{*f}\delta_{*1})}) \\
&\quad-2h^{(2)}_{de}\eta_{eg}\eta_{fg}h_{cf}\delta_{gh}\mathscr{K}_{(m_{*a}h^{(1)}_{*e}h^{(1)}_{*g})(m_{*b}h^{(1)}_{*f}h^{(1)}_{*g})} \\
&\quad+v_{eg}\lambda_e h_{cf}h_{dg}\mathscr{K}_{(m_{*a}h^{(1)}_{*e}h^{(1)}_{*f})(m_{*b}h^{(1)}_{*e}h^{(1)}_{*g})} \\
&\quad+h^{(2)}_{dg}v_{gh}h_{af}h_{ch}(\mathscr{K}_{(m_{*b}h^{(1)}_{*g}h^{(1)}_{*h})(h^{(1)}_{*e}h^{(1)}_{*e}h^{(1)}_{*f})} + \mathscr{K}_{(m_{*b}h^{(1)}_{*e}h^{(1)}_{*f})(h^{(1)}_{*e}h^{(1)}_{*g}h^{(1)}_{*h})}) \\
&\quad+h^{(2)}_{de}\eta_{gh}h_{cf}h_{ah}(\mathscr{K}_{(m_{*b}h^{(1)}_{*g}h^{(1)}_{*h})(h^{(1)}_{*e}h^{(1)}_{*f}h^{(1)}_{*g})} + \mathscr{K}_{(m_{*b}h^{(1)}_{*e}h^{(1)}_{*f})(h^{(1)}_{*g}h^{(1)}_{*g}h^{(1)}_{*h})})\}] \\
&-\frac{1}{6}\sum_{a'b'c'd'e'f'}^{k}\sum_{abcdefghij}^{q} H_{a'a,b'b,c'c,c'd,d'e,d'f}(\boldsymbol{u})h^{(2)}_{fi}v_{gh}h_{ej} \\
&\quad\times\{\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\delta_{e'f'}\delta_{gi}\delta_{hj}\mathscr{K}_{(m_{*a}m_{*b}m_{*c})}\mathscr{K}_{(m_{*d}h^{(1)}_{*g}h^{(2)}_{*h})} \\
&\quad-3\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'}\delta_{e'f'}h^{(2)}_{dg}v_{ij}h_{ch}\mathscr{K}_{(m_{*a}h^{(1)}_{*g}h^{(1)}_{*h})}\mathscr{K}_{(m^{(1)}_{*b}h^{(1)}_{*i}h^{(1)}_{*j})}\},
\end{aligned}
$$

and $R_{S,1}(\boldsymbol{u})$ *and* $R_{S,2}(\boldsymbol{u})$ *are given by* (4.2) *and* (4.3), *respectively.*

Specially, when $X = I_p$ then

$$U_T = Z\left(\frac{1}{n}S\right)^{-1/2}.$$

In this case, $(I_p - P_X) = O$, $H = I_p$ and all the λ_j are 1. Therefore,

$$
\begin{aligned}
R_{T,1}(\boldsymbol{u}) &= -\frac{1}{2}\sum_{a'}^{k}\sum_{ab}^{p}\bar{\chi}_{a'}\kappa_{abb}H_{a'a}(\boldsymbol{u}) \\
&\quad+\frac{1}{6}\sum_{a'b'c'}^{k}\sum_{abc}^{p}(\bar{\chi}_{a'b'c'} - 3\bar{\chi}_{a'}\delta_{b'c'})\kappa_{abc}H_{a'a,b'b,c'c}(\boldsymbol{u}),
\end{aligned}
$$

$$R_{T,2}(\boldsymbol{u}) = \frac{1}{8}\sum_{a'b'}^{k}\sum_{abcd}^{p}\bar{\chi}_{a'}\bar{\chi}_{b'}(\kappa_{acc}\kappa_{bdd} + 3\kappa_{abc}\kappa_{cdd} + 4\kappa_{acd}\kappa_{bcd})H_{a'a,b'b}(\boldsymbol{u})$$

$$+\frac{1}{2}(p+k+1)\sum_{a'}^{k}\sum_{a}^{p}H_{a'a,a'a}(\boldsymbol{u})$$

$$+\frac{1}{24}\sum_{a'b'c'd'}^{k}\left[\sum_{abcd}^{p}\{(\bar{\chi}_{a'b'c'd'} - 3\delta_{a'b'}\delta_{c'd'})\kappa_{abcd} + 6\delta_{a'b'}\delta_{c'd'}\delta_{ac}\delta_{bd}\}\right.$$

$$-2\sum_{abcde}^{p}\{\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}(\kappa_{abc}\kappa_{dee} + 3\kappa_{ade}\kappa_{bce})$$

$$\left.-3\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'}(\kappa_{aee}\kappa_{bcd} + \kappa_{abe}\kappa_{cde} + 2\kappa_{ace}\kappa_{bde})\}\right]H_{a'a,b'b,c'c,d'd}(\boldsymbol{u})$$

$$+\frac{1}{72}\sum_{a'b'c'd'e'f'}^{k}\sum_{abcbdef}^{p}(\bar{\chi}_{a'b'c'}\bar{\chi}_{d'e'f'} - 6\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\delta_{e'f'} + 9\bar{\chi}_{a'}\delta_{b'c'}\bar{\chi}_{d'}\delta_{e'f'})$$

$$\times\kappa_{abc}\kappa_{def}H_{a'a,b'b,c'c,d'd,e'e,f'f}(\boldsymbol{u}).$$

This result coincides with formula in Wakaki, Yanagihara and Fujikoshi (2000).

4.4. Asymptotic expansion of U_{SL}

Let $\tilde{\boldsymbol{a}} = (A'A)^{-1/2}\boldsymbol{a}$ and $\tilde{\boldsymbol{b}} = (X'X)^{-1/2}\boldsymbol{b}$, then U_{SL} can be rewritten as

$$U_{SL} = (\tilde{\boldsymbol{a}}'\tilde{\boldsymbol{a}})^{-1/2}(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1/2}\tilde{\boldsymbol{a}}'U_S\tilde{\boldsymbol{b}}.$$

Then we obtain $\mathrm{E}(U_{SL}) = 0$ and $\mathrm{Var}(U_{SL}) = 1 + \mathrm{o}(1)$. Therefore the characteristic function $C_{U_{SL}}(t)$ of U_{SL} can be written as

$$C_{U_{SL}}(t) = \mathrm{E}[\exp(itU_{SL})] = \mathrm{E}[\exp\{it\,\mathrm{tr}(\Pi'U_S)\}],$$

where $\Pi = (\tilde{\boldsymbol{a}}'\tilde{\boldsymbol{a}})^{-1/2}(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1/2}\tilde{\boldsymbol{a}}\tilde{\boldsymbol{b}}' = [\pi_{a'a}]$ is a $k\times q$ matrix. Note that an asymptotic expansion of $C_{U_{SL}}(t)$ can be obtained from the one of $C_{U_S}(T_3)$ as a special case. Therefore we have the following Theorem 4.3.

Theorem 4.3. *Suppose that the design matrix A and the error matrix $\mathscr{E}$ in* (1.1) *satisfy the assumptions* B1, B2, B3, B4 *and* B5. *Then the distribution function of U_{SL} can be expanded as*

$$\mathrm{P}(U_{SL} \leq x) = \Phi(x) + \frac{1}{\sqrt{n}} R_{SL,1}(x)\phi(x)1 + \frac{1}{n} R_{SL,2}(x)\phi(x) + \mathrm{o}(n^{-1}),$$

where

$$R_{SL,1}(x) = c_1 - \frac{1}{6} c_2 h_2(x), \tag{4.6}$$

$$R_{SL,2}(x) = -\frac{1}{2} c_3 h_1(x) - \frac{1}{24} c_4 h_3(x) - \frac{1}{72} c_5 h_5(x), \tag{4.7}$$

and

$$c_1 = \sum_{a'}^{k} \sum_{a}^{q} \bar{\chi}_{a'} \mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})} \pi_{a'a},$$

$$c_2 = \sum_{a'b'c'}^{k} \sum_{abc}^{q} \bar{\chi}_{a'b'c'} \mathscr{K}_{(m_{*a}m_{*b}m_{*c})} \pi_{a'a}\pi_{b'b}\pi_{c'c},$$

$$\begin{aligned} c_3 = \sum_{a'b'}^{k} \sum_{ab}^{q} [&(p-q)\delta_{a'b'}\delta_{ab} - \delta_{a'b'} \mathscr{K}_{(m_{*a}m_{*b}\rho_{*1}\rho_{*1})} \\ &+ \bar{\chi}_{a'}\bar{\chi}_{b'}\{\mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})(m_{*b}\rho_{*2}\rho_{*2})} + 2\mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(\rho_{*1}\rho_{*2}\rho_{*2})} \\ &+ 3\mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*2})(m_{*b}\rho_{*1}\rho_{*2})}\}]\pi_{a'a}\pi_{b'b}, \end{aligned}$$

$$\begin{aligned} c_4 = \sum_{a'b'c'd'}^{k} \sum_{abcd}^{q} [&12\bar{\chi}_{a'}\bar{\chi}_{b'}\delta_{c'd'} \mathscr{K}_{(m_{*a}m_{*c}\rho_{*1})(m_{*b}m_{*d}\rho_{*1})} \\ &- 4\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\{\mathscr{K}_{(m_{*a}m_{*b}m_{*c})(m_{*d}\rho_{*1}\rho_{*1})} + 3\mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(m_{*c}m_{*d}\rho_{*1})}\} \\ &+ \bar{\chi}_{a'b'c'd'} \mathscr{K}_{(m_{*a}m_{*b}m_{*c}m_{*d})}\}]\pi_{a'a}\pi_{b'b}\pi_{c'c}\pi_{d'd}, \end{aligned}$$

$$\begin{aligned} c_5 = \sum_{a'b'c'd'e'f'}^{k} \sum_{abcdef}^{q} &\bar{\chi}_{a'b'c'}\bar{\chi}_{d'e'f'} \\ &\times \mathscr{K}_{(m_{*a}m_{*b}m_{*c})(m_{*d}m_{*e}m_{*f})} \pi_{a'a}\pi_{b'b}\pi_{c'c}\pi_{d'd}\pi_{e'e}\pi_{f'f}. \end{aligned}$$

Here $\phi(x)$ *and* $\Phi(x)$ *are the probability density function and distribution function of* $\mathrm{N}(0,1)$, *respectively. Further,* $h_j(x)$ *are univariate Hermite polynomials, for example,* $h_1(x) = x$, $h_2(x) = x^2 - 1$, $h_3(x) = x(x^2 - 3)$ *and* $h_5(x) = x(x^4 - 10x^2 + 15)$.

Specially, when $\boldsymbol{\varepsilon}$ is distributed as $\mathrm{N}_p(\mathbf{0}, \Sigma)$, then

$$c_1 = c_2 = c_4 = c_5 = 0, \qquad c_3 = (p-q).$$

Therefore

$$\mathrm{P}(U_{SL} \le x) = \Phi(x) - \frac{1}{2n}(p-q)x\phi(x) + \mathrm{o}(n^{-1}).$$

This result coincides with formula in Fujikoshi (1987, 1993a).

4.5. Asymptotic expansion of U_{TL}

Note that

$$\begin{aligned}\hat{\tau}^2 &= \frac{1}{n}\boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\boldsymbol{b}'(X'S^{-1}X)^{-1}\boldsymbol{b}\\ &= \tilde{\boldsymbol{a}}'\tilde{\boldsymbol{a}}\Bigg[\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}} + \frac{1}{\sqrt{n}}\tilde{\boldsymbol{b}}'M'VM\tilde{\boldsymbol{b}}\\ &\qquad - \frac{1}{n}\tilde{\boldsymbol{b}}'M'\{V(I_p - P_X)V + Z'Z\}M\tilde{\boldsymbol{b}} + \mathrm{O}_p(n^{-3/2})\Bigg].\end{aligned}$$

Therefore, $\hat{\tau}^{-1}$ can be expanded as

$$\begin{aligned}\hat{\tau}^{-1} &= (\tilde{\boldsymbol{a}}'\tilde{\boldsymbol{a}})^{-1/2}(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1/2}\Bigg[1 - \frac{(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1}}{2\sqrt{n}}\tilde{\boldsymbol{b}}'M'VM\tilde{\boldsymbol{b}} + \frac{(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1}}{2n}\tilde{\boldsymbol{b}}'M'\\ &\qquad \times\left\{V\left(I_p - P_X + \frac{3}{4}\Theta\right)V + Z'Z\right\}M\tilde{\boldsymbol{b}} + \mathrm{O}_p(n^{-3/2})\Bigg]\\ &= (\tilde{\boldsymbol{a}}'\tilde{\boldsymbol{a}})^{-1/2}(\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1/2}\left[1 + \frac{\hat{\tau}_1}{\sqrt{n}} + \frac{\hat{\tau}_2}{n} + \mathrm{O}_p(n^{-3/2})\right],\end{aligned} \tag{4.8}$$

where $\Theta = (\tilde{\boldsymbol{b}}'\tilde{\boldsymbol{b}})^{-1}M\tilde{\boldsymbol{b}}\tilde{\boldsymbol{b}}'M'$ is a $p \times p$ matrix whose the (a,b)th element is θ_{ab}. Substituting (4.8) into U_{TL} yields

$$U_{TL} = U_{SL} + \frac{1}{\sqrt{n}}\hat{\tau}_1 U_{SL} + \frac{1}{n}\hat{\tau}_2 U_{SL} + \mathrm{O}_p(n^{-3/2}).$$

So, the characteristic function $C_{U_{TL}}(t)$ of U_{TL} is expressed as follow.

$$\begin{aligned}C_{U_{TL}}(t) &= \mathrm{E}[\exp\{itU_{SL}\}] + \frac{1}{\sqrt{n}}\mathrm{E}[it\hat{\tau}_1 U_{SL}\exp\{itU_{SL}\}]\\ &\quad + \frac{1}{n}\mathrm{E}\left[\left\{it\hat{\tau}_1 U_{SL} + \frac{(it)^2}{2}(\hat{\tau}_1 U_{SL})^2\right\}\exp\{itU_{SL}\}\right] + \mathrm{o}(n^{-1})\\ &= C_{U_{SL}}(t) + \frac{1}{\sqrt{n}}C_{U_{TL}}^{(1)}(t) + \frac{1}{n}C_{U_{TL}}^{(2)}(t) + \mathrm{o}(n^{-1}).\end{aligned}$$

In this expansion, $C_{U_{SL}}(t)$ has been derived in previous subsection. As for the computations of $C_{U_{TL}}^{(1)}(t)$ and $C_{U_{TL}}^{(2)}(t)$, we can get them by using the following equations.

$$C^{(1)}_{U_{TL}}(t) = t\frac{d}{dt}\mathrm{E}[it\hat{\tau}_1 \exp\{itU_{SL}\}],$$

$$C^{(2)}_{U_{TL}}(t) = t\frac{d}{dt}\mathrm{E}[it\hat{\tau}_2 \exp\{itU_{SL}\}] + \frac{t^2}{2}\frac{d^2}{dt^2}\mathrm{E}[it\hat{\tau}_1^2 \exp\{itU_{SL}\}].$$

After simplification, we have the following Theorem 4.4.

Theorem 4.4. *Suppose that the design matrix A and the error matrix $\mathscr{E}$ in* (1.1) *satisfy the assumptions* B1*,* B2*,* B3*,* B4 *and* B5*. Then the distribution function of U_{TL} can be expanded as*

$$\mathrm{P}(U_{TL} \le x) = \Phi(x) + \frac{1}{\sqrt{n}}R_{TL,1}(x)\phi(x)1 + \frac{1}{n}R_{TL,2}(x)\phi(x) + \mathrm{o}(n^{-1}),$$

where

$$\begin{aligned} R_{TL,1}(x) &= R_{SL,1}(x) + \frac{1}{2}c_6(1 + h_2(x)) \\ &= c_1 + \frac{1}{2}c_6 - \frac{1}{6}(c_2 - 3c_6)h_2(x), \\ R_{TL,2}(x) &= R_{SL,2}(x) - \frac{1}{2}c_7h_1(x) - \frac{1}{24}c_8h_3(x) - \frac{1}{24}c_9h_5(x) \\ &= -\frac{1}{2}(c_3 + c_7)h_1(x) - \frac{1}{24}(c_4 + c_8)h_3(x) - \frac{1}{72}(c_5 + 3c_9)h_5(x), \end{aligned}$$

and

$$\begin{aligned} c_6 &= \sum_{a'}^{k}\sum_{a}^{q}\bar{\chi}_{a'}\mathscr{K}_{(m_{*a}\theta_{*1}\theta_{*1})}\pi_{a'a}, \\ c_7 &= (p - q + k) + 2 + \mathscr{K}_{(\theta_{*1}\theta_{*1}\rho_{*1}\rho_{*1})} \\ &\quad + 2\sum_{a'b'}^{k}\sum_{ab}^{q}\bar{\chi}_{a'}\bar{\chi}_{b'}\{\mathscr{K}_{(m_{*a}\theta_{*1}\theta_{*2})(m_{*b}\theta_{*1}\theta_{*2})} + \mathscr{K}_{(m_{*a}\theta_{*1}\rho_{*1})(m_{*b}\theta_{*1}\rho_{*1})} \\ &\qquad - \mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})(m_{*b}\theta_{*1}\theta_{*1})} - \mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(\rho_{*1}\theta_{*1}\theta_{*1})}\}\pi_{a'a}\pi_{b'b}, \\ c_8 &= 6 - 3\mathscr{K}_{(\theta_{*1}\theta_{*1}\theta_{*2}\theta_{*2})} \\ &\quad + 12\sum_{a'b'}^{k}\sum_{ab}^{q}\bar{\chi}_{a'}\bar{\chi}_{b'}\{2\mathscr{K}_{(m_{*a}\theta_{*1}\theta_{*2})(m_{*b}\theta_{*1}\theta_{*2})} + \mathscr{K}_{(m_{*a}\theta_{*1}\rho_{*1})(m_{*b}\theta_{*1}\rho_{*1})} \\ &\qquad - \mathscr{K}_{(m_{*a}m_{*b}\rho_{*1})(\theta_{*1}\theta_{*1}\rho_{*1})} - \mathscr{K}_{(m_{*a}\rho_{*1}\rho_{*1})(m_{*b}\theta_{*1}\theta_{*1})}\}\pi_{a'a}\pi_{b'b} \\ &\quad - 8\sum_{a'b'c'd'}^{k}\sum_{abcd}^{q}\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\mathscr{K}_{(m_{*a}m_{*b}m_{*c})}\mathscr{K}_{(m_{*d}\theta_{*1}\theta_{*1})}\pi_{a'a}\pi_{b'b}\pi_{c'c}\pi_{d'd}, \end{aligned}$$

$$c_9 = 3\sum_{a'b'}^{k}\sum_{ab}^{q}\bar{\chi}_{a'}\bar{\chi}_{b'}\mathscr{K}_{(m_{*a}\theta_{*1}\theta_{*2})}\mathscr{K}_{(m_{*b}\theta_{*1}\theta_{*2})}\pi_{a'a}\pi_{b'b}$$
$$- 2\sum_{a'b'c'd'}^{k}\sum_{abcd}^{q}\bar{\chi}_{a'b'c'}\bar{\chi}_{d'}\mathscr{K}_{(m_{*a}m_{*b}m_{*c})}\mathscr{K}_{(m_{*d}\theta_{*1}\theta_{*1})}\pi_{a'a}\pi_{b'b}\pi_{c'c}\pi_{d'd}.$$

Here $R_{SL,1}(x)$ and $R_{SL,2}(x)$ are defined by (4.6) *and* (4.7), *respectively.*

Specially, when $\boldsymbol{\varepsilon}$ is distributed as $\mathrm{N}_p(\mathbf{0}, \Sigma)$, then

$$c_6 = c_9 = 0, \qquad c_7 = (p - q + k) + 2, \qquad c_8 = 6.$$

Therefore

$$\mathrm{P}(U_{TL} \le x) = \Phi(x) - \frac{1}{4n}\{2(2p - 2q + 2 + k)x + x(x^2 - 3)\}\phi(x) + \mathrm{o}(n^{-1}).$$

The corresponding results in the case when

$$\hat{\tau}^2 = \frac{1}{n-k}\boldsymbol{a}'(A'A)^{-1}\boldsymbol{a}\boldsymbol{b}'(X'S^{-1}X)^{-1}\boldsymbol{b},$$

was derived by Fujikoshi (1993a).

5. Asymptotic expansion of the null distribution of T_G

In this section, we derive an asymptotic expansion of the null distribution of T_G up to the order n^{-1}. Without loss of generality, we may replace $\mathscr{E}$ by $\mathscr{E}\Sigma^{-1/2}$ and X by $\Sigma^{-1/2}X$, in the expressions of T_G. Then $\mathrm{E}[\mathrm{vec}(\mathscr{E})] = \mathbf{0}$ and $\mathrm{Cov}[\mathrm{vec}(\mathscr{E})] = I_{np}$

Suppose that the design matrix A and the distribution of $\boldsymbol{\varepsilon}$ satisfy the assumptions B1, B2, B3, B4 and B5. Note that T_G is a smooth function of U. As $\mathrm{rank}(L) = d$ and $L'L = I_d$, by the same way as in Wakaki, Yanagihara and Fujikoshi (2000) and Bhattacharya and Rao (1976), it can be shown that T_G has a valid expansion up to the order n^{-1} under the assumptions B1, B2, B3, B4 and B5. In the following, we will find an asymptotic expansion of the characteristic function of T_G up to the order n^{-1}, which can be inverted formally. From (2.3), we can write the characteristic function of T_G as

$$C_{T_G}(t) = C_0(t) + \frac{1}{n}C_1(t) + \mathrm{o}(n^{-1}), \tag{5.1}$$

where

$$C_0(t) = \mathrm{E}[\exp\{it\,\mathrm{tr}(U'\Omega U)\}],$$
$$C_1(t) = it\mathrm{E}[\{(r_1 - k - (p - q))\,\mathrm{tr}(U'\Omega U) + r_2(\mathrm{tr}(U'\Omega U))^2\}\exp\{it\,\mathrm{tr}(U'\Omega U)\}].$$

As a method for computing each term in (5.1), we consider to use the results in Theorem 3.1.

By using the integrand function in Theorem 3.1, $C_0(t)$ can be given by

$$C_0(t) = \int_{\mathscr{R}^{kd}} \exp\{it\ \mathrm{tr}(U'\Omega U)\} \times \phi_{kd}(\boldsymbol{u})\left\{1 + \frac{1}{\sqrt{n}} R_1(\boldsymbol{u}) + \frac{1}{n} R_2(\boldsymbol{u})\right\} d\boldsymbol{u} + \mathrm{o}(n^{-1}),$$

where $\boldsymbol{u} = \mathrm{vec}(U)$, and $R_1(\boldsymbol{u})$ and $R_2(\boldsymbol{u})$ are defined by (3.5) and (3.6), respectively. Let $\varphi = (1 - 2it)^{-1}$ and $\Gamma = I_k + (\varphi - 1)\Omega$. Then note that

$$\exp\{it\ \mathrm{tr}(U'\Omega U)\} \exp\left\{-\frac{1}{2}\mathrm{tr}(U'U)\right\} = \exp\left\{-\frac{1}{2}\mathrm{tr}(U'\Gamma^{-1}U)\right\}.$$

Using the transformation from U to $U^* = \Gamma^{-1/2}U$ and the equation $\boldsymbol{u} = \mathrm{vec}(\Gamma^{1/2}U^*) = (I_d \otimes \Gamma^{1/2})\boldsymbol{u}^*$, where $\boldsymbol{u}^* = \mathrm{vec}(U^*)$, $C_0(t)$ is expressed as the expectation with respect to U^*, that is

$$C_0(t) = \mathrm{E}_{U^*}\left[1 + \frac{1}{\sqrt{n}} R_1((I_d \otimes \Gamma^{1/2})\boldsymbol{u}^*) + \frac{1}{n} R_2((I_d \otimes \Gamma^{1/2})\boldsymbol{u}^*)\right] + \mathrm{o}(n^{-1}). \quad (5.2)$$

Here the expectation may be taken with respect to U^* whose columns are independently distributed as $\mathrm{N}_d(\boldsymbol{0}, I_d)$. Let $W = \Gamma^{1/2}U^*$, it is seen that $\boldsymbol{w} = \mathrm{vec}(W) \sim \mathrm{N}_{kd}(\boldsymbol{0}, I_d \otimes \Gamma)$. Therefore the expansion (5.2) can be rewritten as

$$C_0(t) = \varphi^{cd/2}\mathrm{E}_W\left[1 + \frac{1}{\sqrt{n}} R_1(\boldsymbol{w}) + \frac{1}{n} R_2(\boldsymbol{w})\right] + \mathrm{o}(n^{-1}). \quad (5.3)$$

Applying a similar method to $C_1(t)$ yields

$$C_1(t) = \frac{1}{2}(1 - \varphi^{-1})\varphi^{cd/2}\mathrm{E}_W \times [\{(r_1 - k - (p - q))\ \mathrm{tr}(W'\Omega W) + r_2\{\mathrm{tr}(W'\Omega W)\}^2\}] + \mathrm{o}(1). \quad (5.4)$$

Through the calculations of (5.3) and (5.4), we use the following identities which are expectations of the Hermite polynomials, and some relations among the elements of Ω. As for the former, let the (a, b)th elements of Ω and Γ be denoted by ω_{ab} and γ_{ab}, respectively. Note that $\gamma_{ab} = \delta_{ab} + (\varphi - 1)\omega_{ab}$ and

$$\mathrm{E}_W[H_{a'a}(\boldsymbol{w})] = 0, \qquad \mathrm{E}_W[H_{a'a,b'b}(\boldsymbol{w})] = (\varphi - 1)\omega_{a'b'}\delta_{ab},$$

$$\mathrm{E}_W[H_{a'a,b'b,c'c}(\boldsymbol{w})] = 0,$$

$$\mathrm{E}_W[H_{a'a,b'b,c'c,d'd}(\boldsymbol{w})] = (\varphi-1)^2 \sum_{[3]} \omega_{a'b'}\omega_{c'd'}\delta_{ab}\delta_{cd}, \tag{5.5}$$

$$\mathrm{E}_W[H_{a'a,b'b,c'c,d'd,e'e,f'f}(\boldsymbol{w})] = (\varphi-1)^3 \sum_{[15]} \omega_{a'b'}\omega_{c'd'}\omega_{e'f'}\delta_{ab}\delta_{cd}\delta_{ef}.$$

As for the latter, using the property that Ω is an idempotent matrix,

$$\sum_{c'}^{k} \omega_{a'c'}\omega_{b'c'} = \omega_{a'b'},$$

$$\mathrm{tr}(\Omega) = \sum_{a'}^{k} \omega_{a'a'} = c, \qquad \mathrm{tr}(\Omega^2) = \sum_{a'b'}^{k} \omega_{a'b'}^2 = c, \tag{5.6}$$

$$\mathrm{tr}(\Omega^3) = \sum_{a'b'c'}^{k} \omega_{a'b'}\omega_{b'c'}\omega_{a'c'} = c, \qquad \mathrm{tr}(\Omega^4) = \sum_{a'b'c'd'}^{c} \omega_{a'b'}\omega_{b'c'}\omega_{c'd'}\omega_{a'd'} = c.$$

Substituting (5.5) and (5.6) into both of (5.3) and (5.4) yields

$$C_{T_G}(t) = \varphi^{cd/2}\left[1 + \frac{1}{n}\sum_{j=0}^{3} b_j\varphi^j + \mathrm{o}(n^{-1})\right], \tag{5.7}$$

where

$$\begin{aligned}
b_0 = {} & a_1\kappa_{41}^{(1)} - \{a_2 + a_4 - 2a_5 + (c-2)(c+1)a_6\}\kappa_{31}^{(1)} \\
& - \{a_3 - (c+1)a_4 - (c-2)a_5 - (c-2)a_6\}\kappa_{31}^{(2)} \\
& - 2\{3a_4 - 2(2c+1)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2ca_6\}\kappa_{321}^{(2)} \\
& - 2\{3a_5 - 2(c+1)a_6\}\kappa_{322}^{(2)} - 4a_6\{3\kappa_{33}^{(1)} + 2\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\
& + \frac{1}{4}cd\{(c-d-1) - 2r_1\}, \\
b_1 = {} & -2a_1\kappa_{41}^{(1)} + \{3a_2 - 6a_5 + (3c^2 + c - 6)a_6\}\kappa_{31}^{(1)} \\
& + \{3a_3 - (3c+4)a_4 - (3c-2)a_5 + (c+6)a_6\}\kappa_{31}^{(2)} \\
& + 4\{3a_4 - (4c+5)a_6\}\kappa_{32}^{(1)} + 4\{a_4 + a_5 - 2(c+1)a_6\}\kappa_{321}^{(2)} \\
& + 4\{3a_5 - (2c+3)a_6\}\kappa_{322}^{(2)} + 4a_6\{3\kappa_{33}^{(1)} + 2\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\
& - \frac{1}{2}cd\{c - r_1 + r_2(c+d+1)\},
\end{aligned}$$

$$
\begin{aligned}
b_2 &= a_1\kappa_{41}^{(1)} - \{3a_2 - 3a_4 - 6a_5 + (c+2)(3c-1)a_6\}\kappa_{31}^{(1)} \\
&\quad - \{3a_3 - (3c+5)a_4 - (3c+2)a_5 + 5(c+2)a_6\}\kappa_{31}^{(2)} \\
&\quad - 2\{3a_4 - 4(c+2)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2(c+2)a_6\}\kappa_{321}^{(2)} \\
&\quad - 2\{3a_5 - 2(c+2)a_6\}\kappa_{322}^{(2)} + \frac{1}{4}cd(c+d+1)(1+2r_2), \\
b_3 &= \{a_2 - 2a_4 - 2a_5 + (c+1)(c+2)a_6\}\kappa_{31}^{(1)} \\
&\quad + \{a_3 - (c+2)a_4 - (c+2)a_5 + 3(c+2)a_6\}\kappa_{31}^{(2)},
\end{aligned}
\tag{5.8}
$$

and

$$
\begin{aligned}
a_1 &= \frac{1}{24}\sum_{abcd}^{k}\bar{\chi}_{abcd}\sum_{[3]}\omega_{ab}\omega_{cd} - \frac{1}{8}c(c+2), \\
a_2 &= \frac{1}{72}\sum_{abcdef}^{k}\bar{\chi}_{abc}\bar{\chi}_{def}\sum_{[6]}{}'\omega_{ad}\omega_{be}\omega_{cf}, \\
a_3 &= \frac{1}{72}\sum_{abcdef}^{k}\bar{\chi}_{abc}\bar{\chi}_{def}\sum_{[9]}{}'\omega_{ab}\omega_{cd}\omega_{ef}, \\
a_4 &= \frac{1}{12}\sum_{abcd}^{k}\bar{\chi}_{abc}\bar{\chi}_{d}(\omega_{ab}\omega_{cd} + \omega_{ac}\omega_{bd}), \\
a_5 &= \frac{1}{12}\sum_{abcd}^{k}\bar{\chi}_{abc}\bar{\chi}_{d}\omega_{ad}\omega_{bc}, \qquad a_6 = \frac{1}{8}\sum_{ab}^{k}\bar{\chi}_{a}\bar{\chi}_{b}\omega_{ab}.
\end{aligned}
\tag{5.9}
$$

Here $\bar{\chi}_{a_1\ldots a_m}$ is defined by (3.2). Further $\kappa_{41}^{(1)}$, $\kappa_{31}^{(1)}$, $\kappa_{31}^{(2)}$, $\kappa_{32}^{(1)}$, $\kappa_{321}^{(2)}$, $\kappa_{322}^{(2)}$, $\kappa_{33}^{(1)}$, $\kappa_{331}^{(2)}$ and $\kappa_{332}^{(2)}$ are defined by

$$
\kappa_{41}^{(1)} = \sum_{abcd}^{p} q_{ab}q_{cd}\kappa_{abcd}, \qquad \kappa_{31}^{(1)} = \sum_{abcdef}^{p} q_{ad}q_{be}q_{cf}\kappa_{abc}\kappa_{def},
$$

$$
\kappa_{31}^{(2)} = \sum_{abcdef}^{p} q_{ab}q_{cd}q_{ef}\kappa_{abc}\kappa_{def}, \qquad \kappa_{32}^{(1)} = \sum_{abcdef}^{p} \rho_{ad}q_{be}q_{cf}\kappa_{abc}\kappa_{def},
$$

$$
\kappa_{321}^{(2)} = \sum_{abcdef}^{p} \rho_{ab}q_{cd}q_{ef}\kappa_{abc}\kappa_{def}, \qquad \kappa_{322}^{(2)} = \sum_{abcdef}^{p} q_{ab}\rho_{cd}q_{ef}\kappa_{abc}\kappa_{def},
$$

$$
\kappa_{33}^{(1)} = \sum_{abcdef}^{p} q_{ad}\rho_{be}\rho_{cf}\kappa_{abc}\kappa_{def}, \qquad \kappa_{331}^{(2)} = \sum_{abcdef}^{p} q_{ab}\rho_{cd}\rho_{ef}\kappa_{abc}\kappa_{def},
$$

$$\kappa_{332}^{(2)} = \sum_{abcdef}^{p} \rho_{ab} q_{cd} \rho_{ef} \kappa_{abc} \kappa_{def}.$$

Moreover, $\sum_{[6]}' \omega_{ad}\omega_{be}\omega_{cf}$ and $\sum_{[9]}' \omega_{ab}\omega_{cd}\omega_{ef}$ mean the following summations.

$$\sum_{[15]} \omega_{ab}\omega_{cd}\omega_{ef} = \sum_{[6]}{}' \omega_{ad}\omega_{be}\omega_{cf} + \sum_{[9]}{}' \omega_{ab}\omega_{cd}\omega_{ef},$$

$$\sum_{[6]}{}' \omega_{ad}\omega_{be}\omega_{cf} = \omega_{ad}\omega_{be}\omega_{cf} + \omega_{ad}\omega_{bf}\omega_{ce} + \omega_{ae}\omega_{bd}\omega_{cf}$$
$$+ \omega_{ae}\omega_{bf}\omega_{cd} + \omega_{af}\omega_{bd}\omega_{ce} + \omega_{af}\omega_{be}\omega_{cd}.$$

Finally, by inversing for (5.7), we have the following Theorem 5.1.

THEOREM 5.1. *Under the Assumptions* B1, B2, B3, B4 *and* B5, *the null distribution of T_G can be expanded as*

$$\mathrm{P}(T_G \le x) = G_{cd}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{cd+2j}(x) + \mathrm{o}(n^{-1}), \tag{5.10}$$

where G_f is the distribution function of a central chi-squared distribution with f degrees of freedom and the coefficients b_j are given by (5.8).

Note that the final result depends on the cumulants up to the fourth order. By the same way as in the multivariate t-statistic, it is expected that the assumption B4 may be weakened to $\mathrm{E}(\|\varepsilon\|^4) < \infty$.

As for a special case, we assume that $X = I_p$ and $D = I_p$. Then $Q = I_p$ and $(I_p - P_X) = O$. Therefore, $\kappa_{32}^{(1)}$, $\kappa_{321}^{(2)}$, $\kappa_{322}^{(2)}$, $\kappa_{33}^{(1)}$, $\kappa_{331}^{(2)}$ and $\kappa_{332}^{(2)}$ are 0 and $\kappa_{41}^{(1)}$, $\kappa_{31}^{(1)}$ and $\kappa_{31}^{(2)}$ are turned out the multivariate kurtosis and skewnesses (see, Mardia (1970) and Isogai (1985)) which are defined by

$$\kappa_4^{(1)} = \sum_{ab}^{p} \kappa_{aabb}, \qquad \kappa_3^{(1)} = \sum_{abc}^{p} \kappa_{abc}^2, \qquad \kappa_3^{(2)} = \sum_{abc}^{p} \kappa_{aab}\kappa_{bcc}.$$

So, the coefficients b_j are rewritten as

$$b_0 = a_1\kappa_4^{(1)} - \{a_2 + a_4 - 2a_5 + (c-2)(c+1)a_6\}\kappa_3^{(1)}$$
$$- \{a_3 - (c+1)a_4 - (c-2)a_5 - (c-2)a_6\}\kappa_3^{(2)}$$
$$+ \frac{1}{4}cp\{(c-p-1) - 2r_1\},$$

$$
\begin{aligned}
b_1 &= -2a_1\kappa_4^{(1)} + \{3a_2 - 6a_5 + (3c^2 + c - 6)a_6\}\kappa_3^{(1)} \\
&\quad + \{3a_3 - (3c+4)a_4 - (3c-2)a_5 + (c+6)a_6\}\kappa_3^{(2)} \\
&\quad - \frac{1}{2}cp\{c - r_1 + r_2(c+p+1)\}, \\
b_2 &= a_1\kappa_4^{(1)} - \{3a_2 - 3a_4 - 6a_5 + (c+2)(3c-1)a_6\}\kappa_3^{(1)} \\
&\quad - \{3a_3 - (3c+5)a_4 - (3c+2)a_5 + 5(c+2)a_6\}\kappa_3^{(2)} \\
&\quad + \frac{1}{4}cp(c+p+1)(1+2r_2), \\
b_3 &= \{a_2 - 2a_4 - 2a_5 + (c+1)(c+2)a_6\}\kappa_3^{(1)} \\
&\quad + \{a_3 - (c+2)a_4 - (c+2)a_5 + 3(c+2)a_6\}\kappa_3^{(2)}.
\end{aligned}
$$

These results are corresponding with the formula in Wakaki, Yanagihara and Fujikoshi (2000).

Before concluding this section, we state the next corollary which is an alternative of Theorem 5.1.

Corollary 5.1. *Under the same assumptions as in Theorem* 5.1*, the asymptotic expansion* (5.10) *can be written as*

$$
\begin{aligned}
&\mathrm{P}(T_G \le x) \\
&\quad = G_{cd}(x) - \frac{2x}{ncd} g_{cd}(x) \left\{ b_1 + b_2 + b_3 + \frac{(b_2 + b_3)x}{cd+2} + \frac{b_3 x^2}{(cd+2)(cd+4)} \right\} \\
&\qquad + \mathrm{o}(n^{-1}), \qquad (5.11)
\end{aligned}
$$

where $g_f(x)$ is the density function of a central chi-squared distribution with f degrees of freedom and the coefficients b_j are given by (5.8).

6. Robustness and conservativeness

6.1. Transformation of T_G

In this section, we consider certain conditions, which imply robustness and conservativeness of test statistics on effects of nonnormality. Under the condition that n is large enough, our test statistics can be regarded as robust for nonnormality because the limiting distributions in both of normal and nonnormal cases are the same. However, if n is not large enough, we can not regard test statistics as robust for nonnormality, because there are test statistics which can not be ignored an effect of nonnormality like the Hotelling's T^2 test statistic (see, e.g., Chase and Bulgren (1971) and Everitt (1979), etc). Our aim is to get theoretical tendencies of effects for nonnormality as well as numerical ones. On the other side, there is an important investigation on effect of heteroscedastic distribution (see, e.g., Ito (1969, 1980) and Yanagihara (2000)), but this paper is not taken a matter of such case.

We consider the Cornish-Fisher expansion for T_G, which is used as an approximation of the true percentage point. Let $t(u)$ and u denote the true percentage point of T_G and the percentage point of its limiting distribution, respectively, that is

$$\mathrm{P}(T_G \le t(u)) = \mathrm{P}(\chi^2_{cd} \le u),$$

where χ^2_{cd} is a variate of central chi-squared distribution with degrees of freedom cd. Then from (5.11), $t(u)$ can be expanded as

$$\begin{aligned} t(u) &= u + \frac{2u}{ncd}\left\{b_1 + b_2 + b_3 + \frac{(b_2+b_3)u}{cd+2} + \frac{b_3u^2}{(cd+2)(cd+4)}\right\} + \mathrm{o}(n^{-1}) \\ &= t_{\mathrm{E}}(u) + \mathrm{o}(n^{-1}). \end{aligned} \tag{6.1}$$

Through this section, when a correction term for nonnormality in the approximation $t_{\mathrm{E}}(u)$ is sufficiently small, we regard test statistics as robust for nonnormality.

First, we consider improved transformations on chi-squared approximations under normality (see, e.g., Fujikoshi (1997a, 2000), Kakizawa (1996) and Fujisawa (1997)). For example, for T_{HL} and T_{BNP},

$$\tilde{T}_{HL} = \frac{cd+2}{c+d+1}\left\{n + \frac{1}{2}(d-c-1)\right\}\log\left\{1 + \frac{c+d+1}{n(cd+2)}T_{HL}\right\},$$

$$\tilde{T}_{BNP} = T_{BNP} - \frac{c+d+1}{2(n+d)}\left\{1 - \frac{1}{cd+2}T_{BNP}\right\}T_{BNP}.$$

Under normality, the transformed test statistics satisfy

$$\mathrm{P}(\tilde{T}_{HL} \le x) = G_{cd}(x) + \mathrm{o}(n^{-1}), \qquad \mathrm{P}(\tilde{T}_{BNP} \le x) = G_{cd}(x) + \mathrm{o}(n^{-1}).$$

It is without saying that under normality,

$$\mathrm{P}(T_{LR} \le x) = G_{cd}(x) + \mathrm{o}(n^{-1}).$$

For these transformed statistics, the $t_{\mathrm{E}}(u)$ in (6.1) is represented as

$$t_{\mathrm{EN}}(u) = u + \frac{1}{n}\varDelta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)}),$$

where $\boldsymbol{\kappa}_3 = (\kappa_{31}^{(1)}, \kappa_{31}^{(2)}, \kappa_{32}^{(1)}, \kappa_{321}^{(2)}, \kappa_{322}^{(2)}, \kappa_{33}^{(1)}, \kappa_{331}^{(2)}, \kappa_{332}^{(2)})'$. Note that $\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)}) = 0$, when $\boldsymbol{\kappa}_3 = \mathbf{0}$ and $\kappa_{41}^{(1)} = 0$. So, we can regard test statistics as robust under nonnormality, when $\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)})$ is sufficiently small. If $\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)}) \leq 0$, then $t_{\mathrm{EN}}(u) \leq u$. Therefore, $t(u)$ is expected to be smaller than u. So, the test with a critical point u may be conservative more precisely, neglecting the terms of $\mathrm{o}(n^{-1})$,

$$\mathrm{P}(\tilde{T}_G \geq u) \leq \mathrm{P}(\tilde{T}_G \geq t(u)).$$

6.2. The case $\boldsymbol{\kappa}_3 = \mathbf{0}$

We consider the special case, $\boldsymbol{\kappa}_3 = \mathbf{0}$. Then $t_{\mathrm{EN}}(u)$ can be expressed as

$$t_{\mathrm{EN}}(u) = u - \frac{2a_1}{ncd}\kappa_{41}^{(1)} f_1(u), \tag{6.2}$$

where a_1 is defined by (5.9) and

$$f_1(u) = u\left\{1 - \frac{u}{cd+2}\right\}.$$

Therefore,

$$\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)}) = \Delta(u; \kappa_{41}^{(1)}) = -\frac{2a_1}{cd}\kappa_{41}^{(1)} f_1(u).$$

In a testing problem, since percentage points used are in the tail of distribution, for instance upper 10%, 5% and 1% points, so through the following arguments we assume that $u \geq cd + 2$, equivalently $f_1(u) \leq 0$. Actually, as for 10% points, if the degrees of freedom cd greater than 1, $u \geq cd + 2$. On the other hand, $u \geq cd + 2$ always holds in 5% and 1% points. From these equations, $\Delta(u; \kappa_{41}^{(1)}) \leq 0$ holds in the following two cases.

$$\text{(i)} \quad \kappa_{41}^{(1)} \geq 0 \quad \text{and} \quad a_1 \leq 0,$$

$$\text{(ii)} \quad \kappa_{41}^{(1)} < 0 \quad \text{and} \quad a_1 > 0.$$

First, we consider the case $\kappa_{41}^{(1)} \geq 0$ and $a_1 \leq 0$. For example, when T_G is the one-way MANOVA test statistics for testing an equality of mean vectors of k populations with each sample size n_i $(1 \leq i \leq k)$, a_1 becomes

$$a_1 = \frac{1}{8}\left(\sum_{a=1}^{k}\frac{n}{n_a} - k^2 - 2k + 2\right). \tag{6.3}$$

Then $a_1 \leq 0$ is equivalent to $\sum_{a=1}^{k} n/n_a \leq k^2 + 2k - 2$. It means that each sample size is not different extremely. If $\kappa_{41}^{(1)}$ and a_1 satisfy the condition (i),

Table 6.1: Actual test sizes of the one-way ANOVA test for several a_1 and κ_4

		Each value of a_1						
		-0.5	-0.43	-0.25	-0.05	0.06	0.42	1.03
κ_4	Nominal test sizes	Actual test sizes						
-1.2	10%	10.3	10.2	9.8	9.8	10.1*	9.9*	9.1*
	5%	5.3	5.2	5.2	5.2	5.0*	4.9*	4.1*
	1%	1.2	1.2	1.2	1.0	0.9*	0.8*	0.6*
1.5	10%	10.1*	10.3*	9.5*	9.4*	9.7	10.1	10.2
	5%	4.6*	5.0*	4.8*	4.6*	4.8	5.1	5.1
	1%	0.8*	0.9*	1.0*	1.1*	1.0	1.1	1.1
6.0	10%	9.7*	9.9*	9.7*	9.9*	9.7	10.2	10.6
	5%	4.5*	5.0*	4.5*	4.9*	5.0	5.2	5.6
	1%	0.9*	0.9*	0.9*	1.1*	1.0	1.1	1.3
43.2	10%	7.3*	8.3*	8.7*	10.6*	11.4	12.8	14.3
	5%	2.4*	3.3*	3.8*	4.8*	5.6	6.9	9.3
	1%	0.2*	0.2*	0.7*	0.9*	1.2	1.6	2.7
730.16	10%	4.1*	5.3*	8.5*	11.2*	14.1	17.5	17.1
	5%	1.1*	1.7*	2.8*	4.0*	5.0	10.7	13.5
	1%	0.1*	0.1*	0.5*	0.8*	1.2	1.5	5.2

*indicates a test with a critical point to be conservativeness asymptotically.

$t_{\mathrm{EN}}(u) \leq u$. Therefore, we can expect that true percentage point $t(u)$ have a tendency of $t(u) \leq u$, i.e.,

$$\mathrm{P}(\tilde{T}_G \geq u) \leq \mathrm{P}(\tilde{T}_G \geq t(u)).$$

It means that our test statistics may be conservative if $\kappa_{41}^{(1)} \geq 0$.

Next, we consider the case $\kappa_{41}^{(1)} < 0$ and $a_1 > 0$. Then, by the same condition as in the case $\kappa_{41}^{(1)} \geq 0$ and $a_1 \leq 0$, roughly we have, neglecting the terms of $\mathrm{o}(n^{-1})$

$$\mathrm{P}(\tilde{T}_G \geq u) \leq \mathrm{P}(\tilde{T}_G \geq t(u)).$$

However, since $\kappa_{41}^{(1)}$ has a lower bound, the region of $\kappa_{41}^{(1)}$ is narrower than the one in the case $\kappa_{41}^{(1)} \geq 0$ and $a_1 \leq 0$. For a univariate case, $-2 \leq \kappa_4$, where κ_4 is the univariate fourth cumulant.

The results of the simulations are shown in Table 6.1. A test statistic considered is simple, that is the one-way ANOVA test. In this case, a_1 is given by (6.3) and $\kappa_{41}^{(1)}$ is κ_4. We examine several κ_4 and a_1. From Table 6.1, it notes that this test statistics has robustness and conservativeness under condition $a_1 \leq 0$ and $\kappa_4 \geq 0$. Moreover, the cases of $a_1 > 0$ and $\kappa_4 < 0$ are similarly.

As mentioned above, if $\boldsymbol{\kappa}_3 = \mathbf{0}$, $a_1 \leq 0$ and $\kappa_{41}^{(1)} \geq 0$, then our test statistics have a tendency of robustness and conservativeness. Further it is similar for the case that $a_1 > 0$ and $\kappa_{41}^{(1)} < 0$.

6.3. The case $\kappa_3 \neq 0$

In this subsection, we consider the case $\boldsymbol{\kappa}_3 \neq \mathbf{0}$. However, in general form, as there are many factors to determine for a size of $\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)})$, it is difficult to obtain a simple condition for $\Delta(u; \boldsymbol{\kappa}_3, \kappa_{41}^{(1)}) \leq 0$.

Therefore, for simplicity, the one-way ANOVA test is taken up. In this case, $t_{\mathrm{EN}}(u)$ in (6.1) has a simple form as

$$t_{\mathrm{EN}}(u) = u + \frac{1}{n}\left\{a_2\kappa_3^2 - a_1\kappa_4 + \frac{u}{(k+1)}(a_1\kappa_4 - 2a_2\kappa_3^2) + \frac{u^2 a_2 \kappa_3^2}{(k+1)(k+3)}\right\}$$
$$= u + \frac{2u}{n(k-1)}\Delta(u; \kappa_3, \kappa_4),$$

where κ_3 and κ_4 are the third and fourth cumulants in the univariate case, a_1 is given by (6.3) and

$$a_2 = \frac{1}{24}\left(5\sum_{a=1}^{k}\frac{n}{n_a} - 3k^2 - 6k + 4\right).$$

Let

$$f_1(u) = 1 - \frac{u}{k+1}, \qquad f_2(u) = 1 - \frac{2u}{k+1} + \frac{u^2}{(k+1)(k+3)},$$

then the correction term $\Delta(u; \kappa_3, \kappa_4)$ can be expressed as

$$\Delta(u; \kappa_3, \kappa_4) = \frac{2u}{k-1}\{a_2 f_2(u)\kappa_3^2 - a_1 f_1(u)\kappa_4\}.$$

From the same reason in the previous subsection, we assume that $u \geq k+1$, equivalently $f_1(u) \leq 0$. For the condition $\Delta(u; \kappa_3, \kappa_4) \leq 0$, the following two cases are considered.

(1) $f_2(u) \leq 0$,

(2) $f_2(u) > 0$.

First, we consider the case $f_2(u) \le 0$. In this case, from $a_2 \ge 0$ and $\kappa_3^2 \ge 0$ the maximum value of the correction term becomes

$$\Delta_{\max} = \max\{\Delta(u; \kappa_3, \kappa_4)\} = \frac{2u}{k-1}\{-a_1 f_1(u)\kappa_4\}.$$

If $a_1 \le 0$ and $\kappa_4 \ge 0$ or $a_1 > 0$ and $\kappa_4 < 0$, then $\Delta_{\max} \le 0$, equivalently $t_{\mathrm{EN}}(u) \le u$. Therefore, we can expect that the true percentage point $t(u)$ have a tendency of $t(u) \le u$ under condition

$$\text{(i)} \quad \kappa_4 \ge 0, \quad a_1 \ge 0 \quad \text{and} \quad f_2(u) \ge 0,$$

$$\text{(ii)} \quad \kappa_4 < 0, \quad a_1 < 0 \quad \text{and} \quad f_2(u) \ge 0.$$

Next, we consider the case $f_2(u) > 0$. By using inequality $\kappa_3^2 \le \kappa_4 + 2$, the maximum value of $\Delta(u; \kappa_3, \kappa_4)$ can be written as

$$\begin{aligned} \Delta_{\max} &= \frac{2u}{k-1}\{a_2 f_2(u)(\kappa_4 + 2) - a_1 f_1(u)\kappa_4\} \\ &= \frac{2u}{k-1}\{(a_2 f_2(u) - a_1 f_1(u))\kappa_4 + 2a_2 f_2(u)\}. \end{aligned}$$

Therefore, $t_{\mathrm{EN}}(u) \le u$ holds under the following conditions.

$$\text{(iii)} \quad \kappa_4 \ge \frac{2a_2 f_2(u)}{a_1 f_1(u) - a_2 f_2(u)}, \quad a_2 f_2(u) - a_1 f_1(u) \le 0 \quad \text{and} \quad f_2(u) > 0,$$

$$\text{(iv)} \quad \kappa_4 < \frac{2a_2 f_2(u)}{a_1 f_1(u) - a_2 f_2(u)}, \quad a_2 f_2(u) - a_1 f_1(u) > 0 \quad \text{and} \quad f_2(u) > 0,$$

Then, we can roughly say that the test statistic is conservative. However κ_4 has a lower bound, $-2 \le \kappa_4$, the region κ_4 in condition (4) is narrower than the ones in other conditions.

Table 6.2 shows actual test sizes of the one-way ANOVA test statistic in several a_1, a_2, κ_3 and κ_4 by simulation studies. From Table 6.2, it is seen that this test tends to be robust and conservative. However, robustness and conservativeness are not kept in the tail side of percentage points under the conditions which are not included ones (1)$\sim$(4), that is $\kappa_4 > 0$, $a_1 > 0$ and $f_2(u) \le 0$.

In this section, we have seen that the coefficients of asymptotic expansion is useful for deciding a robustness and conservativeness of certain test statistics. For detailed simulation results in the one-way ANOVA and MANOVA test, see Fujikoshi, Ohmae and Yanagihara (1999) and Fujikoshi (2001) respectively.

TABLE 6.2: Actual test sizes of the one-way ANOVA test for several a_1, a_2, κ_3 and κ_4

Each value of $f_2(u)$ [-0.42 (10%), -0.50 (5%), -0.66 (1%)]

		$\kappa_3 = 1.0$, $\kappa_4 = 1.5$			$\kappa_3 = 2\sqrt{2}$, $\kappa_4 = 12.0$			$\kappa_3 = 6.18$, $\kappa_4 = 110.97$		
		Nominal sizes			Nominal sizes			Nominal sizes		
a_1	a_2	10%	5%	1%	10%	5%	1%	10%	5%	1%
−0.50	0.17	10.0*	4.9*	1.0*	8.7*	3.7*	0.7*	8.1*	3.6*	0.6*
−0.43	0.29	9.8*	4.8*	1.0*	8.9*	4.3*	0.8*	8.3*	3.5*	0.6*
−0.25	0.58	9.6*	5.2*	1.1*	8.7*	4.1*	1.0*	9.0*	4.4*	1.0*
−0.05	0.91	9.3*	4.8*	1.0*	8.7*	4.6*	1.1*	9.7*	4.8*	1.2*
0.06	1.10	9.7	4.7	1.1	9.1	5.1	1.3	9.9	5.1	1.5
0.42	1.70	9.7	4.9	1.2	9.5	5.8	1.9	9.8	5.7	2.0
1.03	2.72	9.6	5.2	1.3	10.2	6.7	2.7	11.8	8.1	3.1

*indicates a test with a critical point to be conservativeness asymptotically.

7. Some applications

7.1. Testing for equality on gradients

In this section, we obtain asymptotic expansions of the null distribution for several test statistics by applying Theorem 5.1.

First, we consider testing for equality on gradients. We assume that all the means of k populations are restricted to be linear with respect to time t_j, that is

$$\mu_{ij} = \xi_{i1} + \xi_{i2} t_j \qquad (1 \le i \le k, 1 \le j \le p),$$

Then consider testing for the null hypothesis

$$H_0\colon\ \xi_{12} = \cdots = \xi_{k2}.$$

In other words, this hypothesis means to test for the equality on gradients in the mean structure.

Let n_i be sample size of ith population, $n = \sum_i^k n_i$,

$$A = \begin{bmatrix} \mathbf{1}_{n_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{n_2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1}_{n_k} \end{bmatrix} \quad (n \times k \ \text{matrix}),$$

$$\Xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \vdots & \vdots \\ \xi_{k1} & \xi_{k2} \end{bmatrix} \quad (k \times 2 \text{ matrix}), \qquad X = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_p \end{bmatrix} \quad (p \times 2 \text{ matrix}), \tag{7.1}$$

$$C = \begin{bmatrix} 1 & \dots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & -1 \end{bmatrix} \quad (k-1 \times k \text{ matrix}), \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $\mathbf{1}_n$ is an n-dimensional vector all of whose elements are 1. In this case,

$$\Omega = I_k - \begin{bmatrix} n_1/n & \dots & \sqrt{n_1 n_k}/n \\ \vdots & \ddots & \vdots \\ \sqrt{n_k n_1}/n & \dots & n_k/n \end{bmatrix}.$$

Noting that

$$\sqrt{n}A(A'A)^{-1/2} = \begin{bmatrix} \sqrt{n/n_1}\mathbf{1}_{n_1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \sqrt{n/n_2}\mathbf{1}_{n_2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \sqrt{n/n_k}\mathbf{1}_{n_k} \end{bmatrix},$$

we can derive easily that

$$\bar{\chi}_a = \sqrt{n_a/n}, \qquad \bar{\chi}_{abc} = \begin{cases} \sqrt{n/n_a} & (a=b=c) \\ 0 & (\text{otherwise}) \end{cases}, \qquad \bar{\chi}_{abcd} = \begin{cases} n/n_a & (a=b=c=d) \\ 0 & (\text{otherwise}) \end{cases}. \tag{7.2}$$

Further, using $\omega_{ab} = \delta_{ab} - \sqrt{n_a n_b}/n$, we have

$$\begin{aligned} a_1 &= \frac{1}{8}\left(\sum_{a=1}^{k}\frac{n}{n_a} - k^2 - 2k + 2\right), \\ a_2 &= \frac{1}{12}\left(\sum_{a=1}^{k}\frac{n}{n_a} - 3k + 2\right), \\ a_3 &= \frac{1}{8}\left(\sum_{a=1}^{k}\frac{n}{n_a} - k^2\right), \qquad a_4 = a_5 = a_6 = 0. \end{aligned} \tag{7.3}$$

Next, we consider the assumptions B1, B2 and B3. It is easily shown that all $\|\boldsymbol{a}_j\| = 1$, $n^{-1}\sum_{j=1}^{n}\|\boldsymbol{a}_j\|^4 = 1$ and $n/n_j \le n/\lambda_n$. Therefore the assumptions

B1, B2 and B3 are replaced by

$$n/n_i = \mathrm{O}(1) \qquad (i = 1, 2, \dots, k). \tag{7.4}$$

These results imply an asymptotic expansion.

$$\mathrm{P}(T_G \le x) = G_{k-1}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{k-1+2j}(x) + \mathrm{o}(n^{-1}),$$

where

$$b_0 = a_1\kappa_{41}^{(1)} - (a_2\kappa_{31}^{(1)} + a_3\kappa_{31}^{(2)}) + \frac{1}{4}(k-1)(k-3-2r_1),$$

$$b_1 = -2a_1\kappa_{41}^{(1)} + 3(a_2\kappa_{31}^{(1)} + a_3\kappa_{31}^{(2)}) - \frac{1}{2}(k-1)\{k-1-r_1+r_2(k+1)\},$$

$$b_2 = a_1\kappa_{41}^{(1)} - 3(a_2\kappa_{31}^{(1)} + a_3\kappa_{31}^{(2)}) + \frac{1}{4}(k-1)(k+1)(1+2r_2),$$

$$b_3 = a_2\kappa_{31}^{(1)} + a_3\kappa_{31}^{(2)}.$$

For the special case $\Sigma = I_p$, Q becomes to

$$Q = (s_{(2)})^{-1}\begin{bmatrix} (pt_1 - s_{(1)})^2 & \dots & (pt_1 - s_{(1)})(pt_p - s_{(1)}) \\ \vdots & \ddots & \vdots \\ (pt_p - s_{(1)})(pt_1 - s_{(1)}) & \dots & (pt_p - s_{(1)})^2 \end{bmatrix},$$

where

$$s_{(1)} = \sum_a^p t_a, \qquad s_{(2)} = p^2\sum_a^p t_a^2 - p(s_{(1)})^2.$$

Therefore, $\kappa_{41}^{(1)}$, $\kappa_{31}^{(1)}$ and $\kappa_{31}^{(2)}$ can be rewritten more simple form as

$$\kappa_{41}^{(1)} = (s_{(2)})^{-2}\sum_{abcd}^{p}(pt_a - s_{(1)})(pt_b - s_{(1)})(pt_c - s_{(1)})(pt_d - s_{(1)})\kappa_{abcd},$$

$$\kappa_{31}^{(1)} = \kappa_{31}^{(2)} = (s_{(1)})^{-3}\sum_{abcdef}^{p}(pt_a - s_{(1)})(pt_b - s_{(1)})$$
$$\times (pt_c - s_{(1)})(pt_d - s_{(1)})(pt_e - s_{(1)})(pt_f - s_{(1)})\kappa_{abc}\kappa_{def}.$$

7.2. Testing for effect on quadratic

Secondly, we assume that the mean structure is quadratic, that is

$$\mu_{ij} = \xi_{i1} + \xi_{i2}t_j + \xi_{i3}t_j^2 \qquad (1 \le i \le k, 1 \le j \le p).$$

Then consider to test for the null hypothesis

$$H_0\colon\ \xi_{13} = \cdots = \xi_{k3} = 0.$$

Let the design matrix A be the same as (7.1) and

$$\Xi = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \vdots & \vdots & \vdots \\ \xi_{k1} & \xi_{k2} & \xi_{k3} \end{bmatrix} \quad (k \times 3 \text{ matrix}),$$

$$X = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_p & t_p^2 \end{bmatrix} \quad (p \times 3 \text{ matrix}),$$

$$C = I_k, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

From (7.2) and $\Omega = I_k$, we have

$$a_1 = \frac{1}{8}\left(\sum_{a=1}^{k} \frac{n}{n_a} - k^2 - 2k\right), \qquad a_2 = \frac{1}{12}\sum_{a=1}^{k} \frac{n}{n_a},$$

$$a_3 = \frac{1}{8}\sum_{a=1}^{k} \frac{n}{n_a}, \qquad a_4 = \frac{k}{6}, \qquad a_5 = \frac{k}{12}, \qquad a_6 = \frac{1}{8}.$$

These results imply an asymptotic expansion.

$$\mathrm{P}(T_G \le x) = G_k(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{k+2j}(x) + \mathrm{o}(n^{-1}),$$

where

$$\begin{aligned} b_0 = a_1\kappa_{41}^{(1)} &- \{a_2 + a_4 - 2a_5 + (c-2)(c+1)a_6\}\kappa_{31}^{(1)} \\ &- \{a_3 - (k+1)a_4 - (k-2)a_5 - (k-2)a_6\}\kappa_{31}^{(2)} \\ &- 2\{3a_4 - 2(2k+1)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2ka_6\}\kappa_{321}^{(2)} \\ &- 2\{3a_5 - 2(k+1)a_6\}\kappa_{322}^{(2)} - 4a_6\{3\kappa_{33}^{(1)} + 2\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\ &+ \frac{1}{4}k\{(k-2) - 2r_1\}, \end{aligned}$$

$$
\begin{aligned}
b_1 = & -2a_1\kappa_{41}^{(1)} + \{3a_2 - 6a_5 + (3k^2 + k - 6)a_6\}\kappa_{31}^{(1)} \\
& + \{3a_3 - (3k+4)a_4 - (3k-2)a_5 + (k+6)a_6\}\kappa_{31}^{(2)} \\
& + 4\{3a_4 - (4k+5)a_6\}\kappa_{32}^{(1)} + 4\{a_4 + a_5 - 2(k+1)a_6\}\kappa_{321}^{(2)} \\
& + 4\{3a_5 - (2k+3)a_6\}\kappa_{322}^{(2)} + 4a_6\{3\kappa_{33}^{(1)} + 2\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\
& - \frac{1}{2}k\{k - r_1 + r_2(k+2)\}, \\
b_2 = & \ a_1\kappa_{41}^{(1)} - \{3a_2 - 3a_4 - 6a_5 + (k+2)(3k-1)a_6\}\kappa_{31}^{(1)} \\
& - \{3a_3 - (3k+5)a_4 - (3k+2)a_5 + 5(k+2)a_6\}\kappa_{31}^{(2)} \\
& - 2\{3a_4 - 4(k+2)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2(k+2)a_6\}\kappa_{321}^{(2)} \\
& - 2\{3a_5 - 2(k+2)a_6\}\kappa_{322}^{(2)} + \frac{1}{4}k(k+2)(1+2r_2), \\
b_3 = & \ \{a_2 - 2a_4 - 2a_5 + (k+1)(k+2)a_6\}\kappa_{31}^{(1)} \\
& + \{a_3 - (k+2)a_4 - (k+2)a_5 + 3(k+2)a_6\}\kappa_{31}^{(2)}.
\end{aligned}
$$

Like a previous subsection, we can rewrite the assumptions B1, B2 and B3 as (7.4).

7.3. Testing for hierarchical structure

Thirdly, we consider to divide Ξ as

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix} \quad (k \times q \text{ matrix}),$$

where Ξ_{11} is a $k_1 \times (q - q_1)$ matrix, Ξ_{12} is a $k_1 \times q_1$ matrix, Ξ_{21} is a $(k - k_1) \times (q - q_1)$ matrix and Ξ_{22} is a $(k - k_1) \times q_1$ matrix. Related to a hierarchical structure of mean matrix, we are interested to test for the null hypothesis

$$H_0\colon\ \Xi_{12} = O.$$

Let

$$
\begin{aligned}
C &= [I_{k_1}, O_{k_1, k-k_1}] \quad (k_1 \times k \text{ matrix}), \\
D &= \begin{bmatrix} I_{q_1} \\ O_{q-q_1, q_1} \end{bmatrix} \quad (q \times q_1 \text{ matrix}),
\end{aligned}
$$

where O_{k_1,k_2} is a $k_1 \times k_2$ matrix all of whose elements are 0, then the null hypothesis can be rewritten as $C\Xi D = O$.

In order to be simple form on the coefficient a_j, suppose that the design matrix A is the same as (7.1). Then an asymptotic expansion is as follow.

$$\mathrm{P}(T_G \le x) = G_{k_1 q_1}(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{k_1 q_1 + 2j}(x) + \mathrm{o}(n^{-1}),$$

where

$$\begin{aligned}
b_0 &= a_1 \kappa_{41}^{(1)} - \{a_2 + a_4 - 2a_5 + (c-2)(c+1)a_6\}\kappa_{31}^{(1)} \\
&\quad - \{a_3 - (k_1+1)a_4 - (k_1-2)a_5 - (k_1-2)a_6\}\kappa_{31}^{(2)} \\
&\quad - 2\{3a_4 - 2(2k_1+1)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2k_1 a_6\}\kappa_{321}^{(2)} \\
&\quad - 2\{3a_5 - 2(k_1+1)a_6\}\kappa_{322}^{(2)} - 4a_6\{3\kappa_{33}^{(1)} + 4\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\
&\quad + \frac{1}{4}k_1 q_1\{(k_1 - q_1 - 1) - 2r_1\}, \\
b_1 &= -2a_1 \kappa_{41}^{(1)} + \{3a_2 - 6a_5 + (3k_1^2 + k_1 - 6)a_6\}\kappa_{31}^{(1)} \\
&\quad + \{3a_3 - (3k_1+4)a_4 - (3k_1-2)a_5 + (k_1+6)a_6\}\kappa_{31}^{(2)} \\
&\quad + 4\{3a_4 - (4k_1+5)a_6\}\kappa_{32}^{(1)} + 4\{a_4 + a_5 - 2(k_1+1)a_6\}\kappa_{321}^{(2)} \\
&\quad + 4\{3a_5 - (2k_1+3)a_6\}\kappa_{322}^{(2)} + 4a_6\{3\kappa_{33}^{(1)} + 2\kappa_{331}^{(2)} + \kappa_{332}^{(2)}\} \\
&\quad - \frac{1}{2}k_1 q_1\{k_1 - r_1 + r_2(k_1 + q_1 + 1)\}, \\
b_2 &= a_1 \kappa_{41}^{(1)} - \{3a_2 - 3a_4 - 6a_5 + (k_1+2)(3k_1-1)a_6\}\kappa_{31}^{(1)} \\
&\quad - \{3a_3 - (3k_1+5)a_4 - (3k_1+2)a_5 + 5(k_1+2)a_6\}\kappa_{31}^{(2)} \\
&\quad - 2\{3a_4 - 4(k_1+2)a_6\}\kappa_{32}^{(1)} - 2\{a_4 + a_5 - 2(k_1+2)a_6\}\kappa_{321}^{(2)} \\
&\quad - 2\{3a_5 - 2(k_1+2)a_6\}\kappa_{322}^{(2)} + \frac{1}{4}k_1 q_1 (k_1 + q_1 + 1)(1 + 2r_2), \\
b_3 &= \{a_2 - 2a_4 - 2a_5 + (k_1+1)(k_1+2)a_6\}\kappa_{31}^{(1)} \\
&\quad + \{a_3 - (k_1+2)a_4 - (k_1+2)a_5 + 3(k_1+2)a_6\}\kappa_{31}^{(2)},
\end{aligned}$$

and

$$a_1 = \frac{1}{8}\left(\sum_{a=1}^{k_1}\frac{n}{n_a} - k_1^2 - 2k_1\right), \qquad a_2 = \frac{1}{12}\sum_{a=1}^{k_1}\frac{n}{n_a},$$

$$a_3 = \frac{1}{8}\sum_{a=1}^{k_1}\frac{n}{n_a}, \qquad a_4 = \frac{k_1}{6}, \qquad a_5 = \frac{k_1}{12}, \qquad a_6 = \frac{1}{8}.$$

Then like the previous subsection, we can rewrite the assumptions B1, B2 and B3 as (7.4).

7.4. Generalized Hotelling's T^2 test

Finally, the model considered is defined by

$$\boldsymbol{y}_i = X\boldsymbol{\mu} + \varepsilon_i \qquad (1 \le i \le n),$$

where $\boldsymbol{\mu}$ is a $q \times 1$ unknown parameter vector. Then we deal with a testing for the null hypothesis

$$H_0 : C(\boldsymbol{\mu} - \boldsymbol{\mu}_0)'D = \mathbf{0}',$$

where

$$C = 1, \qquad D = \begin{bmatrix} I_d \\ O_{q-d,d} \end{bmatrix}.$$

This hypothesis means that some elements of an unknown parameter vector $\boldsymbol{\mu}$ are equal to certain values as the elements of $\boldsymbol{\mu}_0$. In this case, if the design matrix $A = \mathbf{1}_n$ and $\Xi = (\boldsymbol{\mu} - \boldsymbol{\mu}_0)'$, we can test for such hypothesis by using a test statistic as

$$T_G = (n - 1 - p + q)\,\mathrm{tr}(S_h S_e^{-1}).$$

Without saying, when $q = p$ and $D = I_q$, then its test statistic becomes to the basic Hotelling's T^2 test statistic

$$T_G = n(\bar{\boldsymbol{y}} - \boldsymbol{\mu}_0)'\left(\frac{1}{n-1}S\right)^{-1}(\bar{\boldsymbol{y}} - \boldsymbol{\mu}_0),$$

where $\bar{\boldsymbol{y}} = n^{-1}\sum_j^n \boldsymbol{y}_i$. As $\Omega = 1$, so $\bar{\chi}_{a_1\dots a_j} = 1$. Therefore

$$a_1 = -\frac{1}{4}, \qquad a_2 = \frac{1}{12}, \qquad a_3 = \frac{1}{8},$$

$$a_4 = \frac{1}{6}, \qquad a_5 = \frac{1}{12}, \qquad a_6 = \frac{1}{8}.$$

Using these coefficients a_j's, we can obtain b_j's as

$$b_0 = -\frac{1}{4}\kappa_{41}^{(1)} + \frac{1}{6}\kappa_{31}^{(1)} + \frac{1}{2}\kappa_{32}^{(1)} - \frac{3}{2}\kappa_{322}^{(2)} - \frac{3}{2}\kappa_{33}^{(1)} - \kappa_{331}^{(2)} - \frac{1}{2}\kappa_{332}^{(2)} - \frac{1}{4}d^2,$$

$$b_1 = \frac{1}{2}\kappa_{41}^{(1)} - \frac{1}{2}\kappa_{31}^{(1)} - \frac{5}{2}\kappa_{32}^{(1)} - \kappa_{321}^{(2)} - \frac{3}{2}\kappa_{322}^{(2)} + \frac{3}{2}\kappa_{33}^{(1)} + \kappa_{331}^{(2)} + \frac{1}{2}\kappa_{332}^{(2)} - \frac{1}{2}d,$$

$$b_2 = -\frac{1}{4}\kappa_{41}^{(1)} - \frac{1}{2}\kappa_{31}^{(2)} + 2\kappa_{32}^{(1)} + \kappa_{321}^{(2)} + \kappa_{322}^{(2)} + \frac{1}{4}d(d+2),$$

$$b_3 = \frac{1}{3}\kappa_{31}^{(1)} + \frac{1}{2}\kappa_{31}^{(2)}.$$

So, the asymptotic expansion based on these ones is given by

$$\mathrm{P}(T_G \le x) = G_d(x) + \frac{1}{n}\sum_{j=0}^{3} b_j G_{d+2j}(x) + \mathrm{o}(n^{-1}).$$

Specially, when $q = p$ and $D = I_q$, then

$$b_0 = -\frac{1}{4}\kappa_4^{(1)} + \frac{1}{6}\kappa_3^{(1)} - \frac{1}{4}p^2, \qquad b_1 = \frac{1}{2}\kappa_4^{(1)} - \frac{1}{2}\kappa_3^{(1)} - \frac{1}{2}p,$$

$$b_2 = -\frac{1}{4}\kappa_4^{(1)} - \frac{1}{2}\kappa_3^{(2)} + \frac{1}{4}p(p+2), \qquad b_3 = \frac{1}{3}\kappa_3^{(1)} + \frac{1}{2}\kappa_3^{(2)}.$$

These coefficients b_j's correspond to the ones in Kano (1995) and Fujikoshi (1997b).

8. Numerical accuracies

8.1. Case of confidence interval

Numerical accuracies are studied on confidence intervals of $\boldsymbol{a}'\Xi\boldsymbol{b}$ and the null distribution of two test statistics.

First, the confidence intervals are taken up. Kabe (1980) proposed a confidence interval for $\boldsymbol{a}'\Xi\boldsymbol{b}$ based on t-distribution. As we consider the nonnormal case, so it is necessary to consider different method in the normal case. Actually, in the nonnormal case, we use the Cornish-Fisher expansion as an approximation of the true percentage point. Let z_α be the α-level standard normal quantile, given by $\Phi(z_\alpha) = \alpha$ and $w_S(z_\alpha)$ and $w_T(z_\alpha)$ denote the true percentage points of U_{SL} and U_{TL}, respectively. In other words,

$$\mathrm{P}(U_{SL} \le w_S(z_\alpha)) = \Phi(z_\alpha),$$

$$\mathrm{P}(U_{TL} \le w_T(z_\alpha)) = \Phi(z_\alpha).$$

From Theorems 4.3 and 4.4, $w_S(z_\alpha)$ and $w_T(z_\alpha)$ can be expanded as

$$\begin{aligned} w_S(z_\alpha) &= z_\alpha + \frac{1}{\sqrt{n}}\psi_{S,1}(z_\alpha) + \frac{1}{n}\psi_{S,2}(z_\alpha) + \mathrm{o}(n^{-1}) \\ &= w_{S,E}(z_\alpha) + \mathrm{o}(n^{-1}), \\ w_T(z_\alpha) &= z_\alpha + \frac{1}{\sqrt{n}}\psi_{T,1}(z_\alpha) + \frac{1}{n}\psi_{T,2}(z_\alpha) + \mathrm{o}(n^{-1}) \\ &= w_{T,E}(z_\alpha) + \mathrm{o}(n^{-1}), \end{aligned}$$

where

$$\begin{aligned} \psi_{S,1} &= -Q_{SL,1}(z_\alpha) \\ &= -c_1 + \frac{1}{6}c_2 h_2(z_\alpha), \\ \psi_{S,2} &= Q_{SL,1}(z_\alpha)Q'_{SL,1}(z_\alpha) - \frac{1}{2}z_\alpha Q_{SL,1}(z_\alpha)^2 - Q_{SL,2}(z_\alpha) \\ &= -\frac{1}{3}c_2 z_\alpha\left\{c_1 - \frac{1}{6}c_2 h_2(z_\alpha)\right\} - \frac{1}{2}z_\alpha\left\{c_1 - \frac{1}{6}c_2 h_2(z_\alpha)\right\}^2 \\ &\quad + \frac{1}{2}c_3 h_1(z_\alpha) + \frac{1}{24}c_4 h_3(z_\alpha) + \frac{1}{72}c_5 h_5(z_\alpha), \\ \psi_{T,1} &= -Q_{TL,1}(z_\alpha) \\ &= -c_1 - \frac{1}{2}c_6 + \frac{1}{6}(c_2 - 3c_6)h_2(z_\alpha), \\ \psi_{T,2} &= Q_{TL,1}(z_\alpha)Q'_{TL,1}(z_\alpha) - \frac{1}{2}z_\alpha Q_{TL,1}(z_\alpha)^2 - Q_{TL,2}(z_\alpha) \\ &= -\frac{1}{3}(c_2 - 3c_6)\left\{c_1 + \frac{1}{2}c_6 - \frac{1}{6}(c_2 - 3c_6)h_2(z_\alpha)\right\} \\ &\quad - \frac{1}{2}z_\alpha\left\{c_1 + \frac{1}{2}c_6 - \frac{1}{6}(c_2 - 3c_6)h_2(z_\alpha)\right\}^2 \\ &\quad + \frac{1}{2}(c_3 + c_7)h_1(z_\alpha) + \frac{1}{24}(c_4 + c_8)h_3(z_\alpha) + \frac{1}{72}(c_5 + 3c_9)h_5(z_\alpha). \end{aligned}$$

Therefore, one-side α level intervals are given by

$$\begin{aligned} \mathscr{I}_1 &= (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \tau w_{S,E}(z_{1-\alpha})), \\ \mathscr{J}_1 &= (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau} w_{T,E}(z_{1-\alpha})). \end{aligned}$$

Similarly, two-side intervals can be expressed as

$$\mathscr{I}_2 = (\boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \tau w_{S,E}(z_{(1+\alpha)/2}), \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \tau w_{S,E}(z_{(1-\alpha)/2})),$$

$$\mathscr{J}_2 = (\boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau} w_{T,E}(z_{(1+\alpha)/2}), \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau} w_{T,E}(z_{(1-\alpha)/2})).$$

In actual use, we use $\hat{\mathscr{I}}_1$, $\hat{\mathscr{I}}_2$, $\hat{\mathscr{J}}_1$ and $\hat{\mathscr{J}}_2$, which are defined from $\mathscr{I}_1$, $\mathscr{I}_2$, $\mathscr{J}_1$ and $\mathscr{J}_2$ by replacing unknown parameters by their estimators, respectively. Let $\hat{w}_{S,E}(\cdot)$ and $\hat{w}_{T,E}(\cdot)$ be the ones defined from $w_{S,E}(\cdot)$ and $w_{T,E}(\cdot)$ by replacing κ_{abc} and κ_{abcd} by their estimators. Moreover, τ has to be replaced by $\hat{\tau}$ in $\hat{w}_{S,E}$. Then

$$\hat{\mathscr{I}}_1 = (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{S,E}(z_{1-\alpha})),$$

$$\hat{\mathscr{J}}_1 = (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{T,E}(z_{1-\alpha})),$$

$$\hat{\mathscr{I}}_2 = (\boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{S,E}(z_{(1+\alpha)/2}), \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{S,E}(z_{(1-\alpha)/2})),$$

$$\hat{\mathscr{J}}_2 = (\boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{T,E}(z_{(1+\alpha)/2}), \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}\hat{w}_{T,E}(z_{(1-\alpha)/2})).$$

Let

$$\tilde{Y} = (\tilde{\boldsymbol{y}}_1, \dots, \tilde{\boldsymbol{y}}_n)' = (Y - A\hat{\Xi}X')\hat{\Sigma}^{-1/2}, \tag{8.1}$$

where

$$\hat{\Sigma} = \frac{1}{n-k}(Y - A\hat{\Xi}X')'(Y - A\hat{\Xi}X'). \tag{8.2}$$

Then the unknown parameters κ_{abc} and κ_{abcd} can be estimated as

$$\begin{aligned} \hat{\kappa}_{abc} &= \frac{n}{(n-1)(n-2)}\sum_{j=1}^{n} \tilde{y}_a^{(j)}\tilde{y}_b^{(j)}\tilde{y}_c^{(j)}, \\ \hat{\kappa}_{abcd} &= \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{j=1}^{n} \tilde{y}_a^{(j)}\tilde{y}_b^{(j)}\tilde{y}_c^{(j)}\tilde{y}_d^{(j)}, \end{aligned} \tag{8.3}$$

where $\tilde{y}_a^{(j)}$ is the ath element of vector $\tilde{\boldsymbol{y}}_j$. For estimators of κ_{abc} and κ_{abcd}, see, e.g., Kaplan (1952), Mardia (1970) and Isogai (1985).

The model considered in simulation studies is $A = \mathbf{1}_n$, $X = I_p$, $\boldsymbol{a} = 1$ and $b = \mathbf{1}_p$. So, $\Pi = p^{-1/2}\mathbf{1}_p$ and $\Theta = p^{-1}\mathbf{1}_p\mathbf{1}_p'$. By using these settings, the coefficients c_j's can be written as

$$c_1 = 0, \qquad c_2 = \frac{1}{p\sqrt{p}} \sum_{abc}^{p} \kappa_{abc}, \qquad c_3 = 0, \qquad c_4 = \frac{1}{p^2} \sum_{abc}^{p} \kappa_{abcd},$$

$$c_5 = \frac{1}{p^3} \sum_{abcdef}^{p} \kappa_{abc}\kappa_{def}, \qquad c_6 = \frac{1}{p\sqrt{p}} \sum_{abc}^{p} \kappa_{abc}, \qquad c_7 = 3 + \frac{2}{p^3} \sum_{abcdef}^{p} \kappa_{abc}\kappa_{def},$$

$$c_8 = 6 - \frac{3}{p^2} \sum_{abc}^{p} \kappa_{abcd} + \frac{16}{p^3} \sum_{abcdef}^{p} \kappa_{abc}\kappa_{def}, \qquad c_9 = \frac{1}{p^3} \sum_{abcdef}^{p} \kappa_{abc}\kappa_{def}.$$

In order to estimate for each coefficient c_j, it is sufficient to do for $\gamma = \sum_{abc}^{p} \kappa_{abc}$ and $\kappa = \sum_{abcd}^{p} \kappa_{abcd}$ as

$$\hat{\gamma} = \sum_{abc}^{p} \hat{\kappa}_{abc}, \qquad \hat{\kappa} = \sum_{abcd}^{p} \hat{\kappa}_{abcd},$$

where $\hat{\kappa}_{abc}$ and $\hat{\kappa}_{abcd}$ are defined by (8.3). Therefore,

$$\hat{c}_2 = \frac{1}{p\sqrt{p}}\hat{\gamma}, \qquad \hat{c}_4 = \frac{1}{p^2}\hat{\kappa}, \qquad \hat{c}_5 = \frac{1}{p^3}\hat{\gamma}^2, \qquad \hat{c}_6 = \frac{1}{p\sqrt{p}}\hat{\gamma},$$

$$\hat{c}_7 = 3 + \frac{2}{p^3}\hat{\gamma}^2, \qquad \hat{c}_8 = 6 - \frac{3}{p^2}\hat{\kappa} + \frac{16}{p^3}\hat{\gamma}^2, \qquad \hat{c}_9 = \frac{1}{p^3}\hat{\gamma}^2.$$

The distributions considered are as follows,

1. *Multivariate Normal Distribution,*
2. *Uniform Distribution*: Each of the p variables is generated independently from a uniform $(-2, 2)$ distribution,
3. *Exponential Distribution*: Each of the p variables is generated independently from an exponential distribution with a mean of unity,
4. *Lognormal Distribution*: Each of the p variables is generated independently from a lognormal distribution such that $\log x \sim \mathrm{N}(0, 1)$.

Table 8.1 gives the following six probabilities on one-side 90%, 95% and 99% confidence intervals.

$$\alpha_1 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \mathscr{I}_1), \qquad \alpha_2 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \mathscr{J}_1),$$

$$\alpha_3 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \hat{\mathscr{I}}_1), \qquad \alpha_4 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \hat{\mathscr{J}}_1),$$

$$\alpha_5 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \mathscr{I}_{1,N}), \qquad \alpha_6 = \mathrm{P}(\boldsymbol{a}'\Xi\boldsymbol{b} \in \mathscr{J}_{1,N}),$$

where $\mathscr{I}_{1,N}$ and $\mathscr{J}_{1,N}$ are confidence intervals with normal error, that is

$$\mathscr{I}_{1,N} = (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}z_{SE,1-\alpha}),$$

$$\mathscr{J}_{1,N} = (-\infty, \boldsymbol{a}'\hat{\Xi}\boldsymbol{b} - \hat{\tau}z_{TE,1-\alpha}).$$

TABLE 8.1: Actual probabilities for confidence intervals of $\boldsymbol{a}'\Xi\boldsymbol{b}$

			Normal Nominal Levels			Uniform Nominal Levels			Exponential Nominal Levels			Log-Normal Nominal Levels		
n	p	α	90%	95%	99%	90%	95%	99%	90%	95%	99%	90%	95%	99%
5	2	α_1	90.0	94.9	98.9	90.7	95.3	99.1	90.0	95.0	98.7	66.2	80.9	99.9
		α_2	88.6	93.4	98.1	88.9	93.1	98.0	87.9	92.9	97.5	96.6	98.5	99.7
		α_3	83.2	88.6	93.7	83.3	88.4	93.3	74.9	79.6	86.4	67.8	72.8	79.8
		α_4	92.1	95.5	97.9	92.0	95.3	97.9	85.0	89.0	93.8	79.0	84.1	90.1
		α_5	84.6	89.4	94.6	84.9	89.6	94.4	76.1	81.1	87.9	69.2	74.0	81.7
		α_6	88.6	93.4	98.1	89.1	93.1	97.7	80.3	86.1	93.1	73.3	79.8	88.6
10	2	α_1	90.1	95.1	99.0	91.1	95.5	99.2	90.4	95.1	99.0	81.8	90.2	99.8
		α_2	89.4	94.5	98.5	90.2	94.8	99.0	89.3	94.1	98.5	96.7	98.8	99.8
		α_3	87.0	92.1	97.1	87.8	92.4	97.3	81.0	85.9	92.0	73.1	78.4	85.9
		α_4	90.4	94.8	98.3	92.0	95.7	98.8	86.8	91.1	95.8	80.5	86.6	93.2
		α_5	87.3	92.3	97.2	88.3	92.8	97.4	81.9	86.7	92.7	73.9	79.6	87.4
		α_6	89.4	94.5	98.5	90.3	94.8	98.9	84.1	89.1	95.3	76.6	82.8	91.2
	4	α_1	90.1	95.2	99.2	89.8	94.8	99.0	90.4	95.2	99.2	86.8	93.0	99.4
		α_2	89.5	94.8	98.8	89.0	94.5	98.7	89.7	94.7	98.9	94.2	97.3	99.7
		α_3	87.2	92.1	97.6	86.2	91.8	97.2	83.1	88.5	94.8	77.4	82.8	90.5
		α_4	90.5	95.0	98.4	90.3	94.7	98.5	87.1	91.9	96.5	81.6	87.4	93.3
		α_5	87.4	92.2	97.5	86.6	91.9	97.2	83.7	88.7	94.7	77.9	83.1	90.4
		α_6	89.5	94.8	98.8	89.0	94.5	98.7	85.9	91.2	97.0	79.7	86.2	93.4
	6	α_1	89.8	94.9	98.8	89.9	95.1	98.9	90.6	95.3	99.1	88.7	94.4	99.3
		α_2	89.6	94.8	98.7	89.2	94.7	98.7	89.9	94.9	98.7	93.4	97.2	99.5
		α_3	86.7	92.2	97.3	86.6	92.3	97.5	84.1	90.0	95.9	79.2	85.4	92.6
		α_4	90.7	95.1	98.4	90.6	95.3	98.3	87.4	91.9	96.6	82.4	87.5	93.4
		α_5	87.2	92.4	97.1	87.1	92.3	97.5	84.6	89.9	95.3	79.4	85.0	91.9
		α_6	89.6	94.8	98.7	89.3	94.7	98.7	86.8	92.3	97.2	81.9	87.9	94.5

Here, $z_{SE,\alpha}$ and $z_{TE,\alpha}$ are Cornish-Fisher expansions of α% point under normal error given by

$$z_{SE,\alpha} = \left\{1 + \frac{1}{2n}(p-q)\right\} z_\alpha,$$

$$z_{TE,\alpha} = z_\alpha + \frac{1}{n}\left\{\frac{1}{2}(2p - 2q + 2 + k)z_\alpha + \frac{1}{4} z_\alpha (z_\alpha^2 - 3)\right\}.$$

From Table 8.1, it seems that using $\mathscr{I}_1$ and $\mathscr{J}_1$ gives considerable improvements for the actual probability. Especially, the region $\mathscr{I}_1$ has high probabilities than $\mathscr{J}_1$. For an actual use, i.e., using the regions $\hat{\mathscr{I}}_1$, $\hat{\mathscr{J}}_1$, $\hat{\mathscr{I}}_{1,N}$ and $\hat{\mathscr{J}}_{1,N}$, Studentized intervals are better than standardized intervals.

However, the estimation problem for κ_{abc}, κ_{abcd} is left over. It will be necessary to find improved estimators. On the other side, it seems that critical points of these intervals do not give good approximations for the true percentage points in the case of log-normal distribution. Particularly, such effects happen in the case of small sample size, $n = 5$. As for a source, it seems that since cumulants κ_{abc} and κ_{abcd} are too much big, the effect of remainder term, e.g., the n^{-2} term in this case, becomes to large. For a study of two-side intervals, we have obtained similar results. Other methods of confidence intervals in a nonnormal ANOVA model have been discussed in Hall (1992).

8.2. Case of test statistic

In this subsection, we examine numerical studies for two test statistics. First, generalized Hotelling's T^2 statistic, which is denoted by T_G in Section 7.4, is taken up. Our purpose is to see how the actual test size closes to the nominal one by using the asymptotic expansion approximations. In the case generalized Hotelling's T^2 statistic, we can use a modified Cornish-Fisher expansion, which give an exact one in the normal error case. It is well known that $(n-d)T_G/d(n-1)$ is distributed as F-distribution with degrees of freedoms d and $n-d$. Using this fact, we can modify $t_{\mathrm{E}}(u)$ as

$$\begin{aligned} t(u) &= \frac{d(n-1)}{n-d}u_F - \frac{2u}{nd}\left\{b_0' - \frac{(b_2'+b_3')u}{d+2} - \frac{b_3'u^2}{(d+2)(d+4)}\right\} + \mathrm{o}(n^{-1}) \\ &= t_{\mathrm{E}}(u) + \mathrm{o}(n^{-1}), \end{aligned}$$

where u_F is the percentage point of F-distribution with degrees of freedoms d and $n-d$ and

$$\begin{aligned} b_0' &= -\frac{1}{4}\kappa_{41}^{(1)} + \frac{1}{6}\kappa_{31}^{(1)} + \frac{1}{2}\kappa_{32}^{(1)} - \frac{3}{2}\kappa_{322}^{(2)} - \frac{3}{2}\kappa_{33}^{(1)} - \kappa_{331}^{(2)} - \frac{1}{2}\kappa_{332}^{(2)}, \\ b_1' &= \frac{1}{2}\kappa_{41}^{(1)} - \frac{1}{2}\kappa_{31}^{(1)} - \frac{5}{2}\kappa_{32}^{(1)} - \kappa_{321}^{(2)} - \frac{3}{2}\kappa_{322}^{(2)} + \frac{3}{2}\kappa_{33}^{(1)} + \kappa_{331}^{(2)} + \frac{1}{2}\kappa_{332}^{(2)}, \\ b_2' &= -\frac{1}{4}\kappa_{41}^{(1)} - \frac{1}{2}\kappa_{31}^{(2)} + 2\kappa_{32}^{(1)} + \kappa_{321}^{(2)} + \kappa_{322}^{(2)}, \\ b_3' &= \frac{1}{3}\kappa_{31}^{(1)} + \frac{1}{2}\kappa_{31}^{(2)}. \end{aligned}$$

Without saying, if the error vectors are distributed as normal distribution, all the coefficients b_j''s are 0. As for the estimation of each cumulant, by using (8.1) and

$$\hat{Q} = \hat{X}(\hat{X}'\hat{X})^{-1}D\{D'(\hat{X}'\hat{X})^{-1}D\}^{-1}D'(\hat{X}'\hat{X})^{-1}\hat{X}',$$

where $\hat{X} = \hat{\Sigma}^{-1/2}X$ and $\hat{\Sigma}$ is defined by (8.2), we can estimate cumulants as follows.

$$\hat{\kappa}_{41}^{(1)} = \frac{n(n+1)}{(n-1)(n-2)(n-3)}\sum_{j=1}^{n}(\tilde{\boldsymbol{y}}_j'\hat{Q}\tilde{\boldsymbol{y}}_j)^2 - d(d+2),$$

$$\hat{\kappa}_{31}^{(1)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)^3,$$

$$\hat{\kappa}_{31}^{(2)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_i)(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)(\tilde{\boldsymbol{y}}_j'\hat{Q}\tilde{\boldsymbol{y}}_j),$$

$$\hat{\kappa}_{32}^{(1)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)^2,$$

$$\hat{\kappa}_{321}^{(2)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_i\}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)(\tilde{\boldsymbol{y}}_j'\hat{Q}\tilde{\boldsymbol{y}}_j),$$

$$\hat{\kappa}_{322}^{(2)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_i)\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\}(\tilde{\boldsymbol{y}}_j'\hat{Q}\tilde{\boldsymbol{y}}_j),$$

$$\hat{\kappa}_{33}^{(1)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\}^2,$$

$$\hat{\kappa}_{331}^{(2)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_i)\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\}\{\tilde{\boldsymbol{y}}_j'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\},$$

$$\hat{\kappa}_{332}^{(2)} = \left\{\frac{n}{(n-1)(n-2)}\right\}^2\sum_{ij}^{n}\{\tilde{\boldsymbol{y}}_i'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_i\}(\tilde{\boldsymbol{y}}_i'\hat{Q}\tilde{\boldsymbol{y}}_j)\{\tilde{\boldsymbol{y}}_j'(I_p - P_{\hat{X}})\tilde{\boldsymbol{y}}_j\}.$$

Table 8.2 gives the actual test sizes for the nominal 10%, 5% and 1% test in several cases of p, q and d. The distributions considered are the same four ones in the case of confidence interval. For each row in table, the top stairs express the actual test sizes based on F-distribution, the next and bottom stairs show the actual sizes by using $t_{\mathrm{E}}(u)$ and $\hat{t}_{\mathrm{E}}(u)$ which is defined from $t_{\mathrm{E}}(u)$ by replacing unknown parameters by their estimators, respectively. From Table 8.2, it seems that using $t_{\mathrm{E}}(u)$ gives a considerable improvement for the actual test size. However, there is a tendency that the approximation tends to be bad when p tends to large. Moreover, the estimation problem for several cumu-

Table 8.2: Actual test sizes of generalized Hotelling T^2 statistic

				Normal Nominal Sizes			Uniform Nominal Sizes			Exponential Nominal Sizes			Log-Normal Nominal Sizes		
n	p	q	d	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
10	2	2	2	9.7	5.1	0.7	11.4	6.5	1.2	19.9	13.8	7.4	28.2	21.9	12.3
				9.7	5.1	0.7	11.3	6.2	1.1	13.3	8.8	2.1	6.1	2.1	0.1
				8.5	3.8	0.4	9.6	5.1	0.9	16.6	10.8	4.7	23.4	16.7	7.6
		2	1	9.5	5.3	1.0	11.3	5.7	0.8	9.7	4.0	0.4	7.6	2.6	0.1
				9.5	5.3	1.0	11.3	5.7	0.8	9.4	4.2	0.4	6.1	2.8	0.2
				8.9	4.6	0.8	10.7	5.2	0.8	8.3	3.3	0.3	5.9	1.8	0.1
		1	1	10.9	5.9	1.3	10.1	5.8	1.2	19.0	13.0	5.2	28.3	20.6	11.2
				10.9	5.9	1.3	10.2	5.6	1.2	12.0	6.0	1.2	3.6	1.0	0.1
				8.6	4.1	0.6	8.3	3.8	0.4	15.3	9.4	2.6	22.9	15.4	7.3
	4	4	4	10.2	5.0	0.8	10.3	5.9	1.5	22.1	14.7	6.0	37.5	28.6	13.3
				10.2	5.0	0.8	10.2	5.8	1.5	15.1	8.8	2.9	7.3	3.3	0.5
				7.4	3.6	0.6	8.5	4.2	0.9	17.5	10.4	3.8	30.8	22.1	8.3
		4	2	10.8	5.5	1.0	11.4	5.9	1.5	15.4	10.4	3.7	23.6	16.7	7.3
				10.8	5.5	1.0	11.4	5.8	1.5	11.6	5.9	1.4	4.4	1.7	0.1
				9.0	4.3	0.5	10.1	4.8	0.9	13.3	8.0	2.4	19.7	12.3	4.3
		4	1	10.6	5.3	1.5	9.3	3.9	0.8	9.0	4.6	0.7	9.6	4.2	0.4
				10.6	5.3	1.5	9.3	3.9	0.8	8.3	4.5	0.7	5.5	3.9	0.3
				9.8	5.1	1.2	8.5	3.8	0.7	8.0	4.0	0.5	8.1	3.7	0.3
		2	2	13.1	7.8	2.1	12.7	7.2	2.0	22.2	15.4	7.1	36.3	26.4	14.1
				13.1	7.8	2.1	12.5	7.0	2.0	12.3	7.3	1.8	4.0	1.6	0.3
				7.4	3.6	0.7	6.9	3.4	0.7	14.4	8.7	2.9	24.6	16.7	7.8
		2	1	9.8	5.1	1.1	12.1	7.0	2.1	19.7	13.0	5.1	29.6	22.6	12.3
				9.8	5.1	1.1	12.1	6.9	2.1	10.1	5.1	1.1	2.0	0.7	0.1
				5.5	2.4	0.3	7.7	4.0	0.7	13.3	7.7	2.6	22.5	15.9	7.1
		1	1	11.5	6.5	2.0	11.4	6.5	2.3	19.6	13.9	5.5	29.5	21.7	10.4
				11.5	6.5	2.0	11.4	6.5	2.1	10.4	5.5	1.8	2.0	0.8	0.1
				5.3	2.4	0.6	5.7	3.0	0.4	12.5	7.6	2.7	19.6	12.8	5.3

lants are left over. As these values tend to be large, it is difficult to obtain good estimators. When $p = q$ and $d = 1$, almost the elements of Q become to small, equivalently each cumulant is small. Since an influence of nonnormality tends to little, we can regard the test statistic as robust in this case. To be not very striking, an effect of nonnormality in the case $q > d$ is small comparatively, based on the same reason in the former case $p = q$ and $d = 1$.

Moreover, it seems that the size of q does not affect the accuracy of actual size. On the other side, the critical points in the case of log-normal distribution do not give good approximations by the same reason in the experiment of confidence interval.

Secondly, we consider the Bartlett-Nanda-Pillai trace criterion. In this time, k populations case is considered because the previous study, generalized Hotelling's T^2 statistic, is the one population case. Further, we consider a simple situation. The between-individuals design matrix A used is defined by (7.1), the within-individuals design matrix X used is given by $\mathbf{1}_p$ and the restricted matrices are assumed by

$$C = \begin{bmatrix} 1 & \dots & 0 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & -1 \end{bmatrix} \quad (k-1 \times k \text{ matrix}), \qquad D = 1.$$

To set up this situation is equivalent to consider the mean structure $\mu_{ij} = \xi_i$ $(1 \le i \le k, 1 \le j \le p)$, and the null hypothesis H_0: $\xi_1 = \cdots = \xi_k$. Further, in this model, the coefficients a_j's are defined by (7.3) and Q becomes to a simple form as

$$Q = (\mathbf{1}_p' \Sigma \mathbf{1}_p)^{-1} \Sigma^{-1/2} \mathbf{1}_p \mathbf{1}_p' \Sigma^{-1/2}.$$

The approximations considered are the limiting distribution and the asymptotic expansions. These simulation studies are carried for $k = 3$, $p = 2$, 3 and 5. Moreover, we take the samples with exponential distribution. The reason why these samples are used is that the cumulants of this distribution are not so big, so it seems that such distribution suits this examination of effect on each cumulant.

Tables 8.3 and 8.4 give $t(u)$, u and $t_{\mathrm{E}}(u)$ and the actual test sizes based on these approximations of percentage points for nominal 5% and 1% test, respectively. Each actual size is given by

$$\alpha_7 = \mathrm{P}(T_G \ge u), \qquad \alpha_8 = \mathrm{P}(T_G \ge t_{\mathrm{E}}(u)), \qquad \alpha_9 = \mathrm{P}(T_G \ge \hat{t}_{\mathrm{E}}(u)).$$

Further, we have tried to study for other statistics, the likelihood ratio statistic and the Lawlye-Hotelling trace criterion, other several variates k and error models, and have obtained similar results. From Tables 8.3 and 8.4, we can see that to use $t_{\mathrm{E}}(u)$ or $\hat{t}_{\mathrm{E}}(u)$ gives a considerable improvement in a comparison with the limiting approximation.

TABLE 8.3: Nominal 5% test of the Bartlett-Nanda-Pillai trace criterion for $k = 3$.

	Sample Sizes			Percentage Points			Actual Sizes		
p	n_1	n_2	n_3	$t(u)$	u	$t_E(u)$	α_7	α_8	α_9
2	5	5	5	5.11	5.99	5.23	2.7	4.6	4.3
	10	10	10	5.75	5.99	5.61	4.3	5.5	5.3
	15	15	15	5.79	5.99	5.74	4.3	5.3	5.0
	5	10	15	5.75	5.99	5.60	4.7	5.5	5.5
	5	5	20	5.53	5.99	5.59	4.1	5.0	5.1
3	5	5	5	5.19	5.99	5.35	2.3	4.5	4.5
	10	10	10	5.69	5.99	5.67	4.5	5.1	5.0
	15	15	15	5.69	5.99	5.78	4.4	4.8	4.7
	5	10	15	5.53	5.99	5.67	3.8	4.5	4.5
	5	5	20	5.76	5.99	5.66	4.2	5.5	5.3
5	5	5	5	5.22	5.99	5.45	2.5	4.3	4.2
	10	10	10	5.65	5.99	5.72	3.9	4.9	4.3
	15	15	15	5.70	5.99	5.81	4.3	4.8	4.8
	5	10	15	5.96	5.99	5.72	5.0	5.8	5.5
	5	5	20	5.68	5.99	5.71	4.2	4.9	4.9

TABLE 8.4: Nominal 1% test of the Bartlett-Nanda-Pillai trace criterion for $k = 3$.

	Sample Sizes			Percentage Points			Actual Sizes		
p	n_1	n_2	n_3	$t(u)$	u	$t_E(u)$	α_7	α_8	α_9
2	5	5	5	7.01	9.21	6.40	0.3	1.7	1.9
	10	10	10	7.89	9.21	7.80	0.4	1.1	0.9
	15	15	15	8.04	9.21	8.27	0.4	0.9	0.9
	5	10	15	8.52	9.21	8.09	0.9	1.3	1.3
	5	5	20	8.17	9.21	8.44	0.8	0.9	0.9
3	5	5	5	6.92	9.21	6.80	0.1	1.2	1.7
	10	10	10	8.48	9.21	8.01	0.6	1.4	1.3
	15	15	15	7.63	9.21	8.41	0.8	1.1	1.1
	5	10	15	8.20	9.21	8.19	0.5	1.1	1.1
	5	5	20	9.18	9.21	8.43	1.0	1.3	1.3
5	5	5	5	6.74	9.21	7.13	0.1	0.7	1.6
	10	10	10	8.32	9.21	8.17	0.6	1.1	0.9
	15	15	15	8.53	9.21	8.52	0.8	1.1	1.0
	5	10	15	8.53	9.21	8.28	0.5	1.3	1.2
	5	5	20	8.32	9.21	8.42	0.7	0.9	1.0

Acknowledgements

I wish to express my deepest gratitude to Prof. Y. Fujikoshi of Hiroshima University for his valuable advices and encouragements and introducing me many fields of mathematical statistics during the academic years 1996–2000. In addition, I would like to thank Prof. S. Kageyama of Hiroshima University for helpful comments and suggestions and educating me primary statistics during 1994–1995. Also, I would like to thank Prof. M. Ohtaki of Hiroshima University for his advices in programming, and I would like to thank Dr. H. Wakaki of Hiroshima University for his recommendation in § 6. Further, I am grateful to Dr. T. Nakayama of Japan Atomic Energy Research Institute, Dr. K. Satoh of Hiroshima University and Dr. K. Kamo of Hiroshima University for their encouragements.

References

[1] Rabi N. Bhattacharya and J. K. Ghosh, On the validity of formal Edgeworth expansion, Ann. Statist. **6** (1978), 434–451.

[2] Rabi N. Bhattacharya and R. Ranga Rao, Normal Approximation and Asymptotic Expansions, John Wiley & Sons, New York, 1976.

[3] G. R. Chase and William G. Bulgren, A Monte Carlo investigations of the robustness of T^2, J. Amer. Statist. Assoc. **66** (1971), 499–502.

[4] Brian S. Everitt, A Monte Carlo investigation of the robustness of Hotelling's one- and two-sample T^2 tests, J. Amer. Statist. Assoc. **74** (1979), 48–51.

[5] Y. Fujikoshi, Asymptotic expansions of the non-null distributions of threestatistics in GMANOVA, Ann. Inst. Statist. Math. **26** (1974), 289–297.

[6] Y. Fujikoshi, Error bounds for asymptotic expansions of the distribution of MLE in a GMANOVA model, Ann. Inst. Statist. Math. **39** (1987), 153–161.

[7] Y. Fujikoshi, Asymptotic expansions oh the standardized and Studentized estimates in the growth curve model, J. Statist. Plann. Inference **36** (1993a), 165–174.

[8] Y. Fujikoshi, The growth curve model theory and its application, Surikagaku **358** (1993b), 60–66. (in Japanese)

[9] Y. Fujikoshi, A method for improving the large-sample chi-squared approximations to some multivariate analysis, Amer. J. Math. Management Sci. **17** (1997a), 15–29.

[10] Y. Fujikoshi, An asymptotic expansion for the distribution of Hotelling's T^2-statistic under nonnormality, J. Multivariate Anal. **61** (1997b), 187–193.

[11] Y. Fujikoshi, Transformations with improved chi-squared approximations, J. Multivariate Anal. **72** (2000), 249–263.

[12] Y. Fujikoshi, Asymptotic expansions for the distributions of multivariate basic statistics and one-way MANOVA tests under nonnormality, J. Statist. Plann. Inference (2001). (to appear)

[13] Y. Fujikoshi, M. Ohmae and H. Yanagihara, Asymptotic approximations of the null distribution of the one-way ANOVA test statistic under nonnormality, J. Japan Statist. Soc. **29** (1999), 147–161.

[14] H. Fujisawa, Improvement on chi-squared approximation by monotone transformation, J. Multivariate Anal. **60** (1997), 84–89.

[15] L. J. Gleser and I. Olkin, Linear models in multivariate analysis, Essays in Probability and Statistics, R. C. Bose, I. M. Chakravarti, P. C. Mahalanobis, C. R. Rao and K. J. C. Smith ed., University of North Carolina Press, Chapel Hill, North Carolina, 1970, 267–292.

[16] P. Hall, Edgeworth expansion for student's t statistic under minimal moment condition, Ann. Probab. **15** (1987), 920–931.

[17] P. Hall, The Bootstrap and Edgeworth Expansion, Springer-Verlag, New York, 1992.

[18] P. J. Huber, Robust regression: asymptotics, conjectures and Monte Calro, Ann. Statist. **1** (1973), 799–821.

[19] T. Isogai, On measures of multivariate skewness and kurtosis, Math. Japon. **28** (1983), 251–261.

[20] K. Ito, On the effect of heteroscedasticity and nonnormality upon some multivariate test procedures, Multivariate Analysis, Vol. 2, Academic Press, New York, 1969.

[21] K. Ito, Robustness of ANOVA and MANOVA test procedures, Handbook of Statistics, Vol. 1, P. R. Krishnaiah, ed., North-Holland Publishing Company, 1980, 199–236.

[22] D. G. Kabe, A remark on a paper by Misra, Biometrical J. **22** (1980), 275–278.

[23] Y. Kakizawa, Higher order monotone Bartlett type adjustment for some multivariate test statistics, Biometrika **83** (1996), 923–927.

[24] Y. Kano, An asymptotic expansion of the distribution of Hotellings T^2-statistic under general condition, Amer. J. Math. Management Sci. **15** (1995), 317–341.

[25] E. L. Kaplan, Tensor notation and the sampling cumulants of k-statistics, Biometrika **39** (1952), 319–323.

[26] T. Kariya, Testing in the Multivariate general Linear Model, Kinokuniya, Tokyo, 1985.

[27] C. G. Khatri, A note on a MANOVA model applied to problems in growth curve, Ann. Inst. Statist. Math. **18** (1966), 75–82.

[28] C. G. Khatri, Robustness study for a linear growth model, J. Multivariate Anal. **24** (1988), 66–87.

[29] Anant M. Kshirsagar and William Boyce Smith, GROWTH CURVES, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.

[30] K. V. Mardia, Measures of multivariate skewness and kurtosis with applications, Biometrika **57** (1970), 519–530.

[31] M. Okamoto and Y. Fujikoshi, Perturbation of a matrix function and its application to multivariate analysis, J. Japan Statist. Soc. **6** (1976), 33–37.

[32] R. F. Potthoff and S. N. Roy, A generalized multivariate analysis of variance model useful especially for growth curve problems, Biometrika **51** (1964), 313–325.

[33] D. von Rosen, The growth curve model: A review, Comm. Statist. Theory Methods **20** (1991), 2791–2822.

[34] D. von Rosen, Distribution and density approximations of a maximum likelihood estimator in the growth curve model, Statistics **29** (1997), 37–59.

[35] Muni S. Srivastava and D. von Rosen, Growth curve models, Multivariate Analysis, Design of Experiments, and Survey Sampling, S. Ghosh ed., Dekker, New York, 547–578, 1999.

[36] H. Wakaki, H. Yanagihara and Y. Fujikoshi, Asymptotic expansion of the null distributions of test statistics for multivariate linear hypothesis under nonnormality, TR No. 00–7, Statistical Research Group, Hiroshima University, 2000.

[37] H. Yanagihara, Asymptotic expansion of the null distribution of test statistic for linear hypothesis in nonnormal linear model, TR No. 99–16, Statistical Research Group, Hiroshima University, 1999.

[38] H. Yanagihara, Asymptotic expansion of the null distribution of one-way ANOVA test statistic for heteroscedastic case under nonnormality, Commun. Statist. Theory Meth. **29** (2000), 463–476.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739-8526, Japan
e-mail: yanagi@math.sci.hiroshima-u.ac.jp

Current address: Department of Statistical Methodology, The Institute of Statistical Mathematics, Minami-Azabu Minato-ku Tokyo 106-8567, Japan
hyanagi@ism.ac.jp