Majorizatiuon and Zipf-Mandelbrot law

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Abstract

In this paper we show how the Zipf-Mandelbrot law is connected to the theory of majorization. Firstly we consider the Csiszár f-divergence for the Zipf-Mandelbrot law and then develop important majorization inequalities for these divergences. We also discuss some special cases for our generalized results by using the Zipf-Mandelbrot law. As applications, we present the majorization inequalities for various distances obtaining by some special convex functions in the Csiszár f-divergence for Z-M law like the Rényi α-order entropy for Z-M law, variational distance for Z-M law, the Hellinger distance for Z-M law, χ^2-distance for Z-M law and triangular discrimination for Z-M law. At the end, we give important applications of the Zipf's law in linguistics and obtain the bounds for the Kullback-Leibler divergence of the distributions associated to the English and the Russian languages.

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1 Introduction and preliminaries

The frequency distribution of words has been a key object of study in statistical linguistics for the past 70 years. This distribution approximately follows a simple mathematical form known as the Zipf's law.

The term "Zipfian distribution" refers to "a distribution of probabilities of occurrence that follows the Zipf's Law" (see [32]). Zipf's law is an experiment law, not a theoretical one; i.e. it describes an occurrence rather than predicting it from some kind of theory. The observation that, in many natural and man-made phenomena, "the probability of occurrence of ... items starts high and tapers off. Thus, a few occur very often while many others occur rarely." The formal definition of this law is: $\mathbf{P_n} = 1/\mathbf{n^a}$, where $\mathbf{P_n}$ is the frequency of occurrence of the $\mathbf{n}$th ranked item and $\mathbf{a}$ is closed to 1.

Applied to language, this means that the rank of a word (in terms of its frequency) is approximately inversely proportional to its actual frequency, and so produces a hyperbolic distribution. To put the George Zipf's Law (1932) in another way: $fr = C$, where: $r =$ the rank of a word, $f =$ the frequency of occurrence of that word, and $C =$ a constant (the value of which depends on the subject under consideration). Essentially this shows an inverse proportional relationship between a word's frequency and its frequency rank. Zipf calls this curve the 'standard curve'. Texts from natural languages do not, of course, behave with such absolute mathematical precision. They can not, because, for one thing, any curve representing empirical data from large texts will be a stepped

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graph, since many non-high-frequency words will share the same frequency. But the overall consensus is that texts match the standard curve significantly well. Li (1992:1842) [17] writes "this distribution, also called the Zipf's law, has been checked for accuracy for the standard corpus of the present-day English [Kučera and Francis, 1967] with very good results. See Miller (1951:91-95) [24] for a concise summary of the match between actual data and the standard curve.

Zipf also studied the relationship between the frequency of occurrence of a word and its length. In The Psycho-Biology of Language (1935), he stated that "it seems reasonably clear that shorter words are distinctly more favoured in language other than words."

Apart from the use of this law in information science and linguistics, Zipf's law is used in economics. This distribution in economics is known as Pareto's law which analyze the distribution of the wealthiest members of the community [10, p.125] . These two laws are the same in the mathematical sense, but they are applied in a different context [11, p.294]. The same type of distribution that we have in Zipf's and Pareto's law, also known as the Power law, can be also found in other scientific disciplines, such as: physics, biology, earth and planetary sciences, computer science, demography and the social sciences [27].

Benoit Mandelbrot in 1966 [18] gave generalization of the Zipf's law, now known as the Zipf-Mandelbrot law, which gave improvement in account for the low-rank words in corpus where $k < 100$ [25]: $f(k) = \frac{C}{(k+t)^s}$ when $t = 0$, we get Zipf's law.
For $n \in \mathbb{N}$, $t \geq 0$, $s > 0$, $k \in \{1, 2, ..., n\}$, in a more clear form, Zipf-Mandelbrot law (probability mass function) is defined with

$$f(k, n, t, s) := \frac{1/(k+t)^s}{H_{n,t,s}}, \tag{1}$$

where,

$$H_{n,t,s} := \sum_{i=1}^{n} \frac{1}{(i+t)^s}. \tag{2}$$

Application of the Zipf-Mandelbrot law can also be found in linguistics [25], information sciences [11, 31] and ecological field studies [26].

In probability theory and statistics, the cumulative distribution function (CDF) of a real-valued random variable X, or just distribution function of X, evaluated at x, is the probability that X will take a value less than or equal to x and we often denote CDF as the following ratio:

$$\text{CDF} := \frac{H_{k,t,s}}{H_{n,t,s}}. \tag{3}$$

The cumulative distribution function is an important application of majorization.

In the case of a continuous distribution, it gives the area under the probability distribution functions are also used to specify the distribution of multivariable random variables. There are various applications of CDF, for example, in learning to rank, the cumulative distribution function (CDF) arises naturally as a probability measure over inequality events of the type $\{X \leq x\}$. The joint CDF lends itself to problems that are easily described in terms of inequality events in which statistical dependence relationships also among events. Examples of this type of problem include web search and document retrieval [5, 6, 14, 33], predicting rating of movies [30] or predicting multiplayer game outcomes with a team structure [13]. In contrast to the canonical problems of classification or regression, in learning to rank we are required to learn some mapping from inputs to inter-dependent output variables so that we may wish to model both stochastic orderings of variable states that statistical dependence relationships between variables.

Now we introduce the main mathematical theory explored in the present work, the theory of majorization. It is a powerful and elegant mathematical tool which can be applied to a wide variety of problems as in quantum mechanics. The theory of majorization is closely related to the notions of 'randomness' and 'disorder'. It indeed allows us to compare two probability distributions, in order for us to know which one of the two is more random. The appearance of Marshall and Olkin's 1979 book on inequalities with special emphasis on majorization generated a surge of interest in potential applications of majorization and Schur convexity in a broad spectrum of fields and then the second volume of this book was published in 2011 [19].
Let us now give the most general definition of majorization. For fixed $n \geq 2$ let

$$\mathbf{x} = (x_1, ..., x_n)\,, \;\; \mathbf{y} = (y_1, ..., y_n)$$

denote two real n-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[n]},$$

$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(n)}$$

denote their ordered components.
Majorization: Let $\mathbf{x} = (x_1, ..., x_n)\,, \mathbf{y} = (y_1, ..., y_n)$ be sequences of real numbers. Then [29, p.319] we say that $\mathbf{y}$ is majorized by $\mathbf{x}$ or $\mathbf{x}$ majorizes $\mathbf{y}$, in symbol, $\mathbf{x} \succ \mathbf{y}$, if we have

$$\sum_{i=1}^{j} y_{[i]} \leq \sum_{i=1}^{j} x_{[i]}, \tag{4}$$

for $j = 1, 2, ..., n-1$ and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

Note that (4) is equivalent to

$$\sum_{i=n-j+1}^{n} y_{(i)} \leq \sum_{i=n-j+1}^{n} x_{(i)},$$

for $j = 1, 2, ..., n-1$.

The following theorem is given in ([28, p.32]):

Theorem 1.1. Let $f : J \to R$ be a continuous convex function on an interval J, $\mathbf{w}$ be a positive n-tuple and $\mathbf{x}$, $\mathbf{y} \in J^n$ satisfying

$$\sum_{i=1}^{k} w_i\, y_i \leq \sum_{i=1}^{k} w_i\, x_i \;\text{ for }\; k = 1, ..., n-1, \tag{5}$$

and

$$\sum_{i=1}^{n} w_i\, y_i \;=\; \sum_{i=1}^{n} w_i\, x_i. \tag{6}$$

(a) If $\mathbf{y}$ is a decreasing n-tuple, then

$$\sum_{i=1}^{n} w_i\, f\left(y_i\right) \;\leq\; \sum_{i=1}^{n} w_i\, f\left(x_i\right). \tag{7}$$

(b) If $\mathbf{x}$ is an increasing n-tuple, then

$$\sum_{i=1}^{n} w_i\, f\left(x_i\right) \;\leq\; \sum_{i=1}^{n} w_i\, f\left(y_i\right). \tag{8}$$

If f is strictly convex and $\mathbf{x} \neq \mathbf{y}$, then (7) and (8) are strict.

One can see the generalizations and refinements of majorization inequality in [1, 2, 3, 4, 16].

Pečarić, Matić and Pearce (1998)[20], (1999)[21], (2000)[22] and (2002)[23] continuously worked on Shannon's inequality and related inequalities in the probability distribution and information science. They studied and discussed in [22, 23] several aspects of Shannon's inequality in discrete as well as in integral forms, by presenting upper estimates of the difference between its two sides. Applications to the bounds in information theory were also given. In [22, p.139-140], they used the majorization inequality to prove the generalized result that is the entropy function achieved its maximum value on the discrete uniform of probability distribution.

Dragomir gave in his monograph (chapter.02) [9] about these divergences like the Kullback-Leibler divergence, variational distance, the Hellinger distance, χ^2-divergence, triangular discrimination and the Rényi α-order entropy.

Motivated the idea in [22] (2000) and [23] (2002), we discuss the behaviour of the results in the form of divergences, majorization and Zipf-Mandelbrot law. We arrange the paper in this manner: in section-2, we consider the Csiszár f-divergence for Zipf-Mandelbrot law and to develop several important majorization inequalities via **CDF** as the condition of majorization. We discuss some special cases of our generalized results. In section-3, we present several applications of our results by constructing distances in the Zipf-Mandelbrot law i.e., the Rényi α-order entropy for Z-M law, variational distance for Z-M law, the Hellinger discrimination for Z-M law, triangular discrimination for Z-M law and χ^2-distance for Z-M law. At the end, in section-4, we give important applications of the Zipf's law in linguistics and obtain the bounds for the Kullback-Leibler divergence of the distributions associated to the English and Russian languages.

2 Main results

We can consider the following two definitions of Csiszár divergence [7, 8] for Zipf-Mandelbrot law:

Definition 2.1. (**Csiszár Divergence for Z-M law**)
Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $n \in \{1, 2, 3, ...\}$, $t_1 \geq 0$, $s_1 > 0$ and also let $q_i > 0$ for $(i = 1, ..., n)$ such that

$$\frac{1}{q_i (i + t_1)^{s_1} H_{n,t_1,s_1}} \in J, \quad i = 1, ..., n, \tag{9}$$

then let

$$\hat{I}_f\left(i,n,t_1,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

Definition 2.2. Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to \mathbb{R}$ be a function. Let $n \in \{1,2,3,...\}$, t_1, t_2 ≥ 0 and $s_1, s_2 > 0$ such that

$$\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \in J, \quad i = 1,...,n, \tag{10}$$

then let

$$\widetilde{I}_f\left(i,n,t_1,t_2,s_1,s_2\right) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{(i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right).$$

Remark 2.3. It is obvious that the second Csiszár divergence for Zipf-Mandelbrot law is a special case of the first one.

We present the following theorem is the connection between Csiszár f-divergence, Zipf-Mandelbrot law and weighted majorization inequality:

Theorem 2.4. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1,2,3,...\}$, t_1, t_2, t_3 ≥ 0 and $s_1, s_2, s_3 > 0$ such that satisfying

$$\frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1,...,n-1, \tag{11}$$

with

$$\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J, \quad (i = 1,...,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,...,n)$, then

$$\begin{aligned} \widetilde{I}_f\left(i,n,t_2,t_3,s_2,s_3\right) &:= \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \widetilde{I}_f\left(i,n,t_1,t_3,s_1,s_3\right) &:= \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{12}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,...,n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\qquad \geq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} f\left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right), \end{aligned} \tag{13}$$

If f is continuous concave function, then the reverse inequalities hold in (12) and (13).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$ for $(i = 1, ..., n)$, then

$$\begin{aligned}\sum_{i=1}^{k} w_i x_i &:= \sum_{i=1}^{k} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}} \\ &= \frac{1}{H_{n,t_1,s_1}} \sum_{i=1}^{k} \frac{1}{(i+t_1)^{s_1}} \\ &= \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1, ..., n-1,\end{aligned}$$

similarly

$$\sum_{i=1}^{k} w_i y_i := \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}}, \ k = 1, ..., n-1.$$

This implies that

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \qquad \Leftrightarrow \qquad \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1, ..., n-1.$$

We can easily check that $\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ is decreasing over $i = 1, ..., n$ and similarly the others too. Now, we investigate the behaviour of y_i for $(i = 1, 2, ..., n)$, take

$$y_i = \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \qquad \text{and} \qquad y_{i+1} = \frac{(i+1+t_3)^{s_3} H_{n,t_3,s_3}}{(i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$y_{i+1} - y_i = \frac{H_{n,t_3,s_3}}{H_{n,t_2,s_2}} \left[\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} - \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}} \right] \leq 0,$$

$$\Leftrightarrow \frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}, \ \ (i = 1, ..., n),$$

which shows that y_i is decreasing.
Therefore, substitute $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{1/(i+t_3)^{s_3} H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}}$ for $(i = 1, ..., n)$ and $f := f$ in Theorem 1(a), then we get (12).
(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is increasing sequence, in Theorem 1.1 (b). Q.E.D.

Theorem 2.5. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$, $(i = 1, ..., n)$ with

$$\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}, \ \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1, ..., n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned} \hat{I}_f(i, n, t_2, s_2, \mathbf{q}) &:= \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\leq \hat{I}_f(i, n, t_1, s_1, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{14}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\qquad \geq \sum_{i=1}^{n} q_i f\left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \end{aligned} \tag{15}$$

If f is continuous concave function, then the reverse inequalities hold in (14) and (15).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i}$, and $w_i = q_i > 0$, $(i = 1, ..., n)$ then we can get as in the previous proof

$$\sum_{i=1}^{k} w_i y_i \leq \sum_{i=1}^{k} w_i x_i \qquad \Leftrightarrow \qquad \frac{H_{k,t_2,s_2}}{H_{n,t_2,s_2}} \leq \frac{H_{k,t_1,s_1}}{H_{n,t_1,s_1}}, \quad k = 1, ..., n-1.$$

Now, we investigate the behaviour of y_i for $(i = 1, 2, ..., n)$, take

$$y_i = \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \qquad \text{and} \qquad y_{i+1} = \frac{1}{q_{i+1} (i+1+t_2)^{s_2} H_{n,t_2,s_2}},$$

$$y_{i+1} - y_i = \frac{1}{H_{n,t_2,s_2}} \left[\frac{1}{q_{i+1}(i+1+t_2)^{s_2}} - \frac{1}{q_i (i+t_2)^{s_2}} \right] \leq 0,$$

$$\Leftrightarrow \frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}, \ \ (i = 1, ..., n),$$

which shows that y_i is decreasing. Therefore, substitute $x_i := \frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}$, $y_i := \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}$, $w_i = q_i > 0$, $(i = 1, ..., n)$ and also $f := f$ in Theorem 1, we get (14).
(b) If we switch the role of y_i into x_i as increasing sequence in the similar fashion as the proof of Part (a), then by using Theorem 1 (b) we get (15). Q.E.D.

Corollary 2.6. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ is a continuous convex function. Let $n \in \{1, 2, 3, ...\}$, t_1, t_2 ≥ 0 and $s_1, s_2 > 0$ such that satisfying (11) with

$$\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}, \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \ \ (i = 1, ..., n),$$

then the following inequality holds

$$\begin{aligned}\hat{I}_f\left(i,n,t_2,s_2,\mathbf{1}\right) &:= \sum_{i=1}^{n} f\left(\frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}}\right)\\ &\leq \hat{I}_f\left(i,n,t_1,s_1,\mathbf{1}\right) := \sum_{i=1}^{n} f\left(\frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\right). \qquad (16)\end{aligned}$$

If f is continuous concave function, then the reverse inequality hold in (16).

Proof. Let us consider $x_i := \frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}}$ and $y_i := \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}}$, we can easily check that $y_i := \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}}$ is decreasing over $i = 1,...,n$. Therefore, substitute $x_i := \frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}}$, $y_i := \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}}$, $q_i = 1$, $(i = 1,...,n)$ and also $f := f$ in (14), we get (16).

Q.E.D.

Theorem 2.7. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1,2,3,...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11) with

$$\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_1)^{s_1}H_{n,t_1,s_1}},\ \frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_2)^{s_2}H_{n,t_2,s_2}} \in J, \quad (i = 1,...,n),$$

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,...,n)$, then

$$\begin{aligned}\widetilde{I}_{id_J\,f}\left(i,n,t_2,t_3,s_2,s_3\right) &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} f\left(\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_2)^{s_2}H_{n,t_2,s_2}}\right)\\ &\leq \widetilde{I}_{id_J\,f}\left(i,n,t_1,t_3,s_1,s_3\right) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} f\left(\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\right). \qquad (17)\end{aligned}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,...,n)$, then

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2}H_{n,t_2,s_2}} f\left(\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_2)^{s_2}H_{n,t_2,s_2}}\right)\\ &\geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1}H_{n,t_1,s_1}} f\left(\frac{(i+t_3)^{s_3}H_{n,t_3,s_3}}{(i+t_1)^{s_1}H_{n,t_1,s_1}}\right). \qquad (18)\end{aligned}$$

If $xf(x)$ is continuous concave function, then the reverse inequalities hold in (17) and (18).

Proof. Let us substitute $x_i := \frac{1/(i+t_1)^{s_1}H_{n,t_1,s_1}}{1/(i+t_3)^{s_3}H_{n,t_3,s_3}}$, $y_i := \frac{1/(i+t_2)^{s_2}H_{n,t_2,s_2}}{1/(i+t_3)^{s_3}H_{n,t_3,s_3}}$, $w_i := \frac{1}{(i+t_3)^{s_3}H_{n,t_3,s_3}}$ for $(i = 1,...,n)$ in Theorem 1(a) and follow the proof of Theorem 2 for function $f(x) := xf(x)$, then we get (17).

(b) We can prove part (b) with the similar substitutions as in Part (a) but switch the role of y_i with x_i that is an increasing sequence, in Theorem 1.1 (b) for function $f(x) := xf(x)$. Q.E.D.

Theorem 2.8. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1, 2, 3, ...\}$, $t_1,\, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$ with

$$\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}},\ \frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \quad (i = 1, ..., n),$$

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned}\hat{I}_{id_J\, f}\,(i, n, t_2, s_2, \mathbf{q}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \hat{I}_{id_J\, f}\,(i, n, t_1, s_1, \mathbf{q}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \qquad (19)\end{aligned}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned}&\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{q_i(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ &\quad \geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{q_i(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \qquad (20)\end{aligned}$$

If $xf(x)$ is continuous concave function, then the reverse inequalities hold in (19) and (20).

Proof. Let us consider $x_i := \frac{1/(i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i}$, $y_i := \frac{1/(i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i}$, and $w_i = q_i > 0$, $(i = 1, ..., n)$ and also $f(x) := xf(x)$ in Theorem 1(a) by follow the proof of Theorem 3(a), we get (19).
(b) If we switch the role of y_i into x_i as an increasing sequence with the similar substitutions as in Part (a), then by using Theorem 1(b) we get (20). Q.E.D.

Corollary 2.9. Let $J \subset \mathbb{R}$ is an interval and $f : J \to \mathbb{R}$ be a function such that $x \to xf(x)$ $(x \in J)$ is a continuous convex function. Let $n \in \{1, 2, 3, ...\}$, $t_1,\, t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) with

$$\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}},\ \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \in J \quad (i = 1, ..., n),$$

then the following inequality holds

$$\begin{aligned}\hat{I}_{id_J\, f}\,(i, n, t_2, s_2, \mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} f\left(\frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}\right) \\ \leq \hat{I}_{id_J\, f}\,(i, n, t_1, s_1, \mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} f\left(\frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}\right). \qquad (21)\end{aligned}$$

If $xf(x)$ is continuous concave function, then the reverse inequality hold in (21).

Proof. Since $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$ is decreasing over $i = 1, ..., n$. Therefore, substitute $x_i := \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}}$ and $y_i := \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}}$, and $q_i = 1$, $(i = 1, ..., n)$ and also $f(x) := xf(x)$ in (19), we get (21).

Q.E.D.

3 Applications

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. All previous results in the Main Section can be easily applied on functions $-\log x$ and $\log x$ to get various results for the Kullback-Leibler divergence for the Zipf-Mandelbrot law as given in [15]. So, in this paper we will give applications on some other known divergences for the Zipf-Mandelbrot law.

The following definitions are the Rényi α-order entropy for the Zipf-Madelbrot law:

Definition 3.1. (**Rényi α- order entropy for Z-M law**)
If $n \in \{1, 2, 3, ...\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$, $(i = 1, ..., n)$, then the Rényi α-order entropy $(\alpha > 1)$ for Zipf-Madelbrot law is defined by

$$\hat{R_\alpha}(i, n, t_1, s_1, \mathbf{q}) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{\alpha-1}.$$

Definition 3.2. If $n \in \{1, 2, 3, ...\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, then for $(\alpha > 1)$

$$\widetilde{R_\alpha}(i, n, t_1, t_2, s_1, s_2) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{\alpha-1}.$$

Application 1. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11),

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1, ..., n)$, then the following inequality holds for $(\alpha > 1)$

$$\begin{aligned}\widetilde{R_\alpha}(i, n, t_2, t_3, s_2, s_3) &:= \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1} \\ &\leq \widetilde{R_\alpha}(i, n, t_1, t_3, s_1, s_3) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1}.\end{aligned} \tag{22}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1, ..., n)$, then the following inequality holds for $(\alpha > 1)$

$$\begin{aligned}&\sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1} \\ &\geq \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} \left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]^{\alpha-1}.\end{aligned} \tag{23}$$

Proof. If we choose $f(t) := t^{\alpha}$, $t \in \mathbb{R}^{+}$ $(\alpha > 1)$, then by using (12) we get

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \right)^{\alpha} \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \right)^{\alpha},$$

we get (22).
(b) Similarly as Part (a), we can prove (23) by using (13) and $f(t) := t^{\alpha}$, $t \in \mathbb{R}^{+}$ $(\alpha > 1)$. Q.E.D.

Application 2. If $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\hat{R_{\alpha}}\left(i, n, t_2, s_2, \mathbf{q}\right) := \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} q_i^{1-\alpha} \leq \hat{R_{\alpha}}\left(i, n, t_1, s_1, \mathbf{q}\right) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{1-\alpha}. \tag{24}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} q_i^{1-\alpha} \geq \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha} q_i^{1-\alpha}. \tag{25}$$

Proof. If we choose $f(t) := t^{\alpha}$, $t \in \mathbb{R}^{+}$ $(\alpha > 1)$, then by using (14) we get

$$\sum_{i=1}^{n} q_i \left[\frac{1}{q_i\,(i+t_2)^{s_2} H_{n,t_2,s_2}} \right]^{\alpha} \leq \sum_{i=1}^{n} q_i \left[\frac{1}{q_i\,(i+t_1)^{s_1} H_{n,t_1,s_1}} \right]^{\alpha},$$

we get (24).
(b) Similarly as Part (a), we can prove (25) by using (15) and $f(t) := t^{\alpha}$, $t \in \mathbb{R}^{+}$ $(\alpha > 1)$. Q.E.D.

Application 3. If $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11), then the following inequality holds

$$\hat{R_{\alpha}}\left(i, n, t_2, s_2, \mathbf{1}\right) := \sum_{i=1}^{n} \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^{-\alpha} \leq \hat{R_{\alpha}}\left(i, n, t_1, s_1, \mathbf{1}\right) := \sum_{i=1}^{n} \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^{-\alpha}. \tag{26}$$

Proof. If we choose $f(t) := t^{\alpha}$, $t \in \mathbb{R}^{+}$ $(\alpha > 1)$, and $q_i := 1$, $(i = 1, ..., n)$ in (24), then we get (26). Q.E.D.

The following definitions are the variational distance for Zipf-Madelbrot law:

Definition 3.3. (**Variational Distance for Z-M law**)
If $n \in \{1,2,3,...\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$, $(i = 1,...,n)$, then the variational distance for Zipf-Mandelbrot law is defined by

$$\hat{V}\left(i,n,t_1,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - q_i \right|.$$

Definition 3.4. If $n \in \{1,2,3,...\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, then

$$\widetilde{V}\left(i,n,t_1,t_2,s_1,s_2\right) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \right|.$$

Application 4. Let $n \in \{1,2,3,...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11),

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1,...,n)$, then

$$\begin{aligned} \widetilde{V}\left(i,n,t_2,t_3,s_2,s_3\right) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \right| \\ \leq \widetilde{V}\left(i,n,t_1,t_3,s_1,s_3\right) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \right|. \end{aligned} \tag{27}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1,...,n)$, then

$$\sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \right| \geq \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \right|. \tag{28}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$, then by using (12) we get

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left| \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1 \right| \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left| \frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1 \right|,$$

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left| \frac{(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \right| \\ &\quad \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left| \frac{(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \right|, \end{aligned}$$

since $(i+t_3)^{s_3} H_{n,t_3,s_3} > 0$, we get (27).
(b) Similarly as Part (a), we can prove (28) by using (13) and $f(t) := |t-1|$, $t \in \mathbb{R}^+$. Q.E.D.

Application 5. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned} \hat{V}\left(i, n, t_2, s_2, \mathbf{q}\right) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - q_i \right| \\ &\leq \hat{V}\left(i, n, t_1, s_1, \mathbf{q}\right) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - q_i \right|. \end{aligned} \tag{29}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - q_i \right| \geq \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - q_i \right|. \tag{30}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$, then by using (14) we get

$$\sum_{i=1}^{n} q_i \left| \frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} - 1 \right| \leq \sum_{i=1}^{n} q_i \left| \frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} - 1 \right|,$$

$$\sum_{i=1}^{n} q_i \left| \frac{1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \right| \leq \sum_{i=1}^{n} q_i \left| \frac{1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} \right|,$$

since $q_i > 0$ $(i = 1, ..., n)$, we get (29).
(b) Similarly as Part (a), we can prove (30) by using (15) and $f(t) := |t-1|$, $t \in \mathbb{R}^+$. Q.E.D.

Application 6. If $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11), then the following inequality hold

$$\begin{aligned} \hat{V}\left(i, n, t_2, s_2, \mathbf{1}\right) &:= \sum_{i=1}^{n} \left| \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1 \right| \\ &\leq \hat{V}\left(i, n, t_1, s_1, \mathbf{1}\right) := \sum_{i=1}^{n} \left| \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1 \right|. \end{aligned} \tag{31}$$

Proof. If we choose $f(t) := |t-1|$, $t \in \mathbb{R}^+$ and $q_i := 1$, $(i = 1, ..., n)$ in (29), then we get (31). Q.E.D.

The following definitions are the Hellinger discrimination for Zipf-Madelbrot law:

Definition 3.5. (**Hellinger Discrimination for Z-M law**)
If $n \in \{1, 2, 3, ...\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, ..., n)$, then the Hellinger discrimination for Zipf-Mandelbrot law is defined by

$$\hat{h}\left(i, n, t_1, s_1, \mathbf{q}\right) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \sqrt{q_i} \right)^2.$$

Definition 3.6. If $n \in \{1, 2, 3, ...\}$, $t_1, t_2 \geq 0$, $s_1, s_2 > 0$, then

$$\widetilde{h}\,(i, n, t_1, t_2, s_1, s_2) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} \right)^2 .$$

Application 7. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11),

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1, ..., n)$, then

$$\begin{aligned} \widetilde{h}\,(i, n, t_2, t_3, s_2, s_3) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2 \\ \leq \widetilde{h}\,(i, n, t_1, t_3, s_1, s_3) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2 . \end{aligned} \tag{32}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1, ..., n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2 \\ &\qquad \geq \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \frac{1}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}} \right)^2 . \end{aligned} \tag{33}$$

Proof. If we choose $f(t) := \frac{1}{2}\left(\sqrt{t} - 1\right)^2$, $t \in \mathbb{R}^+$, then by using (12) we get

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\sqrt{\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - 1 \right)^2 \\ &\qquad \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left(\sqrt{\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - 1 \right)^2 , \end{aligned}$$

$$\begin{aligned} &\sum_{i=1}^{n} \left(\frac{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}} - \sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} \right)^2 \\ &\qquad \leq \sum_{i=1}^{n} \left(\frac{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}} - \sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}}{\sqrt{(i+t_3)^{s_3} H_{n,t_3,s_3}}\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} \right)^2 , \end{aligned}$$

we get (32).

(b) Similarly as Part (a), we can prove (33) by using (13) and $f(t) := \frac{1}{2}\left(\sqrt{t} - 1\right)^2$, $t \in \mathbb{R}^+$. Q.E.D.

Application 8. Let $n \in \{1, 2, 3, ...\}$, $t_1,\ t_2\ \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned}\hat{h}\left(i, n, t_2, s_2, \mathbf{q}\right) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - \sqrt{q_i} \right)^2 \\ &\leq h\left(i, n, t_1, s_1, \mathbf{q}\right) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \sqrt{q_i} \right)^2.\end{aligned} \tag{34}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\begin{aligned}&\sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - \sqrt{q_i} \right)^2 \\ &\geq \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - \sqrt{q_i} \right)^2.\end{aligned} \tag{35}$$

Proof. If we choose $f(t) := \frac{1}{2}\ \left(\sqrt{t} - 1\right)^2$, $t \in \mathbb{R}^+$, then by using (14) we get

$$\sum_{i=1}^{n} \frac{q_i}{2} \left(\frac{1}{\sqrt{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}} - 1 \right)^2 \leq \sum_{i=1}^{n} \frac{q_i}{2} \left(\frac{1}{\sqrt{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}} - 1 \right)^2,$$

$$\sum_{i=1}^{n} q_i \frac{\left(1 - \sqrt{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right)^2}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} \leq \sum_{i=1}^{n} q_i \frac{\left(1 - \sqrt{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right)^2}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}},$$

we get (34).
(b) Similarly as Part (a), we can prove (35) by using (15) and $f(t) := \frac{1}{2}\ \left(\sqrt{t} - 1\right)^2$, $t \in \mathbb{R}^+$. Q.E.D.

Application 9. If $n \in \{1, 2, 3, ...\}$, $t_1,\ t_2\ \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11), then the following inequality holds

$$\begin{aligned}\hat{h}\left(i, n, t_2, s_2, \mathbf{1}\right) &:= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_2)^{s_2} H_{n,t_2,s_2}}} - 1 \right)^2 \\ &\leq \hat{h}\left(i, n, t_1, s_1, \mathbf{1}\right) := \sum_{i=1}^{n} \left(\frac{1}{\sqrt{(i+t_1)^{s_1} H_{n,t_1,s_1}}} - 1 \right)^2.\end{aligned} \tag{36}$$

Proof. Substitute $q_i = 1$, $(i = 1, ..., n)$ in (34), we get (36). Q.E.D.

The following definitions are the Triangular discrimination for Zipf-Madelbrot law:

Definition 3.7. (**Triangular Descrimination in Z-M Law**)
If $n \in \{1, 2, 3, ...\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, ..., n)$, then the Triangular discrimination for Zipf-Mandelbrot law is defined by

$$\hat{\Delta}(i, n, t_1, s_1, \mathbf{q}) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} \right].$$

Definition 3.8. If $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$, $s_1\, s_2 > 0$, then the Triangular discrimination for Zipf-Mandelbrot law is defined by

$$\widetilde{\Delta}(i, n, t_1, t_2, s_1, s_2) := \sum_{i=1}^{n} \frac{1}{[(i+t_1)^{s_1} H_{n,t_1,s_1}][(i+t_2)^{s_2} H_{n,t_2,s_2}]} \left[\frac{\left[(i+t_2)^{s_2} H_{n,t_2,s_2} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{(i+t_2)^{s_2} H_{n,t_2,s_2} + (i+t_1)^{s_1} H_{n,t_1,s_1}} \right].$$

Application 10. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11),

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1, ..., n)$, then

$$\begin{aligned} &\widetilde{\Delta}(i, n, t_2, t_3, s_2, s_3) \\ &\quad := \sum_{i=1}^{n} \frac{1}{[(i+t_2)^{s_2} H_{n,t_2,s_2}]\,[(i+t_3)^{s_3} H_{n,t_3,s_3}]} \left[\frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} + (i+t_2)^{s_2} H_{n,t_2,s_2}} \right] \\ &\quad \leq \widetilde{\Delta}(i, n, t_1, t_3, s_1, s_3) \\ &\quad := \frac{1}{[(i+t_1)^{s_1} H_{n,t_1,s_1}]\,[(i+t_3)^{s_3} H_{n,t_3,s_3}]} \left[\frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} + (i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \end{aligned} \tag{37}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1, ..., n)$, then

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{[(i+t_2)^{s_2} H_{n,t_2,s_2}]\,[(i+t_3)^{s_3} H_{n,t_3,s_3}]} \left[\frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} + (i+t_2)^{s_2} H_{n,t_2,s_2}} \right] \\ &\quad \geq \frac{1}{[(i+t_1)^{s_1} H_{n,t_1,s_1}]\,[(i+t_3)^{s_3} H_{n,t_3,s_3}]} \left[\frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} + (i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \end{aligned} \tag{38}$$

Proof. If we choose $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, then by using (12) we get

$$\begin{aligned} &\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3}/(i+t_2)^{s_2} H_{n,t_2,s_2 - 1}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3}/(i+t_2)^{s_2} H_{n,t_2,s_2}\ +1} \\ &\quad \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3}/(i+t_1)^{s_1} H_{n,t_1,s_1 - 1}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3}/(i+t_1)^{s_1} H_{n,t_1,s_1}\ +1}, \end{aligned}$$

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} \; - \; (i+t_2)^{s_2} H_{n,t_2,s_2}/(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} \; + \; (i+t_2)^{s_2} H_{n,t_2,s_2}/(i+t_2)^{s_2} H_{n,t_2,s_2}}$$
$$\leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} \; - \; (i+t_1)^{s_1} H_{n,t_1,s_1}/(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{(i+t_3)^{s_3} H_{n,t_3,s_3} \; + \; (i+t_1)^{s_1} H_{n,t_1,s_1}/(i+t_1)^{s_1} H_{n,t_1,s_1}},$$

we get (37).
(b) Similarly as Part (a), we can prove (38) by using (13) and $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$. Q.E.D.

Application 11. Let $n \in \{1, 2, 3, ...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\hat{\Delta}\left(i, n, t_2, s_2, \mathbf{q}\right) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1 + q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right]$$
$$\leq \hat{\Delta}\left(i, n, t_1, s_1, \mathbf{q}\right) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right]. \tag{39}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$, then

$$\sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1 + q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right]$$
$$\geq \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1 + q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right]. \tag{40}$$

Proof. If we choose $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$, then by using (14) we get

$$\sum_{i=1}^{n} q_i \frac{\left(1/q_i (i+t_2)^{s_2} H_{n,t_2,s_2} - 1\right)^2}{1/q_i (i+t_2)^{s_2} H_{n,t_2,s_2} + 1} \leq \sum_{i=1}^{n} q_i \frac{\left(1/q_i (i+t_1)^{s_1} H_{n,t_1,s_1} - 1\right)^2}{1/q_i (i+t_1)^{s_1} H_{n,t_1,s_1} + 1},$$

$$\sum_{i=1}^{n} q_i \frac{\left[1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}/q_i (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{1 + q_i (i+t_2)^{s_2} H_{n,t_2,s_2}/q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}$$
$$\leq \sum_{i=1}^{n} q_i \frac{\left[1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}/q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{1 + q_i (i+t_1)^{s_1} H_{n,t_1,s_1}/q_i (i+t_1)^{s_1} H_{n,t_1,s_1}},$$

we get (39).
(b) Similar way as Part (a), we can prove (40) by using (15) and $f(t) := \frac{(t-1)^2}{t+1}$, $t \in \mathbb{R}^+$. Q.E.D.

Application 12. If $n \in \{1,2,3,...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11), then the following inequality holds

$$\hat{\Delta}\left(i,n,t_2,s_2,\mathbf{1}\right) := \sum_{i=1}^{n} \frac{1}{(i+t_2)^{s_2} H_{n,t_2,s_2}} \left[\frac{\left(1-(i+t_2)^{s_2} H_{n,t_2,s_2}\right)^2}{1+(i+t_2)^{s_2} H_{n,t_2,s_2}} \right]$$
$$\leq \hat{\Delta}\left(i,n,t_1,s_1,\mathbf{1}\right) := \sum_{i=1}^{n} \frac{1}{(i+t_1)^{s_1} H_{n,t_1,s_1}} \left[\frac{\left(1-(i+t_1)^{s_1} H_{n,t_1,s_1}\right)^2}{1+(i+t_1)^{s_1} H_{n,t_1,s_1}} \right]. \tag{41}$$

Proof. Substitute $q_i = 1$, $(i = 1, ..., n)$ in (39), we get (41). Q.E.D.

The following definitions are the χ^2-distance (chi-square distance) for Zipf-Madelbrot law:

Definition 3.9. (**χ^2-distance for Z-M law**)
If $n \in \{1,2,3,...\}$, $t_1 \geq 0$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, ..., n)$, then the χ^2-distance for Zipf-Mandelbrot law is defined by

$$\hat{\chi^2}\left(i,n,t_1,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} \frac{\left[1-q_i(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{q_i\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}.$$

Definition 3.10. If $n \in \{1,2,3,...\}$, t_1, $t_2 \geq 0$, s_1, $s_2 > 0$, then

$$\widetilde{\chi^2}\left(i,n,t_1,t_2,s_1,s_2\right) := \sum_{i=1}^{n} \frac{\left[(i+t_2)^{s_2} H_{n,t_2,s_2} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}.$$

Application 13. Let $n \in \{1,2,3,...\}$, t_1, $t_2, t_3 \geq 0$ and $s_1, s_2, s_3 > 0$ such that satisfying (11),

(a) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_2)^{s_2}} \leq \frac{(i+t_3)^{s_3}}{(i+t_2)^{s_2}}$ $(i = 1, ..., n)$, then

$$\widetilde{\chi^2}\left(i,n,t_2,t_3,s_2,s_3\right) := \sum_{i=1}^{n} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2\left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}$$
$$\leq \widetilde{\chi^2}\left(i,n,t_1,t_3,s_1,s_3\right) := \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2\left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}. \tag{42}$$

(b) if $\frac{(i+1+t_3)^{s_3}}{(i+1+t_1)^{s_1}} \geq \frac{(i+t_3)^{s_3}}{(i+t_1)^{s_1}}$ $(i = 1, ..., n)$, then

$$\sum_{i=1}^{n} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2\left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}$$
$$\geq \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2\left[(i+t_3)^{s_3} H_{n,t_3,s_3}\right]}. \tag{43}$$

Proof. If we choose $f(t) := (t-1)^2$, $t \in [0,\infty)$, then by using (12) we get

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left[\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_2)^{s_2} H_{n,t_2,s_2}} - 1\right]^2 \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \left[\frac{(i+t_3)^{s_3} H_{n,t_3,s_3}}{(i+t_1)^{s_1} H_{n,t_1,s_1}} - 1\right]^2,$$

$$\sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2} \leq \sum_{i=1}^{n} \frac{1}{(i+t_3)^{s_3} H_{n,t_3,s_3}} \frac{\left[(i+t_3)^{s_3} H_{n,t_3,s_3} - (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2},$$

we get (42).
(b) Similarly as Part (a), we can prove (43) by using (13) and $f(t) := (t-1)^2$, $t \in [0,\infty)$. Q.E.D.

Application 14. Let $n \in \{1,2,3,...\}$, t_1, $t_2 \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11) and also let $q_i > 0$, $(i = 1,...,n)$,

(a) if $\frac{(i+t_2)^{s_2}}{(i+1+t_2)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,...,n)$, then

$$\hat{\chi^2}(i,n,t_2,s_2,\mathbf{q}) := \sum_{i=1}^{n} \frac{\left[1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{q_i \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2} \leq \hat{\chi^2}(i,n,t_1,s_1,\mathbf{q}) := \sum_{i=1}^{n} \frac{\left[1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{q_i \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}. \tag{44}$$

(b) if $\frac{(i+t_1)^{s_1}}{(i+1+t_1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,...,n)$, then

$$\sum_{i=1}^{n} \frac{\left[1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{q_i \left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2} \geq \sum_{i=1}^{n} \frac{\left[1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{q_i \left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}. \tag{45}$$

Proof. If we choose $f(t) := (t-1)^2$, $t \in [0,\infty)$, then by using (14) we get

$$\sum_{i=1}^{n} q_i \left(\frac{1}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}} - 1\right)^2 \leq \sum_{i=1}^{n} q_i \left(\frac{1}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}} - 1\right)^2,$$

$$\sum_{i=1}^{n} q_i \left(\frac{1 - q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}{q_i (i+t_2)^{s_2} H_{n,t_2,s_2}}\right)^2 \leq \sum_{i=1}^{n} q_i \left(\frac{1 - q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}{q_i (i+t_1)^{s_1} H_{n,t_1,s_1}}\right)^2,$$

we get (44).
(b) Similar way as Part (a), we can prove (45) by using (15) and $f(t) := (t-1)^2$, $t \in [0,\infty)$. Q.E.D.

Application 15. If $n \in \{1,2,3,...\}$, $t_1,\ t_2\ \geq 0$ and $s_1, s_2 > 0$ such that satisfying (11), then the following inequality holds

$$\hat{\chi^2}\left(i,n,t_2,s_2,\mathbf{1}\right) := \sum_{i=1}^{n} \frac{\left[1-(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}{\left[(i+t_2)^{s_2} H_{n,t_2,s_2}\right]^2}$$

$$\leq \hat{\chi^2}\left(i,n,t_1,s_1,\mathbf{1}\right) := \sum_{i=1}^{n} \frac{\left[1-(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}{\left[(i+t_1)^{s_1} H_{n,t_1,s_1}\right]^2}. \tag{46}$$

Proof. Substitute $q_i = 1$, $(i = 1, ..., n)$ in (44), we get (46). Q.E.D.

4 Zipf's Law in Linguistics

For finite n and $t = 0$ the Zipf-Mandelbrot law becomes Zipf's law. Therefore (1) and (2) becomes

$$f\left(k,n,s\right) := \frac{1/k^s}{H_{n,s}}, \qquad \text{where} \quad H_{n,s} := \sum_{i=1}^{n} \frac{1}{i^s}. \tag{47}$$

Gelbukh and Sidorov in [12] observed the difference between the coefficients s_1 and s_2 in Zipf's law for the English and Russian languages. They processed 39 literature texts for each language, chosen randomly from different genres, with the requirement that the size be greater than 10,000 running words each. They calculated coefficients for each of the mentioned texts and as the result they obtained the average of $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language.

The following definitions are the Kullback-Leibler divergence for Zipf's law:

Definition 4.1. (**Kullback-Leibler divergence for Zipf Law**)
If $n \in \{1,2,3,...\}$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, ..., n)$, then the Kullback-Leibler divergence for Zipf's law is defined by

$$\hat{KL}\left(i,n,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} q_i \,\log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right).$$

Definition 4.2. If $n \in \{1,2,3,...\}$, $s_1 > 0$ and also $q_i > 0$ for $(i = 1, ..., n)$, then the Kullback-Leibler divergence for Zipf's law is defined by

$$K\hat{L}_{id}\left(i,n,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \,\log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right).$$

Remark 4.3. The majorization conditions (11) for $t_1 = t_2 = 0$ becomes

$$\frac{H_{k,s_2}}{H_{n,s_2}} \leq \frac{H_{k,s_1}}{H_{n,s_1}}, \quad \text{for} \quad k = 2, ..., n-1, \tag{48}$$

and for $k = 1$, the above inequality becomes

$$H_{n,s_1} \leq H_{n,s_2} \qquad \Leftrightarrow \qquad s_2 \leq s_1, \tag{49}$$

which means that the generalized harmonic number of order n of s_1 is less or equal to the generalized harmonic number of order n of s_2.

Corollary 4.4. Let $n \in \{1,2,3,...\}$ and $s_1, s_2 > 0$ such that $s_2 \leq s_1$ satisfying (48) and also let $q_i > 0$, $(i = 1,...,n)$,

(a) if $\frac{i^{s_2}}{(i+1)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,...,n)$ and the base of log is greater than 1, then

$$\begin{aligned} \hat{KL}\left(i,n,s_2,\mathbf{q}\right) &:= \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_2}\, H_{n,s_2}}\right) \\ &\geq \hat{KL}\left(i,n,s_1,\mathbf{q}\right) := \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{50}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (50).

(b) if $\frac{i^{s_1}}{(i+1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1,...,n)$ and the base of log is greater than 1, then

$$\sum_{i=1}^{n} q_i \,\log\left(\frac{1}{q_i\, i^{s_2}\, H_{n,s_2}}\right) \leq \sum_{i=1}^{n} q_i \,\log\left(\frac{1}{q_i\, i^{s_1}\, H_{n,s_1}}\right). \tag{51}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (51).

Proof. If we choose the function $f(x) := \log x$ and $t_1 = t_2 = 0$ in Theorem 3, we get the required results. Q.E.D.

Corollary 4.5. If $n \in \{1,2,3,...\}$ and $s_1, s_2 > 0$ such that $s_2 \leq s_1$ satisfying (48) and also the base of log is greater than 1, then

$$\begin{aligned} \hat{KL}\left(i,n,s_2,\mathbf{1}\right) &:= \sum_{i=1}^{n} \log\left(\frac{1}{i^{s_2}\, H_{n,s_2}}\right) \\ &\geq \hat{KL}\left(i,n,s_1,\mathbf{1}\right) := \sum_{i=1}^{n} \log\left(\frac{1}{i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{52}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (52).

Proof. If we choose $q_i := 1$, $(i = 1,...,n)$ in (50), then we get (52). Q.E.D.

Application 16. Let $n \in \{1,2,3,...\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (48) and also let $q_i > 0$, $(i = 1,...,n)$,

(a) if $\frac{i^{0,892869}}{(i+1)^{0,892869}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1,...,n)$ and the base of log is greater than 1, then the following bound for the Kullback-Leibler divergence of the distributions associated to the English and Russian languages depending only on the parameter n hold

$$\begin{aligned} 0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ \leq \log\left(n^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \sum_{i=1}^{n} q_i. \end{aligned} \tag{53}$$

If the base of log is in between 0 and 1, then

$$0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \leq \log\left(\frac{H_{n,0,892869}}{H_{n,0,973863}}\right) \sum_{i=1}^{n} q_i. \tag{54}$$

(b) if $\frac{i^{0,973863}}{(i+1)^{0,973863}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$ and the base of log is greater than 1, then (54) holds. If the base of log is in between 0 and 1, then (53) holds.

Proof. (a) Take the difference of Left Hand and the Right Hand sides of (50) and then putting the experimental values of s_1 and s_2, we have

$$0 \leq \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} q_i \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right)$$

$$= \sum_{i=1}^{n} q_i \log\left(i^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \leq \log\left(n^{0,080994}\, \frac{H_{n,0,973863}}{H_{n,0,892869}}\right) \sum_{i=1}^{n} q_i.$$

In the similar fashion, we can prove the other bounds. Q.E.D.

Application 17. If $n \in \{1, 2, 3, ...\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (48) and the base of log is greater than 1, then we give the following bound associated to the English and Russian languages:

$$0 \leq \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) - \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \leq \log\left(n^{0,080994} \frac{H_{n,0,973863}}{H_{n,0,892869}}\right)^{n}. \tag{55}$$

If the base of log is in between 0 and 1, then

$$0 \leq \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) - \sum_{i=1}^{n} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \leq \log\left(\frac{H_{n,0,892869}}{H_{n,0,973863}}\right)^{n}. \tag{56}$$

Corollary 4.6. Let $n \in \{1, 2, 3, ...\}$ and $s_1, s_2 > 0$ such that $s_2 \leq s_1$ satisfying (48) and also let $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{i^{s_2}}{(i+1)^{s_2}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$ and the base of log is greater than 1, then

$$\hat{KL}_{id}(i, n, s_2, \mathbf{q}) := \sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{q_i\, i^{s_2} H_{n,s_2}}\right) \leq \hat{KL}_{id}(i, n, s_1, \mathbf{q}) := \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \tag{57}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (57).

(b) if $\frac{i^{s_1}}{(i+1)^{s_1}} \geq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$ and the base of log is greater than 1, then

$$\sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{q_i\, i^{s_2} H_{n,s_2}}\right) \geq \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{q_i\, i^{s_1} H_{n,s_1}}\right). \tag{58}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (58).

Proof. If we choose the function $xf(x) := x \log x$ and $t_1 = t_2 = 0$ in Theorem 5, we get the required results. Q.E.D.

Corollary 4.7. If $n \in \{1, 2, 3, ...\}$ and $s_1, s_2 > 0$ such that $s_2 \leq s_1$ satisfying (48) and the base of log is greater than 1, then

$$\begin{aligned} \hat{KL}_{id}(i, n, s_2, \mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{i^{s_2} H_{n,s_2}} \log\left(\frac{1}{i^{s_2} H_{n,s_2}}\right) \\ \leq \hat{KL}_{id}(i, n, s_1, \mathbf{1}) &:= \sum_{i=1}^{n} \frac{1}{i^{s_1} H_{n,s_1}} \log\left(\frac{1}{i^{s_1} H_{n,s_1}}\right). \end{aligned} \tag{59}$$

If the base of log is in between 0 and 1, then the reverse inequality holds in (59).

Proof. If we choose $q_i := 1$, $(i = 1, ..., n)$ in (57), then we get (59). Q.E.D.

Application 18. Let $n \in \{1, 2, 3, ...\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (48) and also let $q_i > 0$, $(i = 1, ..., n)$,

(a) if $\frac{i^{0,892869}}{(i+1)^{0,892869}} \leq \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$ and the base of log is greater than 1, then

$$\begin{aligned} 0 &\leq \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ &\quad - \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \\ &\leq \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, H_{n,0,973863}}\right) - \frac{1}{n^{0,892869} H_{n,0,892869}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, n^{0,892869} H_{n,0,892869}}\right). \end{aligned} \tag{60}$$

If the base of log is in between 0 and 1, then

$$\begin{aligned} 0 &\le \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{q_i\, i^{0,892869} H_{n,0,892869}}\right) \\ &\quad - \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{q_i\, i^{0,973863} H_{n,0,973863}}\right) \\ &\le \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, H_{n,0,892869}}\right) - \frac{1}{n^{0,973863} H_{n,0,973863}} \sum_{i=1}^{n} \log\left(\frac{1}{q_i\, n^{0,973863} H_{n,0,973863}}\right). \end{aligned} \tag{61}$$

(b) if $\frac{i^{0,973863}}{(i+1)^{0,973863}} \ge \frac{q_{i+1}}{q_i}$ $(i = 1, ..., n)$ and the base of log is greater than 1, then (61) holds. If the base of log is in between 0 and 1, then (60) holds.

Application 19. If $n \in \{1, 2, 3, ...\}$, $s_1 = 0,973863$ for the English language and $s_2 = 0,892869$ for the Russian language such that satisfying (48) and the base of log is greater than 1, then

$$\begin{aligned} 0 &\le \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \\ &\quad - \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \\ &\le \frac{1}{H_{n,0,973863}} \log\left(\frac{1}{H_{n,0,973863}}\right)^{n} - \frac{n^{0,107131}}{H_{n,0,892869}} \log\left(\frac{1}{n^{0,892869} H_{n,0,892869}}\right). \end{aligned} \tag{62}$$

If the base of log is in between 0 and 1, then

$$\begin{aligned} 0 &\le \frac{1}{H_{n,0,892869}} \sum_{i=1}^{n} \frac{1}{i^{0,892869}} \log\left(\frac{1}{i^{0,892869} H_{n,0,892869}}\right) \\ &\quad - \frac{1}{H_{n,0,973863}} \sum_{i=1}^{n} \frac{1}{i^{0,973863}} \log\left(\frac{1}{i^{0,973863} H_{n,0,973863}}\right) \\ &\le \frac{1}{H_{n,0,892869}} \log\left(\frac{1}{H_{n,0,892869}}\right)^{n} - \frac{n^{0,026137}}{H_{n,0,973863}} \log\left(\frac{1}{n^{0,973863} H_{n,0,973863}}\right). \end{aligned} \tag{63}$$

AUTHOR'S CONTRIBUTION

All authors contributed equally. All authors read and approved the final manuscript.

COMPETING INTERESTS

The authors declare that they have no competing interests.

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