Hankel determinant for starlike and convex functions of order alpha

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\begin{abstract}

The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_2 a_4 - a_3^2|$ for starlike and convex functions of order $\alpha$ ($0 \leq \alpha < 1$), also for the inverse function of $f$, belonging to the class of convex functions of order $\alpha$, using Toeplitz determinants.

2000 Mathematics Subject Classification. 30C45. 30C50.

Keywords. Analytic function, starlike and convex functions, upper bound, second Hankel functional, positive real function, Toeplitz determinants.

\end{abstract}

1 Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $E = \{ z : |z| < 1 \}$. Let $S$ be the subclass of $A$, consisting of univalent functions. In 1976, Noonan and Thomas [15] defined the $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.$$ 

(1.2)

This determinant has been considered by several authors in the literature. For example, Noor [16] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [11]. One can easily observe that the Fekete-Szeg"{o} functional is $H_2(1)$. Fekete-Szeg"{o} then further generalized the estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. Ali [3] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szeg"{o} functional $|\gamma_3 - \tau_2^2|$, where $t$ is real, for the inverse function of $f$ defined as $f^{-1}(w) = w + \sum_{n=3}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order $\alpha$ ($0 < \alpha \leq 1$) denoted by $\tilde{S}T(\alpha)$.

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant

$$\begin{vmatrix}
a_2 & a_3 \\
a_3 & a_4
\end{vmatrix} = |a_2 a_4 - a_3^2|. $$

(1.3)
Janteng, Halim and Darus [10] have considered the functional \( |a_2a_4 - a_3^2| \) and found a sharp bound for the function \( f \) in the subclass \( \mathcal{RT} \) of \( S \), consisting of functions whose derivative has a positive real part studied by Mac Gregor [12]. In their work, they have shown that if \( f \in \mathcal{RT} \) then \( |a_2a_4 - a_3^2| \leq \frac{4}{9} \). These authors [9] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of \( S \), namely, starlike and convex functions denoted by \( \text{ST} \) and \( \text{CV} \) and shown that \( |a_2a_4 - a_3^2| \leq 1 \) and \( |a_2a_4 - a_3^2| \leq \frac{1}{4} \) respectively. Mishra and Gochhayat [13] have obtained the sharp bound to the non-linear functional \( |a_2a_4 - a_3^2| \) for the class of analytic functions denoted by \( R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2}) \), defined as \( \text{Re} \left\{ e^{i\alpha} \frac{2z^2f''(z)}{f'(z)} \right\} > \rho \cos \alpha \), using the fractional differential operator denoted by \( \Omega^2_\lambda \), defined by Owa and Srivastava [17]. These authors have shown that, if \( f \in R_\lambda(\alpha, \rho) \) then \( |a_2a_4 - a_3^2| \leq \left\{ \frac{(1-\rho^2)(2-\lambda^2)(3-\lambda^2)\cos^2\alpha}{9} \right\} \). Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([14], [4], [1]).

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we obtain an upper bound to the functional \( |a_2a_4 - a_3^2| \) for the function \( f \) belonging to the classes starlike and convex functions of order \( \alpha \), denoted by \( \text{ST}(\alpha) \) and \( \text{CV}(\alpha) \), defined as follows.

**Definition 1.1.** Let \( f \) be given by (1.1). Then \( f \in \text{ST}(\alpha) \) \((0 \leq \alpha \leq 1)\), if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha, \quad \forall z \in E. \quad (1.4)
\]

It is observed that for \( \alpha = 0 \), we obtain \( \text{ST}(0) = \text{ST} \). It follows that \( \text{ST}(\alpha) \subset \text{ST} \), for \((0 \leq \alpha < 1)\), \( \text{ST}(1) = \text{z} \) and \( \text{ST}(\alpha) \subset \text{ST}(\beta) \), for \( \alpha \geq \beta \). Robertson [19] obtained that if \( f \in \text{ST}(\alpha) \) \((0 \leq \alpha \leq 1)\), then

\[
|a_n| \leq \left\lfloor \frac{1}{(n-1)!} \prod_{k=2}^{n} (k-2\alpha) \right\rfloor, \quad \text{for} \quad n = 2, 3, ...
\]

(1.5)

The inequality in (1.5) is sharp for the function \( s_\alpha(z) = \left\{ \frac{z}{(1-z)(1-z^2)} \right\} \), for every integer \( n \geq 2 \).

**Definition 1.2.** Let \( f \) be given by (1.1). Then \( f \in \text{CV}(\alpha) \) \((0 \leq \alpha \leq 1)\), if and only if

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \alpha, \quad \forall z \in E. \quad (1.6)
\]

Choosing \( \alpha = 0 \), we get \( \text{CV}(0) = \text{CV} \). It is observed that the sets \( \text{ST}(\alpha) \) and \( \text{CV}(\alpha) \) become smaller as the value of \( \alpha \) increases [6]. Further, from the Definitions 1.1 and 1.2, we observe that, there exists an Alexander type Theorem [2], which relates the classes \( \text{ST}(\alpha) \) and \( \text{CV}(\alpha) \), stated as follows.

\[
f \in \text{CV}(\alpha) \Leftrightarrow zf' \in \text{ST}(\alpha).\]

We first state some preliminary Lemmas required for proving our results.

### 2 Preliminary Results

Let \( P \) denote the class of functions \( p \) analytic in \( E \), for which \( \text{Re}\{p(z)\} > 0 \),

\[
p(z) = (1 + c_1z + c_2z^2 + c_3z^3 + \ldots) = 1 + \sum_{n=1}^{\infty} c_nz^n, \forall z \in E. \quad (2.1)
\]
Lemma 2.1. ([18]) If $p \in P$, then $|c_k| \leq 2$, for each $k \geq 1$.

Lemma 2.2. ([7]) The power series for $p$ given in (2.1) converges in the unit disc $E$ to a function in $P$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots,$$

and $c_{-k} = \tau_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^m \rho_k p_0(\exp(ik)z)$, $\rho_k > 0$, $t_k$ real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m - 1)$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition is due to Caratheodory and Toeplitz can be found in [7].

We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \tau_1 & 2 & c_1 \\ \tau_2 & \tau_1 & 2 \end{vmatrix} = [8 + 2Re(c_1^2c_2) - 2 |c_2|^2 - 4c_1^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some } x, \quad |x| \leq 1. \quad (2.2)$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^2)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \leq 2(4 - c_1^2)^2 - 2|2c_2 - c_1^2|^2 \leq (2c_2 - c_1^2)|z|. \quad (2.3)$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}
\quad \text{for some real value of } z, \quad \text{with } |z| \leq 1. \quad (2.4)$$

3 Main Results

Theorem 3.1. If $f(z) \in ST(\alpha) \,(0 \leq \alpha \leq {\frac{1}{2}}, then

$$|az_2a_4 - a_2^3| \leq (1 - \alpha)^2.$$

Proof. Since $f(z) = z + \sum_{n=2}^\infty a_n z^n \in ST(\alpha)$, from the Definition 1.1, there exists an analytic function $p \in P$ in the unit disc $E$ with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{zf'(z) - \alpha f(z) \right\} \quad \Leftrightarrow \quad \left\{zf'(z) - \alpha f(z) \right\} = \{(1 - \alpha)f(z)p(z)\}. \quad (3.1)$$
Replacing \( f(z) \), \( f'(z) \) and \( p(z) \) with their equivalent series expressions in (3.1), we have

\[
\left\{ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right\} - \alpha \left\{ \sum_{n=2}^{\infty} a_n z^n \right\} = (1 - \alpha) \left[ \sum_{n=2}^{\infty} a_n z^n \right] \times \left[ 1 + \sum_{n=1}^{\infty} c_n z^n \right].
\]

Upon simplification, we obtain

\[
[a_2 z + 2a_3 z^2 + 3a_4 z^3 + ...] = (1 - \alpha)[c_1 z + (c_2 + c_1 a_2)z^2 + (c_3 + c_2 a_2 + c_1 a_3)z^3 + ...] \tag{3.2}
\]

Equating the coefficients of like powers of \( z \), \( z^2 \) and \( z^3 \) respectively in (3.2), after simplifying, we get

\[
[a_2 = (1 - \alpha)c_1; a_3 = \frac{(1 - \alpha)}{2} \{c_2 + (1 - \alpha)c_1^2\}; \\
\]

\[
a_4 = \frac{(1 - \alpha)}{6} \{2c_3 + 3(1 - \alpha)c_1 c_2 + (1 - \alpha)^2 c_1^3\} \tag{3.3}
\]

Substituting the values of \( a_2, a_3 \) and \( a_4 \) from (3.3) in the second Hankel determinant \( |a_2 a_4 - a_3^2| \) for the function \( f \in ST(\alpha) \), we have

\[
|a_2 a_4 - a_3^2| = \left| (1 - \alpha) c_1 \times \frac{(1 - \alpha)}{6} \{2c_3 + 3(1 - \alpha)c_1 c_2 + (1 - \alpha)^2 c_1^3\} \right| - \frac{(1 - \alpha)^2}{4} \{c_2 + (1 - \alpha)c_1^2\}^2.
\]

After simplifying, we get

\[
|a_2 a_4 - a_3^2| = \frac{(1 - \alpha)^2}{12} \times \left| 4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4 \right|. \tag{3.4}
\]

Substituting the values of \( c_2 \) and \( c_3 \) from (2.2) and (2.4) respectively from Lemma 2.2 in the right hand side of (3.4), we have

\[
|4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4| = 4c_1 \times \frac{1}{4} \left| c_1^4 + 2c_1 (4 - c_1^2) x - c_1 (4 - c_1^2) x^2 + 2(4 - c_1^2)(1 - |x|^2) z \right| - 3 \times \frac{1}{4} \left| c_1^2 + x(4 - c_1^2) \right|^2 - (1 - \alpha)^2 c_1^4.
\]

Using the facts that \( |x| < 1 \) and \( |xa + yb| \leq |x||a| + |y||b| \), where \( x, y, a \) and \( b \) are real numbers, after simplifying, we get

\[
4 \left| 4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4 \right| \leq \left| (-4a^2 + 8a - 3)c_1^4 + 8c_1 (4 - c_1^2) \right| + 2c_1^2 (4 - c_1^2)|x| - (c_1 + 2)(c_1 + 6)(4 - c_1^2)|x|^2. \tag{3.5}
\]
Moreover, for fixed $c$, the interior of the closed square $(0, 1) \times (0, 1)$.

From the expressions (3.7) and (3.13), after simplifying, we get

$$
4 \left| 4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4 \right| \leq \left| (-4\alpha^2 + 8\alpha - 3)c_4^4 + 8c_1(4 - c_1^2) + 2c_1^2(4 - c_1^2)|x| - (c_1 - 2)(c_1 - 6)(4 - c_1^2)|x|^2 \right| \quad (3.6)
$$

Choosing $c_1 = c \in [0, 2]$, applying Triangle inequality and replacing $|x|$ by $\mu$ in the right hand side of (3.6), we get

$$
4 \left| 4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4 \right| \leq \left| (4\alpha^2 - 8\alpha + 3)c_4^4 + 8c(4 - c^2) + 2c^2(4 - c^2)\mu + (c - 2)(c - 6)(4 - c^2)\mu^2 \right| = F(c, \mu) \quad (3.7)
$$

Where

$$
F(c, \mu) = [(4\alpha^2 - 8\alpha + 3)c_4^4 + 8c(4 - c^2) + 2c^2(4 - c^2)\mu + (c - 2)(c - 6)(4 - c^2)\mu^2].
$$

We next maximize the function $F(c, \mu)$ on the closed square $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (3.8) partially with respect to $\mu$, we get

$$
\frac{\partial F}{\partial \mu} = 2 \left[ c^2 + (c - 2)(c - 6)\mu \right] \times (4 - c^2) \quad (3.9)
$$

For $0 < \mu < 1$, for fixed $c$ with $0 < c < 2$, from (3.9), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c, \mu)$ is an increasing function of $\mu$ and hence it cannot have a maximum value at any point in the interior of the closed square $[0, 2] \times [0, 1]$.

Moreover, for fixed $c \in [0, 2]$, we have

$$
\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) \quad (3.10)
$$

From the relations (3.8) and (3.10), upon simplification, we obtain

$$
G(c) = \{4\alpha(\alpha - 2)c_4^4 + 48\} \quad (3.11)
$$

$$
G'(c) = \{16\alpha(\alpha - 2)c_3^3\} \quad (3.12)
$$

From the expression (3.12), we observe that $G'(c) \leq 0$ for all values of $0 \leq c \leq 2$ and $0 \leq \alpha \leq \frac{1}{2}$. Therefore, $G(c)$ is a monotonically decreasing function of $c$ in the interval $[0, 2]$ so that its maximum value occurs at $c = 0$. From (3.11), we obtain

$$
\max_{0 \leq c \leq 2} G(0) = 48 \quad (3.13)
$$

From the expressions (3.7) and (3.13), after simplifying, we get

$$
\left| 4c_1 c_3 - 3c_2^2 - (1 - \alpha)^2 c_1^4 \right| \leq 12 \quad (3.14)
$$

From the expressions (3.4) and (3.14), upon simplification, we obtain

$$
|a_2 \alpha_4 - a_3^2| \leq (1 - \alpha)^2 \quad (3.15)
$$
Upon simplification, we obtain

\[ |a_2a_4 - a_3^2| \leq \left[ \frac{(1 - \alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)} \right]. \]

**Remark.** For the choice of \( \alpha = 0 \), we get \( ST(0) = ST \), for which, from (3.15), we get \( |a_2a_4 - a_3^2| \leq 1 \). This inequality is sharp and coincides with that of Janteng, Halim and Darus [9].

**Theorem 3.2.** If \( f(z) \in CV(\alpha) \) \((0 \leq \alpha \leq 1)\), then

\[ \{f'(z) + zf''(z)\} - \alpha f'(z) \]

for the function \( f \) and substituting the values of \( a \) from (3.18) in the second Hankel functional \( 0 \) with \( p(0) = 1 \) and \( \text{Re}\{p(z)\} > 0 \) such that

\[ \left\{ \frac{f'(z) + zf''(z)}{(1 - \alpha)f'(z)} \right\} = p(z) \]

\[ \Leftrightarrow \left\{ (1 - \alpha)f'(z) + zf''(z) \right\} = \left\{ (1 - \alpha)f'(z)p(z) \right\}. \] (3.16)

Replacing \( f'(z), f''(z) \) and \( p(z) \) with their equivalent series expressions in (3.16), we have

\[ \left[ (1 - \alpha) \left\{ 1 + \sum_{n=2}^{\infty} na_n z^{n-1} \right\} + z \left\{ \sum_{n=2}^{\infty} n(n - 1)a_n z^{n-2} \right\} \right] \]

\[ = \left[ (1 - \alpha) \left\{ 1 + \sum_{n=2}^{\infty} na_n z^{n-1} \right\} \times \left\{ 1 + \sum_{n=1}^{\infty} c_n z^{n} \right\} \right]. \]

Upon simplification, we obtain

\[ [2a_2z + 6a_3z^2 + 12a_4z^3 +...] \]

\[ = (1 - \alpha)[c_1z + (c_2 + 2c_1a_2)z^2 + (c_3 + 2c_2a_2 + 3c_1a_2)z^3 + ...]. \] (3.17)

Equating the coefficients of like powers of \( z, z^2 \) and \( z^3 \) respectively in (3.17), after simplifying, we get

\[ a_2 = \frac{(1 - \alpha)}{2} c_1; a_3 = \frac{(1 - \alpha)}{6} \left\{ c_2 + (1 - \alpha)c_1^2 \right\}; \]

\[ a_4 = \frac{(1 - \alpha)}{24} \left\{ 2c_3 + 3(1 - \alpha)c_1 c_2 + (1 - \alpha)^2 c_1^3 \right\}. \] (3.18)

Substituting the values of \( a_2, a_3 \) and \( a_4 \) from (3.18) in the second Hankel functional \( |a_2a_4 - a_3^2| \) for the function \( f \in CV(\alpha) \), upon simplification, we obtain

\[ |a_2a_4 - a_3^2| = \frac{(1 - \alpha)^2}{144} \times [6c_1c_3 - 4c_2^2 + (1 - \alpha)c_1^2 c_2 - (1 - \alpha)^2 c_1^3]. \] (3.19)

Applying the same procedure as described in Theorem 3.1, we get

\[ 2 \left[ 6c_1c_3 - 4c_2^2 + (1 - \alpha)c_1^2 c_2 - (1 - \alpha)^2 c_1^3 \right] \leq \left[ (3\alpha - 2\alpha^2)c_1^2 c_2 - (1 - \alpha)^2 c_1^3 \right], \]

\[ + 6c_1(4 - c_1^2) + (3 - \alpha)c_1^2(4 - c_1^2)|x| - (c_1 + 2)(c_1 + 4)(4 - c_1^2)|x|^2. \] (3.20)
Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in the right hand side of (3.20), upon simplification, we obtain

$$2 \left| 6c_1 c_3 - 4c_2^2 + (1 - \alpha)c_1^2 c_2 - (1 - \alpha)^2 c_1^4 \right| \leq |(3\alpha - 2\alpha^2)c_1^4 + 6\alpha(4 - c^2) + (3 - \alpha)c_1^2(4 - c^2)\mu + (c - 2)(c - 4)(4 - c^2)\mu^2|$$

(3.21)

Applying the same procedure as described in Theorem 3.1, we obtain

$$2 \left| 6c_1 c_3 - 4c_2^2 + (1 - \alpha)c_1^2 c_2 - (1 - \alpha)^2 c_1^4 \right| \leq |(3\alpha - 2\alpha^2)c_1^4 + 6\alpha(4 - c^2) + (3 - \alpha)c_1^2(4 - c^2)\mu + (c - 2)(c - 4)(4 - c^2)\mu^2|$$

(3.22)

Where

$$F(c, \mu) = |(3\alpha - 2\alpha^2)c_1^4 + 6\alpha(4 - c^2) + (3 - \alpha)c_1^2(4 - c^2)\mu + (c - 2)(c - 4)(4 - c^2)\mu^2|.$$  

We next maximize the function $F(c, \mu)$ on the closed square $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (3.23) partially with respect to $\mu$, we get

$$\frac{\partial F}{\partial \mu} = [(3 - \alpha)c_1^2 + 2(c - 2)(c - 4)\mu] \times (4 - c^2).$$

(3.24)

For $0 < \mu < 1$, for fixed $c$ with $0 < c < 2$ and for $(0 \leq \alpha \leq 1)$, from (3.24), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c, \mu)$ is an increasing function of $\mu$ and hence it cannot have a maximum value at any point in the interior of the closed square $[0, 2] \times [0, 1]$. Moreover, for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) \text{(say)}. \quad (3.25)$$

In view of the expression (3.25), replacing $\mu$ by 1 in (3.23), after simplifying, we get

$$G(c) = 2 \left\{ -(\alpha^2 - 2\alpha + 2)c^4 + 2(2 - \alpha)c^2 + 16 \right\}. \quad (3.26)$$

$$G'(c) = 2 \left\{ -4(\alpha^2 - 2\alpha + 2)c^3 + 4(2 - \alpha)c \right\}. \quad (3.27)$$

$$G''(c) = 2 \left\{ -12(\alpha^2 - 2\alpha + 2)c^2 + 4(2 - \alpha) \right\}. \quad (3.28)$$

For Optimum value of $G(c)$, consider $G'(c) = 0$. From (3.27), we get

$$-8c \left\{ (\alpha^2 - 2\alpha + 2)c^2 - (2 - \alpha) \right\} = 0. \quad (3.29)$$

We now discuss the following Cases.

**Case 1)** If $c = 0$, then, from (3.28), we obtain

$$G''(c) = 8(2 - \alpha) > 0, \quad \text{for} \quad 0 \leq \alpha < 1.$$
From the second derivative test, \( G(c) \) has minimum value at \( c = 0 \).

**Case 2)** If \( c \neq 0 \), then, from (3.29), we get
\[
c = \left\{ \frac{(2 - \alpha)}{(\alpha^2 - 2\alpha + 2)} \right\}. \tag{3.30}
\]

Using the value of \( c^2 \) given in (3.30) in (3.28), after simplifying, we obtain
\[
G''(c) = -\left\{ \frac{16(2 - \alpha)}{(\alpha^2 - 2\alpha + 2)} \right\} < 0, \quad \text{for} \quad 0 \leq \alpha < 1.
\]

By the second derivative test, \( G(c) \) has maximum value at \( c \), where \( c^2 \) given by (3.30). Using the value of \( c^2 \) given by (3.30) in (3.26), upon simplification, we obtain
\[
\max_{0 \leq c \leq 2} G(c) = 2 \left[ \frac{(17\alpha^2 - 36\alpha + 36)}{(\alpha^2 - 2\alpha + 2)} \right]. \tag{3.31}
\]

Considering the maximum value of \( G(c) \) at \( c \), where \( c^2 \) is given by (3.30), from (3.22) and (3.31), after simplifying, we get
\[
|a_2a_4 - a_3^2| \leq \left[ \frac{(1 - \alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)} \right]. \tag{3.32}
\]

From the expressions (3.19) and (3.32), we obtain
\[
|a_2a_4 - a_3^2| \leq \left[ \frac{(1 - \alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)} \right]. \tag{3.33}
\]

This completes the proof of our Theorem 3.2.

**Remark.** Choosing \( \alpha = 0 \), we have \( CV(0) = CV \), for which, from (3.33), we get \( |a_2a_4 - a_3^2| \leq \frac{1}{8} \). This inequality is sharp and coincides with that of Janteng, Halim and Darus [9].

**Theorem 3.3.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(\alpha)(0 \leq \alpha < \frac{\pi}{3}) \) and \( f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n \) near \( w = 0 \), is the inverse function of \( f \), then
\[
|t_2t_4 - t_3^2| \leq \left[ \frac{(57\alpha^2 - 84\alpha + 36)}{288} \right].
\]

**Proof.** Since \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(\alpha) \), from the definition of inverse function of \( f \), we have
\[
w = f \left\{ f^{-1}(w) \right\}. \tag{3.34}
\]

Using the expression for \( f(z) \), the relation (3.34) is equivalent to
\[
w = f \left\{ f^{-1}(w) \right\} = \left[ f^{-1}(w) + \sum_{n=2}^{\infty} a_n \left\{ f^{-1}(w) \right\}^n \right] = \left\{ f^{-1}(w) \right\} + a_2 \left\{ f^{-1}(w) \right\}^2 + a_3 \left\{ f^{-1}(w) \right\}^3 + ... \tag{3.35}
\]
Using the expression for $f^{-1}(w)$ in (3.35), we have
\[
w = \left\{ (w + t_2w^2 + t_3w^3 + ...) + a_2(w + t_2w^2 + t_3w^3 + ...)^2 + a_3(w + t_2w^2 + t_3w^3 + ...)^3 + a_4(w + t_2w^2 + t_3w^3 + ...)^4 + ... \right\}.
\]
Upon simplification, we obtain
\[
\{(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + ... \} = 0. \tag{3.36}
\]
Equating the coefficients of like powers of $w^2$, $w^3$ and $w^4$ on both sides of (3.36) respectively, we have
\[
\{(t_2 + a_2) = 0; (t_3 + 2a_2t_2 + a_3) = 0; (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4) = 0 \}.
\]
After simplifying, we get
\[
\begin{align*}
t_2 &= -a_2; t_3 = \{-a_3 + 2a_2^2\}; t_4 = \{-a_4 + 5a_2a_3 - 5a_2^2\}. \tag{3.37}
\end{align*}
\]
Using the values of $a_2$, $a_3$ and $a_4$ in (3.18) along with (3.37), upon simplification, we obtain
\[
\begin{align*}
\{t_2 &= -\frac{(1 - \alpha)c_1}{2}; t_3 = -\frac{(1 - \alpha)}{6}\{c_2 - 2(1 - \alpha)c_1^2\}; \\
t_4 &= -\frac{(1 - \alpha)}{24}\{2c_3 - 7(1 - \alpha)c_1c_2 + 6(1 - \alpha)^2c_1^3\} \tag{3.38}
\end{align*}
\]
Substituting the values of $t_2$, $t_3$ and $t_4$ from (3.38) in the second Hankel functional $|t_2t_4 - t_3^2|$ for the inverse function $f \in CV(\alpha)$, after simplifying, we get
\[
|t_2t_4 - t_3^2| = \frac{(1 - \alpha)^2}{144} \times |6c_1c_3 - 5(1 - \alpha)c_1^2c_2 - 4c_2^2 + 2(1 - \alpha)^2c_1^4|. \tag{3.39}
\]
Substituting the values of $c_2$ and $c_3$ from (2.2) and (2.4) respectively from Lemma 2.2 in the right hand side of (3.39), using the same procedure as described in Theorem 3.1, upon simplification, we obtain
\[
\begin{align*}
&2|6c_1c_3 - 5(1 - \alpha)c_1^2c_2 - 4c_2^2 + 2(1 - \alpha)^2c_1^4| \leq \{(3\alpha - 4\alpha^2)c_1^4 \\
&+ 6c_1(4 - c_2^2) + (3 - 5\alpha)c_1^2(4 - c_2^2)|x| - (c_1 + 2)(c_1 + 4)(4 - c_2^2)|x|^2\}. \tag{3.40}
\end{align*}
\]
Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in the right hand side of (3.40), applying the same procedure as described in Theorem 3.1, after simplifying, we get
\[
\begin{align*}
&2|6c_1c_3 - 5(1 - \alpha)c_1^2c_2 - 4c_2^2 + 2(1 - \alpha)^2c_1^4| \leq \left[(3\alpha - 4\alpha^2)c_1^4 + 6c(4 - c^2) + (3 - 5\alpha)c_1^2(4 - c^2)\mu - (c - 2)(c - 4)(4 - c^2)\mu^2\right] \\
&= F(c, \mu)(say), \quad \text{with } 0 \leq \mu = |x| \leq 1. \tag{3.41}
\end{align*}
\]
Where
\[
F(c, \mu) = \left[(3\alpha - 4\alpha^2)c^4 + 6c(4 - c^2) + (3 - 5\alpha)c_1^2(4 - c^2)\mu + (c - 2)(c - 4)(4 - c^2)\mu^2\right]. \tag{3.42}
\]
We next maximize the function \( F(c, \mu) \) on the closed square \([0, 2] \times [0, 1]\). Differentiating \( F(c, \mu) \) in (3.42) partially with respect to \( \mu \), we obtain

\[
\frac{\partial F}{\partial \mu} = [(3 - 5\alpha)c^2 + 2(c - 2)(c - 4)\mu] \times (4 - c^2). \tag{3.43}
\]

For \( 0 < \mu < 1 \), for fixed \( c \) with \( 0 < c < 2 \) and for \( 0 \leq \alpha \leq 1 \), from (3.43), we observe that \( \frac{\partial F}{\partial \mu} > 0 \). Consequently, \( F(c, \mu) \) is an increasing function of \( c \) and hence it cannot have a maximum value at any point in the interior of the closed square \([0, 2] \times [0, 1]\). Moreover, for fixed \( c \in [0, 2] \), we have

\[
\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) \text{(say)}. \tag{3.44}
\]

Replacing \( \mu \) by 1 in (3.42), after simplifying, we get

\[
G(c) = \{-4(1 - \alpha)^2c^4 + 4(2 - 5\alpha)c^2 + 32\}. \tag{3.45}
\]

\[
G'(c) = \{-16(1 - \alpha)^2c^3 + 8(2 - 5\alpha)c\}. \tag{3.46}
\]

\[
G''(c) = \{-48(1 - \alpha)^2c^2 + 8(2 - 5\alpha)\}. \tag{3.47}
\]

For maximum or minimum value of \( G(c) \), consider \( G'(c) = 0 \). From (3.46), we get

\[
-8c \{2(1 - \alpha)^2c^2 - (2 - 5\alpha)\} = 0. \tag{3.48}
\]

We now discuss the following Cases.

**Case 1)** If \( c = 0 \), then, from (3.47), we obtain

\[
G''(c) = \{8(2 - 5\alpha)\} > 0, \quad \text{for} \quad 0 \leq \alpha < \frac{2}{5}.
\]

From the second derivative test, \( G(c) \) has minimum value at \( c = 0 \).

**Case 2)** If \( c \neq 0 \), then, from (3.48), we get

\[
c^2 = \left\{ \frac{(2 - 5\alpha)}{2(1 - \alpha)^2} \right\}. \tag{3.49}
\]

Using the value of \( c^2 \) given in (3.49) in (3.47), after simplifying, we obtain

\[
G''(c) = -\{16(2 - 5\alpha)\} < 0, \quad \text{for} \quad 0 \leq \alpha < \frac{2}{5}.
\]

By the second derivative test, \( G(c) \) has maximum value at \( c \), where \( c^2 \) given in (3.49). Using the value of \( c^2 \) given by (3.49) in (3.45), upon simplification, we obtain

\[
\max_{0 \leq c \leq 2} G(c) = \left[ \frac{57\alpha^2 - 84\alpha + 36}{(1 - \alpha)^2} \right]. \tag{3.50}
\]

Considering, the maximum value of \( G(c) \) at \( c \), where \( c^2 \) is given by (3.49), from (3.41) and (3.50), after simplifying, we get

\[
\left| 6c_1c_3 - 5(1 - \alpha)c_1^2c_2 - 4c_2^2 + 2(1 - \alpha)^2c_1^4 \right| \leq \left[ \frac{57\alpha^2 - 84\alpha + 36}{2(1 - \alpha)^2} \right]. \tag{3.51}
\]
From the expressions (3.39) and (3.51), upon simplification, we obtain

\[ |t_2t_4 - t_3^2| \leq \left( \frac{(57\alpha^2 - 84\alpha + 36)}{288} \right). \]  

(3.52)

This completes the proof of our Theorem 3.3.

**Remark.1** Choosing \( \alpha = 0 \), we get \( CV(0) = CV \), class of convex functions, for which, from (3.52), we get

\[ |t_2t_4 - t_3^2| \leq \frac{1}{4}. \]

**Remark.2** For the function \( f \in CV \), we have

\[ |a_2a_4 - a_2^2| \leq \frac{1}{4} \]

and

\[ |t_2t_4 - t_3^2| \leq \frac{1}{4}. \]

From these two results, we conclude that the upper bound to the second Hankel determinant of a convex function and its inverse is the same.

**Acknowledgements.** The authors would like to thank the esteemed Referee for his/her valuable suggestions and comments in the preparation of this paper.

**References**


