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Geometric growth for stochastic difference equations with application to branching populations

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We investigate the asymptotic behaviour of discrete-time processes that satisfy a stochastic difference equation. We provide conditions to guarantee geometric growth on the whole set where these processes go to infinity. The class of processes considered includes homogeneous Markov chains. The results are of interest in population dynamics. In this work they are applied to two branching populations.

Keywords: branching processes; discrete-time processes; homogeneous Markov chains; stochastic difference equations

1. Introduction

Within the scientific literature on populations dynamics, branching processes arouse great interest, as has been made clear in recent monographs, such as Kimmel and Axelrod (2002), Pakes (2003) and Haccou *et al.* (2005). The study of the asymptotic behaviour of such processes has been a major focus of attention, in particular the determination of growth rates in certain populations, and the search for conditions to guarantee their geometric growth. For instance, some conditions for geometric growth on the corresponding whole explosion set of the process (i.e., on the whole set where the process goes to infinity) have been established for the standard Bienaymé–Galton–Watson process (see Jagers 1975), and for the population-size dependent branching process (see Küster 1985; Pierre Loti Viaud 1994). However, there are still many classes of branching models for which, although conditions for geometric growth are known, they do not guarantee such growth on the whole explosion set. Examples are the asexual controlled branching process with random control functions (see González *et al.* 2003), and the bisexual Galton–Watson process for which only particular cases have been investigated (see Haccou *et al.* 2005). In addition to the intrinsic interest of solving this problem, its solution will help in the application of inferential results concerning these branching processes.

In this paper we consider this problem in a more general context. Actually, we investigate the asymptotic behaviour of a large class of processes $\{Z_n\}_{n\geqslant 0}$ that satisfy a stochastic

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difference equation. In particular, we focus our attention on the problem of determining conditions to guarantee geometric growth of $\{Z_n\}_{n\geqslant 0}$ on the whole set $\{Z_n \to \infty\}$. This class includes homogeneous branching models as particular cases.

The paper is organized as follows. In Section 2, we present the probabilistic setting, provide the necessary working assumptions, and state the main results. Section 3 deals with their application to two classes of branching models: asexual controlled branching processes with random control functions, and bisexual branching processes. Finally, for ease of reading, the proofs are relegated to Section 4.

2. Probabilistic setting and main results

Let $\{\mathcal{F}_n\}_{n\geqslant 0}$ be a non-decreasing sequence of σ-algebras on some probability space, and let us denote by $\{Z_n\}_{n\geqslant 0}$ a sequence of non-negative integer-valued random variables such that $Z_0 > 0$, Z_n is $\mathcal{F}_n$-measurable, $n \geqslant 0$, and the following stochastic difference equation is almost surely satisfied:

$$Z_{n+1} = Z_n + h(Z_n) + \xi_{n+1}, \qquad n = 0, 1, \ldots, \tag{1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}$ is a measurable function, $\mathbb{R}^+$ denoting the non-negative real numbers, $h(0) = 0$, and $\{\xi_n\}_{n\geqslant 1}$ is a zero-mean martingale difference sequence with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$.

Remark 1. Note that any discrete-time homogeneous Markov chain $\{Z_n\}_{n\geqslant 0}$ can be rewritten in the form (1) by considering $h(k) := \mathrm{E}[Z_{n+1}|Z_n = k] - k$ and $\xi_{n+1} := Z_{n+1} - \mathrm{E}[Z_{n+1}|Z_n]$, $n \geqslant 0$.

As the motivation for this study comes from populations dynamics, we keep the usual terminology in branching theory. Thus, Z_n can be viewed as representing the size of some randomly growing population at time n.

Using the fact that $h(0) = 0$, it follows that $\{Z_n = 0\} \subseteq \{Z_k = 0\}$, $k \geqslant n$. Consequently, when the process reaches the zero state it stays in it for ever. We focus our attention on the set $\{Z_n \to \infty\}$, namely, the explosion event associated with the process. This is obviously included in the complement of $\{Z_n \to 0\}$, which is called the extinction event since it indicates the non-existence of individuals in the population. The aim of this work is to investigate conditions guaranteeing geometric growth for $\{Z_n\}_{n\geqslant 0}$ on the whole explosion set and not just on a part of it.

Using the fact that Z_n is $\mathcal{F}_n$-measurable and $\mathrm{E}[\xi_{n+1}|\mathcal{F}_n] = 0$ almost surely, it follows that $\mathrm{E}[\xi_{n+1}|Z_n] = 0$ almost surely, $n \geqslant 0$. Therefore,

$$\tau_k := \mathrm{E}[Z_{n+1}Z_n^{-1}|Z_n = k] = 1 + k^{-1}h(k), \qquad k = 1, 2, \ldots .$$

By convention, when necessary, we define τ_0 as an arbitrary positive constant. Notice that τ_k can be intuitively interpreted as the expected growth rate per individual when, in a certain generation, there are k progenitors in the population. These expected values will play a major role in this paper. In order to obtain geometric growth for $\{Z_n\}_{n\geqslant 0}$ on its explosion set, it will be assumed that, for a certain $\delta > 0$, $h(x) \sim \delta x$ as $x \nearrow \infty$, hence $\tau_k \sim \tau$ as $k \nearrow \infty$, where $\tau := 1 + \delta$. We prove (Theorem 4(ii)) that, on the whole set $\{Z_n \to \infty\}$, the process $\{W'_n\}_{n \geq 0}$, where $W'_n := Z_n \tau^{-n}$, converges almost surely to a positive and finite random variable. To this end, we use, suitably adapted, some probabilistic techniques considered by Pierre Loti Viaud (1994) for the particular class of asexual population-size dependent branching processes.

Since dealing with this problem directly seems to be difficult, we consider a different approach. Notice that, rewriting Z_n in the form $Z_n = Z_0 \prod_{k=0}^{n-1} Z_{k+1} Z_k^{-1}$, a natural approximation to Z_n is given by $Z_0 \prod_{k=0}^{n-1} \mathrm{E}[Z_{k+1} Z_k^{-1} | Z_k] = Z_0 \prod_{k=0}^{n-1} \tau_{Z_k}$. Consequently, the geometric growth of Z_n can be established through the relation

$$W'_n = W_n \nu_n \tau^{-n} \qquad \text{a.s.,} \tag{2}$$

where $W_0 := W'_0$ and $W_n := Z_n \nu_n^{-1}$ with $\nu_n := \prod_{k=0}^{n-1} \tau_{Z_k}$, $n \geq 1$.

Thus, by (2), to prove that Z_n grows like τ^n on the whole explosion set it is sufficient to verify that, in such an event, the two sequences $\{W_n\}_{n \geq 0}$ and $\{\nu_n \tau^{-n}\}_{n \geq 1}$ are almost surely convergent to positive and finite limits. It is easily checked that the randomly normed sequence, $\{W_n\}_{n \geq 0}$, is a non-negative martingale. Hence, from the martingale convergence theorem, it converges almost surely to a non-negative and finite limit W. It remains to prove that $P(W > 0) > 0$, and thence that $\{W > 0\} = \{Z_n \to \infty\}$ almost surely, and finally that $\{\nu_n \tau^{-n}\}_{n \geq 1}$ converges almost surely on $\{Z_n \to \infty\}$ to a positive and finite limit. To this purpose, we introduce, for $\alpha > 0$, the α-order absolute variation rates $\gamma_{k,\alpha} := \mathrm{E}^{1/\alpha}[|Z_{n+1} Z_n^{-1} - \tau_{Z_n}|^\alpha | Z_n = k]$, $k \geq 1$, and consider the following working assumptions:

(A1) For some $\alpha \in [1, 2]$, there exist a positive constant M and a non-increasing positive function $\eta(x)$ satisfying $\sum_{n=1}^\infty \eta(n) n^{-1} < \infty$, such that $\gamma_{k,\alpha} \leq M \tau_k \eta(k)$, $k \geq 0$. (We assume the existence of $\gamma_{k,\alpha}$, $k \geq 1$, for such an α and take $\gamma_{0,\alpha} := 0$.)
(A2) There exists $\tilde{\tau}$ such that $\tau_k \geq \tilde{\tau} > 1$, $k \geq 0$.
(A3) $|\tau_k - \tau| \leq \lambda(k)$, $k \geq 1$, where $\lambda(x)$ is a positive and non-increasing function such that $\sum_{n=1}^\infty \lambda(n) n^{-1} < \infty$.

Remark 2. Using the lemma in Klebaner (1985: 52), from the function $\eta(x)$ in (A1) we can construct another function $\tilde{\eta}(x)$ that is non-increasing, $\tilde{\eta}(x) \geq \eta^\alpha(x)$, $\sum_{n=1}^\infty \tilde{\eta}(n) n^{-1} < \infty$, and such that $x \tilde{\eta}^\alpha(x^{1/\alpha})$ is a concave and non-decreasing function on $(0, \infty)$. Therefore, there is no loss of generality in assuming that $x \eta^\alpha(x^{1/\alpha})$ is concave and non-decreasing on $(0, \infty)$. Note that, by (A2), $\nu_n \geq \tilde{\tau}^n$, $n \geq 1$. Requirement (A3) concerns the speed of convergence of $\{\tau_k\}_{k \geq 1}$ to τ, as $k \nearrow \infty$. In particular, for homogeneous Markov chains, (A3) is a sufficient condition for the almost sure convergence of $\{W'_n\}_{n \geq 0}$, and a necessary condition for the almost sure limit to be a non-degenerate-at-zero random variable. See Cohn and Klebaner (1986) for details.

Henceforth, α remains fixed by requirement (A1).

Next, we state the main results of the present work. The first theorem establishes the L^α-convergence of $\{W_n\}_{n \geq 0}$ to its almost sure limit W.

Theorem 1. *Assume* (A1) *and* (A2). Then,

(i) $\{W_n\}_{n\geqslant 0}$ *converges in* L^{α} *to* W,
(ii) $\|W_n - W\|_{\alpha} \leqslant K\int_{\tilde{\tau}^n}^{\infty} \eta(x)x^{-1}\,\mathrm{d}x$, for some positive constant K, where $\|X\|_{\alpha} := \mathrm{E}^{1/\alpha}[|X|^{\alpha}]$.

Remark 3. Taking into account that $\mathrm{E}[W_n] = \mathrm{E}[Z_0] > 0$, $n \geqslant 0$, as a consequence of Theorem 1, we deduce that $P(W > 0) > 0$, and therefore (A1) and (A2) are conditions which guarantee that $P(Z_n \to \infty) > 0$. Kersting (1986) used other probabilistic procedures and also assumed the existence of moments with a higher order than that considered in (A1) to derive a positive probability for non-extinction.

It is clear that $\{W > 0\} \subseteq \{Z_n \to \infty\}$. A first step in determining the behaviour of $\{W_n\}_{n\geqslant 0}$ on the whole explosion set is the following theorem.

Theorem 2. *Assume* (A1) *and* (A2). *Then, on* $\{Z_n \to \infty\}$, $\{W_{n+1}W_n^{-1}\}_{n\geqslant 0}$ *converges almost surely to 1.*

From the previous results, we can state the following theorem.

Theorem 3. *Assume* (A1) *and* (A2). *Then* $\{W > 0\} = \{Z_n \to \infty\}$ *almost surely.*

Remark 4. At this point, we know that, under (A1) and (A2), $\{W_n\}_{n\geqslant 0}$ converges almost surely and in L^{α} to W with $P(W > 0) > 0$, and also that $\{W > 0\} = \{Z_n \to \infty\}$ almost surely. Note that (A3) was not used to obtain these results. It will be required, however, to determine the limiting behaviour of $\{\nu_n\tau^{-n}\}_{n\geqslant 1}$ and to obtain, finally, that $\{Z_n\}_{n\geqslant 0}$ grows geometrically like $\{\tau^n\}_{n\geqslant 0}$ on the whole explosion set.

Theorem 4. *Assume* (A1), (A2), *and* (A3). *Then,*

(i) $\{\nu_n\tau^{-n}\}_{n\geqslant 1}$ *converges almost surely to a positive and finite random variable on* $\{Z_n \to \infty\}$,
(ii) $\{W'_n\}_{n\geqslant 0}$ *converges almost surely to* W' *with* $P(W' > 0) > 0$, *and* $\{W' > 0\} = \{Z_n \to \infty\}$ *almost surely.*

Remark 5. In the framework of homogeneous Markov chains, this result is connected with that derived in Cohn and Klebaner (1986) who considered $\alpha = 1$. Indeed, that paper proved the almost sure convergence of the normed process to a non-degenerate-at-zero limit. We have completed their result here by showing that such a limit is non-degenerate at zero on the whole explosion set. Also, using a method similar to that considered in Fujimagari (1976), one could weaken requirements (A1) and (A2) by imposing that they are satisfied from some k_0 onwards.

3. Applications to branching populations

The results clearly are applicable to the great family of homogeneous branching processes owing to their Markovian property. But it is now our intention to particularize them to two branching models of especial importance in the recent literature. We first consider asexual controlled branching processes with random control functions, and then bisexual branching processes.

3.1. Asexual controlled branching processes with random control functions

First studied in Yanev (1975), the asexual controlled branching process with random control functions is defined as follows:

$$Z_0 = N,$$

$$Z_{n+1} = \sum_{j=1}^{\phi_{n,Z_n}} X_{n,j}, \qquad n = 0, 1, \ldots \tag{3}$$

where the empty sum is considered to be 0, N is a positive integer, and $\{X_{n,j} : n \geqslant 0;\, j \geqslant 1\}$, $\{\phi_{n,k} : n, k \geqslant 0\}$ are independent sets of non-negative integer-valued random variables defined on the same probability space. The variables $X_{n,j}$ are independent and identically distributed; their common probability law is called the offspring probability distribution. For $n \geqslant 0$, $\{\phi_{n,k}\}_{k \geqslant 0}$ are independent stochastic processes such that $\phi_{n,k}$, $n \geqslant 0$, are identically distributed. Intuitively, $X_{n,j}$ is the number of offspring of the jth individual in the nth generation, and Z_{n+1} represents the total number of individuals in the $(n+1)$th generation. The individuals give rise to descendants independently with the same offspring probability distribution for each generation, but, when in a certain nth generation there are k individuals, the random variable $\phi_{n,k}$ controls the process in such a way that, if $\phi_{n,k} = j$, then j progenitors will take part in the reproduction process that will determine Z_{n+1}.

Let us write $m := \mathrm{E}[X_{0,j}]$, $m_\alpha := \mathrm{E}[|X_{0,j} - m|^\alpha]$, $\varepsilon_k := \mathrm{E}[\phi_{0,k}]$, and $\varepsilon_{k,\alpha} := \mathrm{E}[|\phi_{0,k} - \varepsilon_k|^\alpha]$, $k \geqslant 0$, $\alpha > 0$. Then, from Theorem 4, we can establish the following result.

Proposition 1. *Assume:*

(a) $\tau_k = mk^{-1}\varepsilon_k \sim \tau > 1$, *as* $k \nearrow \infty$, *with* $\tau_k > 1$, $k \geqslant k_0 \geqslant 1$;

(b) $|\tau_k - \tau| \leqslant \lambda(k)$, $k \geqslant 1$, *where* $\lambda(x)$ *is a positive and non-increasing function such that* $\sum_{n=1}^{\infty} \lambda(n) n^{-1} < \infty$;

(c) *the existence of a positive and non-increasing function* $\eta(x)$ *with* $\sum_{n=1}^{\infty} \eta(n) n^{-1} < \infty$, *and some* $\alpha \in [1, 2]$, *such that for* $k \geqslant k_0$,

$$(m_\alpha \varepsilon_k)^{1/\alpha} \leqslant m\varepsilon_k \eta(k) \quad \text{and} \quad (\varepsilon_{k,\alpha})^{1/\alpha} \leqslant \varepsilon_k \eta(k).$$

Then $\{Z_n \tau^{-n}\}_{n \geqslant 0}$ *converges almost surely, as* $n \nearrow \infty$, *to a finite and non-degenerate-at-zero random variable* W', *with* $\{W' > 0\} = \{Z_n \to \infty\}$ *almost surely.*

Indeed, using the C_α-inequality, namely $|a+b|^\alpha \leqslant 2^{\alpha-1}(|a|^\alpha + |b|^\alpha)$, $\alpha \in [1, 2]$, and the Von Bahr–Esseen inequality (see Von Bahr and Esseen 1965), it follows that $\gamma_{k,\alpha} \leqslant 2k^{-1} m\varepsilon_k \eta(k)$. Hence, as a consequence of Theorem 4(ii), the proposition follows.

As was noted in the Introduction, the results concerning the inference problems arising from asexual controlled branching processes (see González *et al.* 2004, 2005) have improved applicability. In those papers, most of the asymptotic results were established on the set $\{W' > 0\}$, but a doubt was raised there as to how one can possibly know that one is in such a set when only a finite sample is available. From Proposition 1, $\{W' > 0\} = \{Z_n \to \infty\}$ almost surely, and, using Lemma 2.3 of Guttorp (1991), those asymptotic results are also obtained on $\{Z_n > 0\}$, providing a verifiable condition that resolves the doubt.

3.2. Bisexual branching processes

Introduced in Daley (1968), the bisexual Galton–Watson process is a bivariate sequence $\{(F_n, M_n)\}_{n\geqslant 1}$ defined recursively in the form

$$Z_0 = N, \quad (F_{n+1}, M_{n+1}) = \sum_{i=1}^{Z_n} (f_{n,i}, m_{n,i}), \quad Z_{n+1} = L(F_{n+1}, M_{n+1}), \qquad n = 0, 1, \ldots,$$

with the empty sum considered to be (0, 0), where N is a positive integer, $\{(f_{n,i}, m_{n,i}) : i \geqslant 1;\ n \geqslant 0\}$ is a sequence of independent and identically distributed non-negative and integer-valued random variables, and $L : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a function assumed to be monotonic non-decreasing in each argument, integer-valued on the integers, and such that $L(x, y) \leqslant xy$. Intuitively, the process starts with $Z_0 = N \geqslant 1$ couples (female–male mating units), and the variables $f_{n,i}$ and $m_{n,i}$ represent the number of females and males descending from the ith couple of the nth generation, respectively. Consequently, F_{n+1} and M_{n+1} are, respectively, the number of females and males in the $(n+1)$th generation, $n \geqslant 0$, which form $Z_{n+1} = L(F_{n+1}, M_{n+1})$ couples. These couples reproduce independently with the same offspring probability distribution for each generation. Recent reviews of this branching model are provided in Hull (2003) and Haccou *et al.* (2005).

A bisexual Galton–Watson process is said to be superadditive if $L(x_1 + x_2, y_1 + y_2) \geqslant L(x_1, y_1) + L(x_2, y_2)$, $x_1, x_2, y_1, y_2 \in \mathbb{R}^+$. For superadditive bisexual branching processes, Daley *et al.* (1986) proved that $\lim_{k\to\infty} \tau_k = \sup_{k\geqslant 1} \tau_k =: \tau$. Again, from Theorem 4 we establish the following result.

Proposition 2. *Assume:*

(a) $\tau > 1$ *and* $|\tau_k - \tau| \leqslant \lambda(k)$, $k \geqslant 1$, *where* $\lambda(x)$ *is a positive and non-increasing function such that* $\sum_{n=1}^{\infty} \lambda(n) n^{-1} < \infty$*;*

(b) $\{\gamma_{k,1}\}_{k\geqslant 1}$ *is non-increasing and* $\sum_{k=1}^{\infty} \gamma_{k,1} k^{-1} < \infty$*.*

Then $\{Z_n \tau^{-n}\}_{n\geqslant 0}$ *converges almost surely to a finite and non-degenerate-at-zero limit* W'*, with* $\{W' > 0\} = \{Z_n \to \infty\}$ *almost surely.*

Indeed, it is easy to verify that requirements (a) and (b) imply assumptions (A1), (A2) and (A3) in Theorem 4(ii).

Conditions (a) and (b) were introduced in González and Molina (1996) to prove the L^1-convergence of $\{Z_n\tau^{-n}\}_{n\geqslant 0}$ to W', but it was not established in that work that $\{W'>0\}=\{Z_n\to\infty\}$ almost surely. The importance of investigating conditions which guarantee that equality is pointed out by G. Alsmeyer in Haccou *et al*. (2005: 177). Proposition 2 solves this question.

Recently, a generalization of this model called the 'bisexual branching process with population-size dependent mating' was investigated in Molina *et al.* (2002, 2004). This branching model considers a sequence $\{L_k\}_{k\geqslant 0}$ of mating functions, in which the mating not only depends on the number of females and males in the generation but also on the size of the previous generation. It can be verified that, under the conditions imposed in Molina *et al.* (2004), requirements (A1), (A2) and (A3) hold, so that again we deduce the geometric growth of $\{Z_n\}_{n\geqslant 0}$ on the whole explosion set.

As was noted above for the asexual controlled branching process, these results complete the inferential theory developed for bisexual branching processes (see González *et al.* 2001).

4. Proofs

Proof of Theorem 1. (i) It is sufficient to show that $\{W_n\}_{n\geqslant 0}$ is a Cauchy sequence in L^α. First, let us prove that $\sup_{n\geqslant 0}\|W_n\|_\alpha<\infty$. Using (A1) and (A2), we have almost surely

$$\begin{aligned}\mathrm{E}^{1/\alpha}[|W_{n+1}-W_n|^\alpha|\mathcal{F}_n] = W_n\tau_{Z_n}^{-1}\gamma_{Z_n,\alpha} &\leqslant MW_n\eta(Z_n)\\ &\leqslant MW_n\eta((W_n^\alpha\tilde{\tau}^{\alpha n})^{1/\alpha}), \qquad n\geqslant 0.\end{aligned} \tag{4}$$

Since $\eta^*(x):=x\eta^\alpha(x^{1/\alpha})$ is concave (see Remark 2), applying Jensen's inequality, we have from (4) that

$$\|W_{n+1}-W_n\|_\alpha\leqslant M\|W_n\|_\alpha\eta(\|W_n\|_\alpha\tilde{\tau}^n), \tag{5}$$

and therefore

$$\|W_{n+1}\|_\alpha\leqslant\|W_{n+1}-W_n\|_\alpha+\|W_n\|_\alpha\leqslant\|W_n\|_\alpha(1+M\eta(\|W_n\|_\alpha\tilde{\tau}^n)). \tag{6}$$

Since $\eta(x)$ is non-increasing and $\|W_n\|_\alpha\geqslant\mathrm{E}[Z_0]>0$, from (6),

$$\|W_n\|_\alpha\leqslant\|Z_0\|_\alpha\prod_{j=0}^{n-1}(1+M\eta(\tilde{\tau}^j\mathrm{E}[Z_0])).$$

Hence $\sup_{n\geqslant 0}\|W_n\|_\alpha<\infty$ if and only if $\prod_{j=0}^\infty(1+M\eta(\tilde{\tau}^j\mathrm{E}[Z_0]))<\infty$ or, equivalently, $\sum_{j=0}^\infty\eta(\tilde{\tau}^j\mathrm{E}[Z_0])<\infty$. But the latter is true by (A1).

We now verify that $\{W_n\}_{n\geqslant 0}$ is a Cauchy sequence in L^α. Since $\eta^*(x)$ is an increasing function on $(0,\infty)$, and from (5), we have for $n>m>0$,

$$\|W_m - W_n\|_\alpha \leqslant \sum_{j=m}^{n-1} \|W_{j+1} - W_j\|_\alpha \leqslant M \sum_{j=m}^{n-1} L\eta(L\tilde{\tau}^j),$$

where L denotes a constant satisfying $L \geqslant \sup_{n\geqslant 0}\|W_n\|_\alpha$. Finally, again using (A1), we deduce that $\|W_m - W_n\|_\alpha \to 0$ as $n, m \to \infty$.

(ii) First, observe that

$$W - W_n = \sum_{k=0}^{\infty} (W_{n+k+1} - W_{n+k}), \qquad n \geqslant 0.$$

Hence

$$\begin{aligned}\|W - W_n\|_\alpha &\leqslant \sum_{k=0}^{\infty} \|W_{n+k+1} - W_{n+k}\|_\alpha \leqslant ML \sum_{k=0}^{\infty} \eta(L\tilde{\tau}^{n+k}) \\ &= ML \frac{\tilde{\tau}}{\tilde{\tau} - 1} \sum_{k=0}^{\infty} \frac{\eta(L\tilde{\tau}^{n+k})}{L\tilde{\tau}^{n+k}} (L\tilde{\tau}^{n+k} - L\tilde{\tau}^{n+k-1}) \\ &\leqslant ML \frac{\tilde{\tau}}{\tilde{\tau} - 1} \int_{L\tilde{\tau}^{n-1}}^{\infty} \eta(x) x^{-1}\, \mathrm{d}x.\end{aligned}$$

Thus, choosing $L \geqslant \tilde{\tau}$, the proof is complete. □

Proof of Theorem 2. For simplicity we assume that $P(Z_n \to \infty) = 1$. Note that $\lim_{n\to\infty} W_{n+1} W_n^{-1} = 1$ almost surely on $\{Z_n \to \infty\}$ is equivalent to

$$P\left(\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} B_n\right) = 1, \qquad \text{where } B_n := \{|W_{n+1} W_n^{-1} - 1| \leqslant \varepsilon\}$$

for any sufficiently small $\varepsilon > 0$.

For $N > 0$, we define $K_N := \min\{n : Z_n \geqslant N\}$. Since $P(Z_n \to \infty) = 1$ it is clear that $P(K_N < \infty) = 1$. We have

$$P\left(\bigcup_{k=0}^{\infty} \bigcap_{n=k}^{\infty} B_n\right) \geqslant P\left(\bigcap_{n=K_N}^{\infty} B_n\right) = \sum_{k=0}^{\infty} P\left(\bigcap_{n=K_N}^{\infty} B_n | K_N = k\right) P(K_N = k).$$

Now

$$\begin{aligned}P\left(\bigcap_{n=K_N}^{\infty} B_n | K_N = k\right) &= P\left(\bigcap_{n=K_N}^{\infty} B_n | Z_t < N,\ t = 0, 1, \dots, k-1;\ Z_k \geqslant N\right) \\ &= P\left(\bigcap_{n=K_N}^{\infty} B_n | Z_k \geqslant N\right)\end{aligned}$$

and consequently

$$P\left(\bigcup_{k=0}^{\infty}\bigcap_{n=k}^{\infty}B_n\right) \geqslant \sum_{k=0}^{\infty}P(K_N=k) - \sum_{k=0}^{\infty}\inf_{t\geqslant N}P\left(\bigcup_{n=k}^{\infty}B_n^c|Z_k=t\right)P(K_N=k). \tag{7}$$

Note that

$$P\left[\bigcup_{n=k}^{\infty}B_n^c\middle|Z_n=k\right] = \sum_{n=k}^{\infty}P[C_n|Z_k=t],$$

where

$$C_k := B_k^c,$$

$$C_n := B_n^c \cap B_{n-1}\cap\ldots\cap B_k, \qquad n=k+1,\, k+2, \ldots.$$

It follows that

$$\begin{aligned} P[C_n Z_k=t] &= \mathrm{E}[\mathbf{1}_{B_n^c\cap B_{n-1}\cap\ldots\cap B_k}|Z_k=t] \\ &= \mathrm{E}[\mathrm{E}[\mathbf{1}_{B_n^c\cap B_{n-1}\cap\ldots\cap B_k}|\mathcal{F}_n]|Z_k=t] \\ &= \mathrm{E}[\mathbf{1}_{B_{n-1}\cap\ldots\cap B_k}\mathrm{E}[\mathbf{1}_{\{|Z_{n+1}Z_n^{-1}-\tau_{Z_n}|\geqslant\tau_{Z_n}\varepsilon\}}|\mathcal{F}_n]|Z_k=t], \end{aligned}$$

where $\mathbf{1}_C$ denotes the indicator function of the set C. Now, for some constant $M^*>0$,

$$P(|Z_{n+1}Z_n^{-1}-\tau_{Z_n}|\geqslant\tau_{Z_n}\varepsilon|\mathcal{F}_n) \leqslant \left(\frac{\gamma_{Z_n,\alpha}}{\tau_{Z_n}\varepsilon}\right)^{\alpha} \leqslant M^*\eta^\alpha(Z_n).$$

Moreover, since $\cap_{j=k}^{n-1}\{|W_{j+1}-W_j|<\varepsilon\}\subseteq\{Z_n<Z_k\tilde{\tau}^{n-k}\}$ and $\eta(x)$ is non-increasing, it follows that $P(C_n|Z_k=t)\leqslant M^*\eta^\alpha(\tilde{\tau}^{n-k}t)$.

Thus, from (7),

$$P\left(\bigcup_{k=0}^{\infty}\bigcap_{n=k}^{\infty}B_n\right) \geqslant 1-M^*\sum_{k=0}^{\infty}\inf_{t\geqslant N}\sum_{n=k}^{\infty}\eta^\alpha(\tilde{\tau}^{n-k}t)P(K_N=k).$$

By considering the properties of $\eta(x)$, we deduce that $\sum_{n=1}^{\infty}\eta^\alpha(n)n^{-1}<\infty$ and therefore $\lim_{t\to\infty}\sum_{n=k}^{\infty}\eta^\alpha(\tilde{\tau}^{n-k}t)=0$, which concludes the proof. □

Proof of Theorem 3. Clearly $\{W>0\}\subseteq\{Z_n\to\infty\}$. Now consider the other inclusion. Let $\mathcal{A}:=\{\sum_{n=0}^{\infty}\mathrm{E}[|\zeta_{n+1}|^\alpha|\mathcal{F}_n]<\infty\}$, where $\zeta_{n+1}:=(W_{n+1}W_n^{-1}-1)\mathbf{1}_{\{Z_n>0\}}$. We prove that $\{Z_n\to\infty\}\subseteq\mathcal{A}\subseteq\{W>0\}$ almost surely. Since $\sum_{n=0}^{\infty}\mathrm{E}[|\zeta_{n+1}|^\alpha|\mathcal{F}_n]\leqslant M^\alpha\sum_{n=0}^{\infty}\eta^\alpha(Z_n)$, it is sufficient to show that

$$\sum_{n=0}^{\infty}\eta^\alpha(Z_n)<\infty \qquad \text{a.s. on } \{Z_n\to\infty\}. \tag{8}$$

By Theorem 2, and since $Z_{n+1}Z_n^{-1}\geqslant W_{n+1}W_n^{-1}\tilde{\tau}$, it follows that

$$\liminf_{n\to\infty}Z_{n+1}Z_n^{-1}>1 \qquad \text{a.s. on } \{Z_n\to\infty\}.$$

Hence, given the requirements imposed on the function $\eta(x)$, (8) holds.

We now prove that $\mathcal{A} \subseteq \{W > 0\}$. Applying Theorem 2.17 of Hall and Heyde (1980), we have

$$\sum_{n=1}^{\infty} \zeta_n \text{ converges a.s. on } \mathcal{A}. \tag{9}$$

Moreover, on $\mathcal{A}$,

$$\sum_{k=0}^{\infty} \mathrm{E}[\,\big||\zeta_{k+1}|^{\alpha} - \mathrm{E}[|\zeta_{k+1}|^{\alpha}|\mathcal{F}_k]\big|\,|\mathcal{F}_k] \leqslant 2\sum_{k=0}^{\infty} \mathrm{E}[|\zeta_{k+1}|^{\alpha}|\mathcal{F}_k] < \infty.$$

Again using Theorem 2.17 of Hall and Heyde (1980),

$$\sum_{k=0}^{\infty} (|\zeta_{k+1}|^{\alpha} - \mathrm{E}[|\zeta_{k+1}|^{\alpha}|\mathcal{F}_k]) \text{ converges a.s. on } \mathcal{A}.$$

Therefore $\sum_{n=1}^{\infty}|\zeta_n|^{\alpha}$ converges almost surely on $\mathcal{A}$, and, since $\alpha \leqslant 2$,

$$\sum_{n=1}^{\infty} |\zeta_n|^2 \text{ also converges a.s. on } \mathcal{A}. \tag{10}$$

Note that

$$W_n = W_0 \prod_{i=1}^{n}(1 + \zeta_i), \qquad n \geqslant 1.$$

Finally, taking into account (9) and (10), and applying Theorem 7.30 of Stromberg (1981), we complete the proof. □

Proof of Theorem 4. (i) The proof follows the steps of Lemma 10 in Pierre Loti Viaud (1994) for population-size dependent branching processes. Let us consider $\gamma \in (1, \tilde{\tau})$ ($\tilde{\tau}$ given in (A2)) and, for each $\ell \geqslant 0$, let $\Omega_\ell = \{Z_n \geqslant \gamma^n,\ n \geqslant \ell\}$. From Theorem 3,

$$\{Z_n \to \infty\} = \bigcup_{\ell \geqslant 0} \Omega_\ell \qquad \text{a.s.}$$

Thus it suffices to show, for each $\ell \geqslant 0$, that $\{\nu_n \tau^{-n}\}_{n \geqslant 1}$ converges almost surely to a positive and finite limit on Ω_ℓ.

Let $B \in (1, \tau)$ be fixed, and $L \geqslant \ell$ be such that $\inf\{\tau_k,\ k \geqslant \gamma^L\} \geqslant B$. We have almost surely on Ω_ℓ, for $n \geqslant L$,

$$\frac{\nu_n \tau^L}{\tau^n \nu_L} = \exp\left\{\sum_{k=L}^{n-1} \log(1 + (\tau_{Z_k} - \tau)\tau^{-1})\right\} \leqslant \exp\left\{B^{-1}\sum_{k=L}^{\infty} |\tau_{Z_k} - \tau|\right\} \tag{11}$$

and

$$\frac{\nu_L \tau^n}{\tau^L \nu_n} = \exp\left\{\sum_{k=L}^{n-1} \log(1 + (\tau_{Z_k} - \tau)\tau_{Z_k}^{-1})\right\} \leqslant \exp\left\{B^{-1}\sum_{k=L}^{\infty} |\tau_{Z_k} - \tau|\right\}. \tag{12}$$

By the properties of $\lambda(x)$ and the definition of Ω_ℓ, we have almost surely on Ω_ℓ that

$$\sum_{k=L}^{\infty} |\tau_{Z_k} - \tau| \leqslant \sum_{k=L}^{\infty} \lambda(Z_k) \leqslant \sum_{k=L}^{\infty} \lambda(\gamma^k) \to 0 \qquad \text{as } L \to \infty. \tag{13}$$

Therefore from (11), (12) and (13), we deduce that, almost surely on Ω_ℓ, $\{\nu_n \tau^{-n}\}_{n\geqslant 1}$ is a Cauchy sequence in $(0, \infty)$. Hence there exists the almost sure limit of $\{\nu_n \tau^{-n}\}_{n\geqslant 1}$ on Ω_ℓ, and (11) and (13) imply that $\sup_{n\geqslant 1} \nu_n \tau^{-n} < \infty$ almost surely on Ω_ℓ, while (12) and (13) imply that $\sup_{n\geqslant 1} \tau^n \nu_n^{-1} < \infty$ almost surely on Ω_ℓ.

(ii) The proof is straightforward using (2) and Theorems 1, 3 and 4(i). □

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