Bernoulli **11**(3), 2005, 471–509

Recurrent extensions of self-similar Markov processes and Cramér's condition

VÍCTOR RIVERO

Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 175, rue du Chevaleret, F-75013 Paris, France. E–mail: rivero@ccr.jussieu.fr

Let ξ be a real-valued Lévy process that satisfies Cramér's condition, and X a self-similar Markov process associated with ξ via Lamperti's transformation. In this case, X has 0 as a trap and satisfies the assumptions set out by Vuolle-Apiala. We deduce from the latter that there exists a unique excursion measure $\mathbf{n}$, compatible with the semigroup of X and such that $\mathbf{n}(X_{0+} > 0) = 0$. Here, we give a precise description of $\mathbf{n}$ via its associated entrance law. To this end, we construct a self-similar process $X^{\natural}$, which can be viewed as X conditioned never to hit 0, and then we construct $\mathbf{n}$ similarly to the way in which the Brownian excursion measure is constructed via the law of a Bessel(3) process. An alternative description of $\mathbf{n}$ is given by specifying the law of the excursion process conditioned to have a given length. We establish some duality relations from which we determine the image under time reversal of $\mathbf{n}$.

Keywords: description of excursion measures; Lévy processes; self-similar Markov process; weak duality

1. Introduction

Let $X = (X_t, t \geqslant 0)$ be a strong Markov process with values in $[0, \infty[$ and, for $x \geqslant 0$, denote by $\mathbb{P}_x$ its law starting from x. Assume that X possesses the following scaling property: there exists some $\alpha > 0$ such that

$$\text{the law of } (cX_{tc^{-1/\alpha}},\ t \geqslant 0) \text{ under } \mathbb{P}_x \text{ is } \mathbb{P}_{cx}, \tag{1}$$

for any $x \geqslant 0$ and $c > 0$. Such processes were introduced by Lamperti (1972) under the name of semi-stable processes; nowadays they are called α-self-similar Markov processes. We refer to Embrechts and Maejima (2002) for a recent account of self-similar processes.

Lamperti established that for each fixed $\alpha > 0$, there exists a one-to-one correspondence between α-self-similar Markov processes on $[0, \infty[$ and real-valued Lévy processes which we now sketch. Let $(\mathbb{D}, \mathcal{D})$ be the space of cadlag paths $\omega : [0, \infty[\to] - \infty, \infty[$ endowed with the σ-algebra generated by the coordinate maps and the natural filtration $(\mathcal{D}'_t, t \geqslant 0)$, satisfying the usual conditions of right continuity and completeness. Let $\check{\mathbf{P}}$ be a probability measure on $\mathcal{D}'$ such that under $\check{\mathbf{P}}$ the coordinate process $\check{\xi}$ is a Lévy process. Throughout out this paper we will refer to this process as the unkilled Lévy process. Let $\mathbf{P}$ be the law of the Lévy process ξ which is obtained by killing $\check{\xi}$ at a rate $\mathbf{k} \geqslant 0$, that is, $\check{\xi}$ is killed at an independent exponential time $\mathbb{e}$ with parameter $\mathbf{k}$. We denote by ζ the lifetime of ξ, and

$(\mathcal{D}_t,\ t \geqslant 0)$ its filtration. If $\mathbf{k} = 0$ we assume furthermore that ξ drifts to $-\infty$, i.e. $\lim_{s\to\infty}\xi_s = -\infty$, $\mathbf{P}$–almost surely. Set, for $t \geqslant 0$,

$$\tau(t) = \inf\left\{s > 0,\ \int_0^s \exp\{\xi_r/\alpha\}\mathrm{d}r > t\right\},$$

with the usual convention that $\inf\{\varnothing\} = \infty$. For an arbitrary $x > 0$, let $\mathbb{P}_x$ be the distribution on $\mathbb{D}^+ = \{\omega : [0, \infty[\to [0, \infty[;\ \text{cadlag}\}$, of the time-transformed process

$$X_t = x\exp(\xi_{\tau(tx^{-1/\alpha})}), \qquad t \geqslant 0,$$

where the above quantity is assumed to be 0 when $\tau(tx^{-1/\alpha}) = \infty$. We define $\mathbb{P}_0$ as the law of the process identical to 0. Classical results on time transformation yield that under $(\mathbb{P}_x,\ x \geqslant 0)$ the process X is Markovian with respect to the filtration $(\mathcal{G}_t = \mathcal{D}_{\tau(t)},\ t \geqslant 0)$. Furthermore, X has scaling property (1). Thus, X is a self-similar Markov process on $[0, \infty[$ having 0 as a trap or absorbing point. Conversely, any self-similar Markov process that has 0 as a trap can be constructed in this way (Lamperti 1972).

Let T_0 be the first hitting time of 0 for X, that is,

$$T_0 = \inf\{t > 0 : X_{t-} = 0 \text{ or } X_t = 0\}.$$

It should be clear that the distribution of T_0 under $\mathbb{P}_x$ is the same as that of $x^{1/\alpha}I$ under $\mathbf{P}$, with I the so-called Lévy exponential functional associated with ξ and α, that is,

$$I = \int_0^\zeta \exp\{\xi_s/\alpha\}\mathrm{d}s. \tag{2}$$

If $\mathbf{k} > 0$, or if $\mathbf{k} = 0$ and ξ drifts to $-\infty$, we have that $I < \infty$, $\mathbf{P}$-a.s. As a consequence, we have that if $\mathbf{k} > 0$ then

$$\mathbb{P}_x(X_{T_0-} > 0,\ T_0 < \infty) = 1, \qquad \text{for all } x > 0,$$

whereas if $\mathbf{k} = 0$ and ξ drifts to $-\infty$,

$$\mathbb{P}_x(X_{T_0-} = 0,\ T_0 < \infty) = 1, \qquad \text{for all } x > 0.$$

Denote by P_t and V_q respectively the semigroup and resolvent for the process X killed at time T_0, say (X, T_0),

$$P_t f(x) = \mathbb{E}_x(f(X_t),\ t < T_0), \qquad x > 0,$$

$$V_q f(x) = \int_0^\infty \mathrm{e}^{-qt} P_t f(x)\mathrm{d}t, \qquad x > 0,$$

for non-negative or bounded measurable functions f. It is customary to refer to (X, T_0) as the minimal process.

Given that the preceding construction enables us to describe the behaviour of the self-similar Markov process X until the first time it hits 0, Lamperti (1972) raised the following question: what are the self-similar Markov processes $\tilde{X}$ on $[0, \infty[$ which behave like (X, T_0) up to the time $\tilde{T}_0$? Lamperti solved this problem in the case where the minimal process is a Brownian motion killed at 0. Then Vuolle-Apiala (1994) tackled this problem using excursion theory for Markov processes and assuming that the following hypotheses hold: there exists $\kappa > 0$ such that

(H1a) the limit

$$\lim_{x\to 0} \frac{\mathbb{E}_x(1 - \mathrm{e}^{-T_0})}{x^\kappa}$$

exists and is strictly positive, and

(H1b) the limit

$$\lim_{x\to 0} \frac{V_q f(x)}{x^\kappa}$$

exists for all $f \in C_K]0, \infty[$ and is strictly positive for some such functions,

with $C_K]0, \infty[= \{f : \mathbb{R} \to \mathbb{R}$, continuous and with compact support on $]0, \infty[\}$. The main result of Vuolle-Apiala (1994) is the existence of a unique entrance law $(\mathbf{n}_s, s > 0)$ such that

$$\lim_{s\to 0} \mathbf{n}_s B^c = 0,$$

for every neighourhood B of 0 and

$$\int_0^\infty \mathrm{e}^{-s}\mathbf{n}_s 1 \mathrm{d}s = 1.$$

This entrance law is determined by its q-potential via the formula

$$\int_0^\infty \mathrm{e}^{-qs}\mathbf{n}_s f \mathrm{d}s = \lim_{x\to 0} \frac{V_q f(x)}{\mathbb{E}_x(1 - \mathrm{e}^{-T_0})}, \qquad q > 0, \tag{3}$$

for $f \in C_K]0, \infty[$. Then, using the results of Blumenthal (1983), Vuolle-Apiala proved that associated with the entrance law $(\mathbf{n}_s, s > 0)$ there exists a unique recurrent Markov process $\tilde{X}$ having scaling property (1) which is an extension of the minimal process (X, T_0), hence, $\tilde{X}$ killed at time $\tilde{T}_0$ is equivalent to (X, T_0) and 0 is a recurrent regular state for $\tilde{X}$, that is,

$$\tilde{\mathbb{P}}_x(T_0 < \infty) = 1, \quad \forall x > 0, \quad \tilde{\mathbb{P}}_0(T_0 = 0) = 1,$$

with $\tilde{\mathbb{P}}$ the law on $\mathbb{D}^+$ of $\tilde{X}$. Furthermore, the results of Blumenthal (1983) ensure that there exists a unique excursion measure, say $\mathbf{n}$, on $(\mathbb{D}^+, \mathcal{G}_\infty)$ compatible with the semigroup P_t such that its associated entrance law is $(\mathbf{n}_s, s > 0)$; the property $\lim_{s\to 0}\mathbf{n}_s B^c = 0$, for any B-neighourhood of 0, is equivalent to $\mathbf{n}(X_{0+} > 0) = 0$, that is, the process leaves 0 continuously under $\mathbf{n}$. Then the excursion measure $\mathbf{n}$ is the unique excursion measure having the properties $\mathbf{n}(X_{0+} > 0) = 0$ and $\mathbf{n}(1 - \mathrm{e}^{-T_0}) = 1$. See Section 2.1 for the definitions.

The first aim of this paper is to provide a more explicit description of the excursion measure $\mathbf{n}$ and its associated entrance law $(\mathbf{n}_s, s > 0)$. To this end, we shall mimic a well-known construction of the Brownian excursion measure via the Bessel(3) process that we next sketch for ease of reference. Let P (R) be a probability measure on $(\mathbb{D}^+, \mathcal{G}_\infty)$ under which the coordinate process is a Brownian motion killed at 0 (a Bessel(3) process). The probability measure R appears as the law of the Brownian motion conditioned never to hit 0. More precisely, for $u > 0$, $x > 0$,

$$\lim_{t\to\infty} P_x(A|T_0 > t) = R_x(A),$$

for any $A \in \mathcal{G}_u$, (e.g. McKean 1963). Moreover, the function $h(x) = x^{-1}$, $x > 0$, is excessive for the semigroup of the Bessel(3) process and its h-transform is the semigroup of the Brownian motion killed at 0. Let n be the h-transform of R_0 via the function $h(x) = x^{-1}$, that is, n is the unique measure on $(\mathbb{D}^+, \mathcal{G}_\infty)$ with support in $\{T_0 > 0\}$ such that under n the coordinate process is Markovian with semigroup that of Brownian motion killed at 0, and for every $\mathcal{G}_t$-stopping time T and any positive $\mathcal{G}_T$-measurable functional F_T,

$$\mathrm{n}(F_T,\, T < T_0) = R_0\left(\frac{F_T}{X_T}\right).$$

Then the measure n is a multiple of Itô's excursion measure for Brownian motion (Imhof 1984, Section 4).

In order to carry out this programme we will make the following hypotheses on the Lévy process ξ:

(H2a) ξ is not arithmetic, that is, the state space is not a subgroup of $k\mathbb{Z}$ for any real number k.

(H2b) There exists $\theta > 0$ such that $\mathbf{E}(\mathrm{e}^{\theta\xi_1},\, 1 < \zeta) = 1$.

(H2c) $\mathbf{E}(\xi_1^+ \mathrm{e}^{\theta\xi_1},\, 1 < \zeta) < \infty$, with $a^+ = a \vee 0$.

Condition (H2c) can be stated in terms of the Lévy measure Π of ξ as

(H2c′) $\int_{\{x>1\}} x\mathrm{e}^{\theta x}\Pi(\mathrm{d}x) < \infty$

(cf. Sato 1999, theorem 25.3). Such hypotheses are satisfied by a wide class of Lévy processes, in particular by those associated, via Lamperti's transformation, with self-similar diffusions and stable processes. We will refer to these hypotheses as (H2).

Condition (H2b) is called *Cramér's condition* for the Lévy process ξ and, in the case $\mathbf{k} = 0$, forces ξ to drift to $-\infty$ or equivalently $\mathbf{E}(\xi_1) < 0$. Thus if the (H2) hypotheses hold we will refer to the case where $\mathbf{k} = 0$ and ξ drifts to $-\infty$ as the case $\mathbf{k} = 0$. Cramér's condition enables us to construct a law $\mathbf{P}^\natural$ on $\mathbb{D}$, such that under $\mathbf{P}^\natural$ the coordinate process $\xi^\natural$ is a Lévy process that drifts to ∞ and $\mathbf{P}^\natural|_{\mathcal{D}_t} = \mathrm{e}^{\theta\xi_t}\mathbf{P}|_{\mathcal{D}_t}$. Then we will show that the self-similar Markov process $X^\natural$ associated with the Lévy process $\xi^\natural$ plays the rôle of a Bessel(3) process in our construction of the excursion measure $\mathbf{n}$.

The rest of this paper is organized as follows. In Section 2.1 we recall Itô's programme as established by Blumenthal (1983). The excursion measure $\mathbf{n}$ that interests us is the unique (up to a multiplicative constant) excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. Nevertheless, this is not the only excursion measure compatible with the semigroup of the minimal process, which is why in Section 2.2 we review some properties that should be satisfied by any excursion measure corresponding to a self-similar extension of the minimal process. There we also obtain necessary and sufficient conditions for the existence of an excursion measure n^j such that $n^j(X_{0+} = 0) = 0$, which are valid for any self-similar Markov process having 0 as a trap. In Section 2.3 we construct a self-similar Markov process $X^\natural$ which is related to (X, T_0) in an analogous way to that in which the Bessel(3) process is related to Brownian motion killed at 0. We also prove that conditions (H1) are satisfied under hypotheses (H2), give a more explicit expression for the limit in equation (3) and show that hypotheses (H1) imply the conditions (H2b) and (H2c). Next, in

Section 3 we give our main description of the excursion measure $\mathbf{n}$ and give an answer to the question raised by Lamperti that can be sketched as follows: given a Lévy process ξ satisfying (H2), then an α-self-similar Markov process X associated with ξ admits a recurrent extension that leaves 0 continuously a.s. if and only if $0 < \alpha\theta < 1$. The purpose of Section 4 is to give an alternative description of the measure $\mathbf{n}$ by determining the law of the excursion process conditioned by its length (for Brownian motion this corresponds to the description of the Itô excursion measure via the law of a Bessel(3) bridge). In Section 5 we study some duality relations for the minimal process and in particular we determine the image under time reversal of $\mathbf{n}$. Finally, in Appendix A we establish that the extensions of any two minimal processes which are in weak duality, are still in weak duality as might be expected.

Sometimes it will be necessary to distinguish between the case $\mathbf{k} > 0$ and $\mathbf{k} = 0$ in order to obtain our results. However, given that the methods are quite similar in both cases we have chosen to only present the complete proofs when $\mathbf{k} = 0$. We will indicate the places where changes are necessary to obtain the results in the case $\mathbf{k} > 0$.

Note, finally, that the development of this work uses Doob's theory of h-transforms (see Sharpe 1988) without further reference.

2. Preliminaries and first results

This section contains several parts. In Section 2.1, we recall Itô's programme and the results in Blumenthal (1983). The purpose of Section 2.2 is to study the excursion measures compatible with the semigroup of the minimal process (X, T_0). Finally, in Section 2.3 we establish the existence of a self-similar Markov process $X^\natural$ which bears the same relation to the minimal process (X, T_0) as the Bessel(3) process does to Brownian motion killed at 0. The results in Sections 2.1 and 2.2 do not require (H2).

2.1. Some general facts on recurrent extensions of Markov processes

A measure n on $(\mathbb{D}^+, \mathcal{G}_\infty)$ having infinite mass is called a *pseudo-excursion measure* compatible with the semigroup P_t if the following conditions are satisfied:

(i) n is carried by

$$\{\omega \in \mathbb{D}+ \mid 0 < T_0 < \infty \text{ and } X_t = 0, \forall t \geqslant 0\};$$

(ii) for every bounded $\mathcal{G}_\infty$-measurable H and each $t > 0$ and $\Lambda \in \mathcal{G}_t$,

$$n(H \circ \theta_t, \Lambda \cap \{t < T_0\}) = n(\mathbb{E}_{X_t}(H), \Lambda \cap \{t < T_0\}),$$

where θ_t denotes the shift operator.

If, moreover,

(iii) $n(1 - \mathrm{e}^{-T_0}) < \infty$, we will say that n is an *excursion measure*. A normalized excursion measure n is an excursion measure n such that $n(1-\mathrm{e}^{-T_0})=1$. The role played by condition (iii) will be explained below.

The entrance law associated with a pseudo-excursion measure n is defined by

$$n_s(\mathrm{d}y) := n(X_s \in \mathrm{d}y,\ s < T_0), \qquad s > 0.$$

A partial converse holds: given an entrance law $(n_s, s>0)$ such that

$$\int_0^\infty (1-\mathrm{e}^{-s})\mathrm{d}n_s 1 < \infty,$$

there exists a unique excursion measure n such that its associated entrance law is $(n_s, s>0)$, (see Blumenthal 1983).

It is well known in the theory of Markov processes that one way to construct recurrent extensions of a Markov process is Itô's programme or pathwise approach that can be described as follows. Assume that there exists an excursion measure n compatible with the semigroup of the minimal process P_t. Realize a Poisson point process $\Delta = (\Delta_s, s>0)$ on $\mathbb{D}^+$ with characteristic measure n. Thus each atom Δ_s is a path and $T_0(\Delta_s)$ denotes its lifetime:

$$T_0(\Delta_s) = \inf\{t > 0 : \Delta_s(t) = 0\}.$$

Set

$$\sigma_t = \sum_{s \leqslant t} T_0(\Delta_s), \qquad t \geqslant 0.$$

Since $n(1-\mathrm{e}^{-T_0}) < \infty$, $\sigma_t < \infty$ a.s. for every $t>0$. It follows that the process $\sigma = (\sigma_t, t \geqslant 0)$ is an increasing cadlag process with stationary and independent increments, that is, a subordinator. Its law is characterized by its Laplace exponent ϕ, defined by

$$\mathbf{E}(\mathrm{e}^{-\lambda\sigma_1}) = \mathrm{e}^{-\phi(\lambda)}, \qquad \lambda > 0,$$

and $\phi(\lambda)$ can be expressed thanks to the Lévy–Khinchine formula as

$$\phi(\lambda) = \int_{]0,\infty[} (1-\mathrm{e}^{-\lambda s})\nu(\mathrm{d}s),$$

with ν a measure such that $\int_{]0,\infty[}(s\wedge 1)\nu(\mathrm{d}s) < \infty$, called the Lévy measure of σ (see Bertoin 1996, Chapter 3, for background). An application of the exponential formula for Poisson point processes gives

$$\mathbf{E}(\mathrm{e}^{-\lambda\sigma_1}) = \mathrm{e}^{-n(1-\mathrm{e}^{-\lambda T_0})}, \qquad \lambda > 0,$$

that is, $\phi(\lambda) = n(1-\mathrm{e}^{-\lambda T_0})$ and the tail of the Lévy measure is given by

$$\nu[s, \infty[\ = n(s < T_0) = n_s 1, \qquad s > 0.$$

Observe that if we assume $\phi(1) = n(1-\mathrm{e}^{-T_0}) = 1$ then ϕ is uniquely determined. Since n has infinite mass, σ_t is strictly increasing in t. Let L_t be the local time at 0, that is, the continuous inverse of σ:

$$L_t = \inf\{r > 0 : \sigma_r > t\} = \inf\{r > 0 : \sigma_r \geqslant t\}.$$

Define a process $(\tilde{X}_t,\ t \geqslant 0)$ as follows. For $t \geqslant 0$, let $L_t = s$; then $\sigma_{s-} \leqslant t \leqslant \sigma_s$. Set

$$\tilde{X}_t = \begin{cases} \Delta_s(t - \sigma_{s-}), & \text{if } \sigma_{s-} < \sigma_s \\ 0, & \text{if } \sigma_{s-} = \sigma_s \text{ or } s = 0. \end{cases} \tag{4}$$

That the process so constructed is a Markov process has been established in full generality by Salisbury (1986a; 1986b) and under some regularity hypotheses on the semigroup of the minimal process by Blumenthal (1983). See also Rogers (1983) for its analytical counterpart. In our setting, the hypotheses of Blumenthal (1983) are satisfied, as is shown by the following lemma.

Lemma 1. *Let* $C_0]0, \infty[$*, be the space of continuous functions on* $]0, \infty[$ *vanishing at* 0 *and* ∞.

(i) *If* $f \in C_0]0, \infty[$*, then* $P_t f \in C_0]0, \infty[$ *and* $P_t f \to f$ *uniformly as* $t \to 0$.
(ii) $\mathbb{E}_x(\mathrm{e}^{-qT_0})$ *is continuous in* x *for each* $q > 0$ *and*

$$\lim_{x\to 0} \mathbb{E}_x(\mathrm{e}^{-T_0}) = 1 \quad \text{and} \quad \lim_{x\to\infty} \mathbb{E}_x(\mathrm{e}^{-T_0}) = 0.$$

This lemma is an easy consequence of Lamperti's transformation. Alternatively, a proof can be found in Vuolle-Apiala (1994, pp. 549–550).

Therefore we have from Blumenthal (1983) that $\tilde{X}$ is a Markov process with Feller semigroup and its resolvent $\{U_q,\ q > 0\}$ satisfies

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(\mathrm{e}^{-qT_0}) U_q f(0), \qquad x > 0,$$

for $f \in C_b(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R}, \text{ continuous and bounded}\}$. That is, $\tilde{X}$ is an extension of the minimal process. Furthermore, if $\{X'_t,\ t \geqslant 0\}$ is a Markov process extending the minimal one with Itô's excursion measure n and local time at 0, say $\{L'_t,\ t \geqslant 0\}$, such that

$$\mathbb{E}'\left(\int_0^\infty \mathrm{e}^{-s}\mathrm{d}L'_s\right) = 1,$$

where $\mathbb{E}'$ is the law of X'. Then the process $\tilde{X}$ and X' are equivalent and Itô's excursion measure for $\tilde{X}$ is n.

Thus, the results in Blumenthal (1983) establish a one-to-one correspondence between excursion measures and recurrent extensions of Markov processes. Given an excursion measure n, we will say that the associated extension of the minimal process leaves 0 continuously a.s. if $n(X_{0+} > 0) = 0$ or, equivalently, in terms of its entrance law, $\lim_{s\to 0} n_s(B^c) = 0$ for every neighbourhood B of 0, (Blumenthal 1983); if n is such that $n(X_{0+} = 0) = 0$, we will say that the extension leaves 0 by jumps a.s. The latter condition on n is equivalent to the existence of a jumping-in measure η, that is, η is a σ–finite measure on $]0, \infty[$ such that the entrance law associated with n can be expressed as

$$n_s f = n(f(X_s),\, s < T_0) = \int_{]0,\infty[} \eta(\mathrm{d}x) P_s f(x), \qquad s > 0,$$

for every $f \in C_b(\mathbb{R}^+)$ (Meyer 1971).

Finally, observe that if n is a pseudo-excursion measure that does not satisfy condition (iii), we can still realize a Poisson point process of excursions on $(\mathbb{D}^+, \mathcal{G}_\infty)$ with characteristic measure n but we cannot form a process extending the minimal one by sticking together the excursions because the sum of lengths $\sum_{s\leqslant t} T_0(\Delta_s)$ is infinite $\mathbb{P}$-a.s. for every $t > 0$.

2.2. Some properties of excursion measures for self-similar Markov process

Next, we deduce necessary and sufficient conditions that must be satisfied by an excursion measure in order that the associated recurrent extension of the minimal process be self-similar. For $c \in \mathbb{R}$, let H_c be the dilatation $H_c f(x) = f(cx)$.

Lemma 2. *Let n be an excursion measure and $\tilde{X}$ the associated recurrent extension of the minimal process. The following are equivalent:*

(i) *The process $\tilde{X}$ has the scaling property.*
(ii) *There exists $\gamma \in]0, 1[$ such that, for any $c > 0$,*

$$n\left(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s\right) = c^{(1-\gamma)/\alpha} n\left(\int_0^{T_0} \mathrm{e}^{-(qc^{1/\alpha}s)} H_c f(X_s)\mathrm{d}s\right),$$

for $f \in C_b(\mathbb{R}^+)$.
(iii) *There exists $\gamma \in]0, 1[$ such that, for any $c > 0$,*

$$n_s f = c^{-\gamma/\alpha} n_{s/c^{1/\alpha}} H_c f, \qquad \text{for all } s > 0,$$

for $f \in C_b(\mathbb{R}^+)$.

Remark. If conditions (i)–(iii) in the preceding lemma hold, the subordinator σ which is the inverse local time of $\tilde{X}$ is a stable subordinator of parameter γ, with γ determined by condition (ii) or (iii).

Proof. (ii) $\Longleftrightarrow$ (iii) is straightforward.

(i) $\Rightarrow$ (ii). Suppose that there exists an excursion measure n such that the associated recurrent extension $\tilde{X}$ has scaling property (1). Let $\mathcal{M}$ be the random set of zeros of the process $\tilde{X}$, that is, $\mathcal{M} = \{t \geqslant 0 | \tilde{X}(t) = 0\}$. By construction $\mathcal{M}$ is the closed range of the subordinator $\sigma = (\sigma_t,\, t \geqslant 0)$, that is, $\mathcal{M}$ is a regenerative set. The recurrence of $\tilde{X}$ implies that $\mathcal{M}$ is unbounded a.s. By the scaling property for $\tilde{X}$, we have that

$$\mathcal{M} \stackrel{d}{=} c\mathcal{M}, \qquad \text{for each } c > 0,$$

that is, $\mathcal{M}$ is self-similar. Thus the subordinator should have the scaling property and since the only Lévy processes that have the scaling property are the stable processes it follows that σ is a stable subordinator of parameter γ for some $\gamma \in]0, 1[$ or, in terms of its Laplace exponent, $\phi(\lambda) = n(1 - \mathrm{e}^{-\lambda T_0}) = \lambda^\gamma$, $\lambda > 0$. Recall that the scaling property for the extension can be stated in terms of its resolvent by saying that, for any $c > 0$,

$$U_q f(x) = c^{1/\alpha} U_{qc^{1/\alpha}} H_c f(x/c), \qquad \text{for all } x \geqslant 0, \tag{5}$$

for $f \in C_b(\mathbb{R}^+)$. Using the compensation formula for Poisson point processes, we obtain that

$$U_q f(0) = \frac{n(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s)}{n(1 - \mathrm{e}^{-qT_0})}, \tag{6}$$

From equation (5) we have that the measure n should be such that

$$\frac{n(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s)}{n(1 - \mathrm{e}^{-qT_0})} = c^{1/\alpha} \frac{n(\int_0^{T_0} \mathrm{e}^{-qc^{1/\alpha}s} H_c f(X_s)\mathrm{d}s)}{n(1 - \mathrm{e}^{-qc^{1/\alpha}T_0})},$$

and therefore we conclude that

$$n\left(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s\right) = c^{(1-\gamma)/\alpha} n\left(\int_0^{T_0} \mathrm{e}^{-(qc^{1/\alpha}s)} H_c f(X_s)\mathrm{d}s\right).$$

(ii) $\Rightarrow$ (i). The scaling property of $\tilde{X}$ is obtained by means of (5). In fact, the only thing that needs to be checked is that equation (5) holds for $x = 0$, since we have the identity

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(\mathrm{e}^{-qT_0}) U_q f(0), \qquad x > 0,$$

and the scaling property of the minimal process stated in terms of its resolvent V_q, that is,

$$V_q f(x) = c^{1/\alpha} V_{qc^{1/\alpha}} H_c f(x/c), \qquad x > 0,\ c > 0,\ q > 0.$$

Indeed, by construction it follows that formula (6) holds and hypothesis (ii) implies that $n(1 - \mathrm{e}^{-qT_0}) = q^\gamma$, $q > 0$; the conclusion is immediate. □

In the following lemma we give a description of the sojourn measure of $\tilde{X}$ and a necessary condition for the existence of a excursion measure n such that one of the conditions in Lemma 2 holds.

Lemma 3. *Let n be a normalized excursion measure and $\tilde{X}$ the associated extension of the minimal process (X, T_0). Assume that one of the conditions (i)–(iii) in Lemma 2 holds. Then*

$$n\left(\int_0^{T_0} 1_{\{X_s \in \mathrm{d}y\}} \mathrm{d}s\right) = C_{\alpha,\gamma} y^{(1-\alpha-\gamma)/\alpha} \mathrm{d}y, \qquad y > 0,$$

with γ determined by (ii) of Lemma 2 and $C_{\alpha,\gamma} \in]0, \infty[$ a constant. As a consequence, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha \mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$, where I denotes the exponential functional (2).

Proof. Recall that the sojourn measure

$$n\left(\int_0^{T_0} 1_{\{X_s\in \mathrm{d}y\}}\mathrm{d}s\right) = \int_0^\infty n_s(\mathrm{d}y)\mathrm{d}s$$

is a σ-finite measure on $]0, \infty[$ and is the unique excessive measure for the semigroup of the process $\tilde{X}$ (Dellacherie *et al*. 1992, XIX.46). Next, using result (iii) in Lemma 2 and Fubini's theorem, we obtain the following representation of the sojourn measure, for $f \geqslant 0$ measurable:

$$\begin{aligned}\int_0^\infty n_s f\,\mathrm{d}s &= \int_0^\infty s^{-\gamma} n_1(H_{s^\alpha} f)\mathrm{d}s\\ &= \int n_1(\mathrm{d}z)\int_0^\infty s^{-\gamma} f(s^\alpha z)\mathrm{d}s\\ &= C_{\alpha,\gamma}\int_0^\infty u^{(1-\alpha-\gamma)/\alpha} f(u)\mathrm{d}u,\end{aligned}$$

with $0 < C_{\alpha,\gamma} = \alpha^{-1}\int n_1(\mathrm{d}z)z^{-(1-\gamma)/\alpha} < \infty$. This proves the first part of the lemma.

We now prove that $\mathbf{E}(I^{-(1-\gamma)}) < \infty$. On the one hand, the function $\varphi(x) = \mathbb{E}_x(\mathrm{e}^{-T_0})$ is integrable with respect to the sojourn measure. To see this, use the Markov property under n to obtain

$$\begin{aligned}n\left(\int_0^{T_0}\varphi(X_s)\mathrm{d}s\right) &= \int_0^\infty n(\varphi(X_s),\, s < T_0)\mathrm{d}s\\ &= \int_0^\infty n(\mathrm{e}^{-T_0}\circ\theta_s,\, s < T_0)\mathrm{d}s\\ &= \int_0^\infty n(\mathrm{e}^{-(T_0-s)},\, s < T_0)\mathrm{d}s\\ &= n(1-\mathrm{e}^{-T_0}) = 1.\end{aligned}$$

On the other hand, using the representation of the sojourn measure, Fubini's theorem and the scaling property, we have that

$$\begin{aligned}C_{\alpha,\gamma}\int_0^\infty \mathbb{E}_y(\mathrm{e}^{-T_0})y^{(1-\alpha-\gamma)/\alpha}\mathrm{d}y &= C_{\alpha,\gamma}\int_0^\infty \mathbf{E}(\mathrm{e}^{-y^{1/\alpha}I})y^{(1-\alpha-\gamma)/\alpha}\mathrm{d}y\\ &= C_{\alpha,\gamma}\alpha\mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma).\end{aligned}$$

Therefore, $\mathbf{E}(I^{-(1-\gamma)}) < \infty$ and $C_{\alpha,\gamma} = (\alpha\mathbf{E}(I^{-(1-\gamma)})\Gamma(1-\gamma))^{-1}$. □

We next study the extensions $\tilde{X}$ that leave 0 a.s. by jumps. Using only scaling property (1) it can be verified that the only possible jumping-in measures such that the associated excursion measure satisfies (ii) in Lemma 2 are of the type

$$\eta(\mathrm{d}x) = b_{\alpha,\beta}x^{-(1+\beta)}\mathrm{d}x, \qquad x > 0,\ 0 < \alpha\beta < 1,$$

with a constant $b_{\alpha,\beta} > 0$, depending on α and β, (Vuolle-Apiala 1994). This having been said, we can state an elementary but satisfactory result on the existence of extensions of the minimal process that leave 0 by jumps a.s.

Proposition 1. *Let $\beta \in]0, 1/\alpha[$. The following are equivalent:*

(i) $\mathbf{E}(I^{\alpha\beta}) < \infty$.
(ii) *The pseudo-excursion measure $n^j = \mathbb{P}^\eta$, based on the jumping-in measure $\eta(\mathrm{d}x) = x^{-(1+\beta)}\mathrm{d}x$, $x > 0$, is an excursion measure.*
(iii) *The minimal process (X, T_0) admits an extension $\tilde{X}$, that is a self-similar recurrent Markov process, and leaves 0 by jumps a.s. according to the jumping-in measure $\eta(\mathrm{d}x) = b_{\alpha,\beta}x^{-(1+\beta)}\mathrm{d}x$, with $b_{\alpha,\beta} = \beta/\mathbf{E}(I^{\alpha\beta})\Gamma(1-\alpha\beta)$.*

If one of these conditions holds then γ in (ii) in Lemma 2 is equal to $\alpha\beta$.

Condition (i) in Proposition 1 is easily verified under weak technical assumptions. That is to say, if we assume (H2), the aforementioned condition is verified for every $\beta \in]0, (1/\alpha) \wedge \theta[$; this will be deduced from Lemma 4 below. On the other hand, the condition is verified in other settings, as can be seen in the following example.

Example 1 *Generalized self-similar sawtooth processes.* Let $\alpha > 0$, $\mathbf{k} = 0$, $\tilde{\sigma}$ be a subordinator such that $\mathbf{E}(\tilde{\sigma}_1) < \infty$, and X the α-self-similar process associated with the Lévy process $\xi = -\tilde{\sigma}$. Then ξ is a Lévy process with infinite lifetime that drifts to $-\infty$, X has a finite lifetime T_0 and X decreases from its starting point until the time T_0, when it is absorbed at 0. Furthermore, it was proved by Carmona *et al*. (1997) that the Lévy exponential functional $I = \int_0^\infty \exp\{-\tilde{\sigma}_s/\alpha\}\mathrm{d}s$ has finite integral moments of all orders. It follows that condition (i) in Proposition 1 is satisfied by every $\beta \in]0, 1/\alpha[$. Thus for each $\beta \in]0, 1/\alpha[$ the α-self-similar extension $\tilde{X}$ that leaves 0 by jumps according to the jumping-in measure in (iii) of Proposition 1 is a process having sample paths that looks like a saw with 'rough' teeth. These are all the possible extensions of X, that is, it is impossible to construct an excursion measure such that its associated extension of (X, T_0) leaves 0 continuously a.s., since we know that the process X decreases to 0.

Proof of Proposition 1. Let $\eta(\mathrm{d}x) = x^{-(1+\beta)}\mathrm{d}x$, $x > 0$, and n^j be the pseudo-excursion measure $n^j = \mathbb{P}^\eta$. By definition, the entrance law associated with n^j is

$$n^j_s f = \int_0^\infty \mathrm{d}x\, x^{-(1+\beta)} P_s f(x), \qquad s > 0.$$

Thus, for n^j to be an excursion measure, the only condition it needs to satisfy is $n^j(1 - \mathrm{e}^{-T_0}) < \infty$. This follows from the elementary calculation

$$\int_0^\infty \mathrm{d}x\, x^{-(1+\beta)}\mathbb{E}_x(1-\mathrm{e}^{-T_0}) = \int_0^\infty \mathrm{d}x\, x^{-(1+\beta)}\mathbf{E}(1-\mathrm{e}^{-x^{1/\alpha}I})$$

$$= \alpha\mathbf{E}\left(\int \mathrm{d}y\, y^{-\alpha\beta-1}(1-\mathrm{e}^{-yI})\right)$$

$$= \mathbf{E}(I^{\alpha\beta})\frac{\Gamma(1-\alpha\beta)}{\beta}.$$

That is, $n^j(1-\mathrm{e}^{-T_0}) < \infty$ if and only if $\mathbf{E}(I^{\alpha\beta}) < \infty$, which proves the equivalence between the assertions in (i) and (ii). If (ii) holds it follows from the results in Blumenthal (1983) and Lemma 2 that associated with the normalized excursion measure $n^{j'} = b_{\alpha,\beta}\mathbb{P}^\eta$ there exists a unique extension of the minimal process (X, T_0) which is a self-similar Markov process and which leaves 0 by jumps according to the jumping-in measure $b_{\alpha,\beta}x^{-(1+\beta)}\mathrm{d}x$, $x>0$, which establishes (iii). Conversely, if (iii) holds the Itô excursion measure of $\tilde{X}$ is $n^{j'} = b_{\alpha,\beta}\mathbb{P}^\eta$ and the statement in (ii) follows. □

2.3. The process $X^\natural$ analogous to the Bessel(3) process

Here we shall establish the existence of a self-similar Markov process $X^\natural$ that can be viewed as the self-similar Markov process (X, T_0) conditioned never to hit 0. In the case where (X, T_0) is a Brownian motion killed at 0, $X^\natural$ corresponds to the Bessel(3) process. To this end, we next recall some facts on Lévy processes and density transformations and deduce some consequences for self-similar Markov processes. We henceforth assume (H2).

The law of a Lévy process ξ obtained by killing at a rate $\mathbf{k}$ is characterized by a function $\Psi: \mathbb{R} \to \mathbb{C}$ defined by the relation

$$\mathbf{E}(\mathrm{e}^{\mathrm{i}u\xi_1}, 1<\zeta) = \exp\{-\Psi(u)\}, \qquad u \in \mathbb{R}.$$

The function Ψ is called the characteristic exponent of the Lévy process ξ and can be expressed thanks to the Lévy–Khinchine formula as

$$\Psi(u) = \mathbf{k} - \mathrm{i}au + \frac{\sigma^2u^2}{2} + \int_{\mathbb{R}}(1-\mathrm{e}^{\mathrm{i}ux} + \mathrm{i}ux1_{\{|x|<1\}})\Pi(\mathrm{d}x),$$

where Π is a measure on $\mathbb{R}\backslash\{0\}$ such that $\int(|x|^2\wedge 1)\Pi(\mathrm{d}x)<\infty$. The measure Π is called the Lévy measure, σ^2 the Gaussian coefficient and $\mathbf{k}$ the killing rate. Conditions (H2b) and (H2c) imply that the Lévy exponent of ξ admits an analytic extension to the complex strip $\mathfrak{I}(z)\in[-\theta, 0]$. Thus we can define a function $\psi: [0,\theta]\to\mathbb{R}$ by

$$\mathbf{E}(\mathrm{e}^{\lambda\xi_1}, 1<\zeta) = \mathrm{e}^{\psi(\lambda)} \quad \text{and} \quad \psi(\lambda) = -\Psi(-\mathrm{i}\lambda), \quad 0\leqslant\lambda\leqslant\theta.$$

Holder's inequality implies that ψ is a convex function and that θ is the unique solution to the equation $\psi(\lambda)=0$ for $\lambda>0$. This happens if and only if there exists a $\theta>0$ such that

$$\mathbf{k} = a\theta + \sigma^2\theta^2/2 + \int_{\mathbb{R}}(\mathrm{e}^{\theta x} - 1 - \theta x1_{\{|x|<1\}})\Pi(\mathrm{d}x).$$

Furthermore, the function $h(x) = \mathrm{e}^{\theta x}$ is invariant for the semigroup of ξ. Let $\mathbf{P}^{\natural}$ be the h-transform of $\mathbf{P}$ via the invariant function $h(x) = \mathrm{e}^{\theta x}$. That is, the measure $\mathbf{P}^{\natural}$ is the unique measure on $(\mathbb{D}, \mathcal{D})$ such that, for every finite $\mathcal{D}_t$-stopping time T and each $A \in \mathcal{D}_T$,

$$\mathbf{P}^{\natural}(A) = \mathbf{P}(\mathrm{e}^{\theta \xi_T} A \cap \{T < \zeta\}).$$

Under $\mathbf{P}^{\natural}$ the process $(\xi^{\natural}_t, t \geqslant 0)$ is a Lévy process with infinite lifetime, characteristic exponent

$$\Psi^{\natural}(u) = \Psi(u - \mathrm{i}\theta), \qquad u \in \mathbb{R},$$

and drifts to ∞; more precisely,

$$0 < m^{\natural} := \mathbf{E}^{\natural}(\xi_1) = \psi'(\theta-) < \infty.$$

For a proof of these facts and more about this change of measure, see Sato (1999, Section 33).

Let $\mathbb{P}^{\natural}_x$ denote the law on $\mathbb{D}^+$ of the self-similar Markov process starting at $x > 0$ and associated with the Lévy process $\xi^{\natural}$ via Lamperti's transformation. In what follows it will be implicit that the superscript $\natural$ refers to the measure $\mathbb{P}^{\natural}$ or $\mathbf{P}^{\natural}$. We now establish a relation between the probability measures $\mathbb{P}$ and $\mathbb{P}^{\natural}$ analogous to that between the law of a Brownian motion killed at 0 and the law of a Bessel(3) process (McKean 1963). Informally, the law $\mathbb{P}^{\natural}_x$ can be interpreted as the law under $\mathbb{P}_x$ of X conditioned never to hit 0.

Proposition 2. (i) *Let $x > 0$ be arbitrary. Then we have that $\mathbb{P}^{\natural}_x$ is the unique measure such that, for every $\mathcal{G}_t$-stopping time T, we have*

$$\mathbb{P}^{\natural}_x(A) = x^{-\theta}\mathbb{P}_x(AX^{\theta}_T, T < T_0),$$

for any $A \in \mathcal{G}_T$. In particular, the function $h^* : [0, \infty[\to [0, \infty[$ defined by $h^*(x) = x^{\theta}$ is invariant for the semigroup P_t.

(ii) *For every $x > 0$ and $t > 0$, we have*

$$\mathbb{P}^{\natural}_x(A) = \lim_{s\to\infty} \mathbb{P}_x(A | T_0 > s),$$

for any $A \in \mathcal{G}_t$.

The proof of (i) in Proposition 2 is a straightforward consequence of the fact that $\mathbf{P}^{\natural}$ is the h-transform of $\mathbf{P}$ and that for every $\mathcal{G}_t$-stopping time T we have that $\tau(T)$ is a $\mathcal{D}_t$-stopping time. To prove (ii) in Proposition 2 we need the following lemma that provides us with a tail estimate for the law of the Lévy exponential functional I associated with ξ as defined in (2).

Lemma 4. *Under conditions* (H2) *we have that*

$$\lim_{t\to\infty} t^{\alpha\theta}\mathbf{P}(I > t) = C,$$

where

$$0 < C = \frac{\alpha}{m^\natural} \int t^{\alpha\theta - 1} (\mathbf{P}(I > t) - \mathbf{P}(\mathrm{e}^{\xi_1'} I > t)) \mathrm{d}t < \infty,$$

with $\xi_1' \stackrel{d}{=} \xi_1$ *and independent of* I*. If* $0 < \alpha\theta < 1$*, then*

$$C = \frac{\alpha}{m^\natural} \mathbf{E}(I^{-(1-\alpha\theta)}).$$

Two proofs of this result have been given in a slightly restrictive setting by Mejane (2002). However, one of these proofs can be extended to our case and in fact it is an easy consequence of a result on random equations originally due to Kesten (1973) who in turn uses a difficult result on random matrices. A simpler proof of Kesten's result was given by Goldie (1991).

Sketch of proof of Lemma 4. We study first the case $\mathbf{k} = 0$. In this case $\zeta = \infty$ a.s. and $I = \int_0^\infty \exp\{\xi_s/\alpha\} \mathrm{d}s$. It is straightforward that the Lévy exponential functional I satisfies in law the equation

$$I \stackrel{d}{=} \int_0^1 \mathrm{e}^{\xi_s/\alpha} \mathrm{d}s + \mathrm{e}^{\xi_1/\alpha} I' = Q + MI',$$

with I' the Lévy exponential functional associated with $\xi' = \{\xi_t' = \xi_{1+t} - \xi_1,\ t \geqslant 0\}$, a Lévy process independent of $\mathcal{D}_1$ and with the same distribution as ξ. Thus, according to Kesten (1973) and Goldie (1991), if the conditions (i)–(iv) below are satisfied then there exists a strictly positive constant C such that

$$\lim_{t \to \infty} t^{\alpha\theta} \mathbf{P}(I > t) = C.$$

The hypotheses of Kesten's theorem are:

(i) M is not arithmetic.
(ii) $\mathbf{E}(M^{\alpha\theta}) = 1$.
(iii) $\mathbf{E}(M^{\alpha\theta} \ln^+(M)) < \infty$.
(iv) $\mathbf{E}(Q^{\alpha\theta}) < \infty$.

Assuming conditions (H2), the only thing that needs to be verified is that (iv) holds. Indeed,

$$\begin{aligned} \mathbf{E}(Q^{\alpha\theta}) &\leqslant \mathbf{E}(\sup\{\mathrm{e}^{\theta\xi_s} : s \in [0, 1]\}) \\ &\leqslant \frac{\mathrm{e}}{\mathrm{e} - 1} (1 + \theta \sup\{\mathbf{E}(\xi_s^+ \mathrm{e}^{\theta\xi_s}) : s \in [0, 1]\}) < \infty. \end{aligned}$$

The second inequality is obtained using the fact that $(\mathrm{e}^{\theta\xi_t},\ t \geqslant 0)$ is a positive martingale and Doob's inequality. The first formula for the value of the limit, $C = \lim_{t\to\infty} t^{\alpha\theta} \mathbf{P}(I > t)$, is a consequence of Goldie (1991, Lemma 2.2 and Theorem 4.1). That the latter limit exists implies that $\mathbf{E}(I^a) < \infty$, for all $0 < a < \alpha\theta$. Now, to obtain the expression for C when $0 < \alpha\theta < 1$, we will use the following formula for the moments of I:

$$\mathbf{E}(I^a) = \frac{a}{-\psi(a/\alpha)}\mathbf{E}(I^{a-1}), \qquad \text{for } 0 < a < \alpha\theta, \tag{7}$$

which can be proved with arguments similar to those given by Bertoin and Yor (2002b, Proposition 2). We will also use the well-known identity

$$\lambda^a = \frac{a}{\Gamma(1-a)}\int_0^\infty (1 - \mathrm{e}^{-\lambda x})x^{-(1+a)}\mathrm{d}x, \qquad \lambda > 0,\ a \in]0, 1[.$$

On the one hand, since $0 < \alpha\theta < 1$, we have from Bingham *et al*. (1989, Corollary 8.1.7) that

$$\lim_{s\to 0}\frac{\mathbf{E}(1 - \mathrm{e}^{-sI})}{s^{\alpha\theta}} = C\Gamma(1 - \alpha\theta).$$

On the other hand, by equation (7) we have

$$\begin{aligned}
\mathbf{E}(I^{-(1-\alpha\theta)})\alpha\theta &= \lim_{a\to\alpha\theta}\mathbf{E}(I^{a-1})a \\
&= \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\lim_{a\uparrow\alpha\theta}(-\psi(a/\alpha))\int_0^\infty s^{-(1+a)}\mathbf{E}(1 - \mathrm{e}^{-sI})\mathrm{d}s \\
&= C\alpha\theta\lim_{a\uparrow\alpha\theta}\frac{-\psi(a/\alpha)}{\alpha\theta - a} \\
&= C\theta\psi'(\theta-),
\end{aligned} \tag{8}$$

which proves the claimed result in the case $\mathbf{k} = 0$. In the case $\mathbf{k} > 0$, the Lévy exponential functional I has the same law as $A_{\mathbb{e}} = \int_0^{\mathbb{e}} \exp\{\check{\xi}_s/\alpha\}\mathrm{d}s$, with $\check{\xi}$ the unkilled Lévy process and $\mathbb{e}$ an exponential random variable with parameter $\mathbf{k}$ and independent of $\check{\xi}$. Using the memorylessness of the exponential law we can easily verify that $A_{\mathbb{e}}$ satisfies the equation in law $A_{\mathbb{e}} \stackrel{d}{=} Q + MA'_{\mathbb{e}'}$ with $Q = \int_0^1 \exp\{\check{\xi}_s\}1_{\{s<\mathbb{e}\}}\mathrm{d}s$, $M = \mathrm{e}^{\check{\xi}_1/\alpha}1_{\{1<\mathbb{e}\}}$ and $A'_{\mathbb{e}'}$ with the same law as $A_{\mathbb{e}}$ and independent of (Q, M) and $\mathbb{e}$. Next we verify, in the same way as in the case $\mathbf{k} = 0$, that the random variables (Q, M) satisfy hypotheses (i)–(iv) of Kesten's theorem. Finally, to estimate the value of the constant C when $0 < \alpha\theta < 1$, we use an identity similar to (7) for $A_{\mathbb{e}}$,

$$\check{\mathbf{E}}(A^a_{\mathbb{e}}) = \frac{a}{-\psi(a/\alpha)}\check{\mathbf{E}}(A^{a-1}_{\mathbb{e}}), \qquad 0 < a < \alpha\theta,$$

which is obtained in Carmona *et al*. (1997, Proposition 3.1(i)). □

The proof of Proposition 2 follows by standard arguments.

Proof of (ii) in Proposition 2. Recall that the law of T_0 under $\mathbb{P}_x$ is that of $x^{1/\alpha}I$ under $\mathbf{P}$. Thus we deduce from Lemma 4 that, for every $x > 0$,

$$\lim_{s\to\infty} s^{\alpha\theta}\mathbb{P}_x(T_0 > s) = x^\theta C.$$

Using the Markov property and a dominated convergence argument, we obtain that

$$\mathbb{P}_x(A|T_0 > s) = \mathbb{P}_x(A1_{\{t<T_0\}}\mathbb{P}_{X_t}(T_0 > s-t)/\mathbb{P}_x(T_0 > s))$$
$$\underset{s\to\infty}{\longrightarrow} x^{-\theta}\mathbb{P}_x(AX_t^{\theta}1_{\{t<T_0\}}).$$

□

By Proposition 2, the semigroup of X under $\mathbb{P}_x^{\natural}$ is given by

$$P_s^{\natural}f(x) := \mathbb{E}_x^{\natural}(f(X_s)) = x^{-\theta}\mathbb{E}_x(f(X_s)X_s^{\theta}1_{\{s<T_0\}}), \qquad \text{for } x > 0,$$

with f a positive or bounded measurable function. Let J be the Lévy exponential functional associated with the process $\xi^{\natural}$, that is,

$$J = \int_0^{\infty} \exp\{-\xi_s^{\natural}/\alpha\}\mathrm{d}s, \tag{9}$$

which is finite $\mathbf{P}^{\natural}$-a.s. since $\xi^{\natural}$ drifts to ∞. Now, since under $\mathbf{P}^{\natural}$ the process $(\xi_s^{\natural}, s \geqslant 0)$ is a non-arithmetic Lévy process with $0 < m^{\natural} < \infty$, the measure $\mathbb{P}_x^{\natural}$ converges in the sense of finite-dimensional distributions to a probability measure $\mathbb{P}_{0+}^{\natural}$ as $x \to 0+$ (Bertoin and Yor 2002a, Theorem 1). Moreover, the law of X_s under $\mathbb{P}_{0+}^{\natural}$ is an entrance law for the semigroup $P_t^{\natural}$ and is related to the law of the Lévy exponential functional J under $\mathbf{P}^{\natural}$ by the formula

$$\mathbb{E}_{0+}^{\natural}(f(X_s^{1/\alpha})) = \frac{\alpha}{m^{\natural}}\mathbf{E}^{\natural}(f(s/J)/J), \qquad s > 0, \tag{10}$$

for f measurable and positive. Recall also that $m^{\natural}/\alpha = \mathbf{E}^{\natural}(1/J) < \infty$. See Bertoin and Yor (2002a) for a proof of these facts.

The next result states that under (H2) conditions (H1) hold, and gives a first description of the entrance law $(\mathbf{n}_s, s > 0)$.

Proposition 3. *Assume hypotheses* (H2).

(i) *If* $0 < \alpha\theta < 1$, *then hypotheses* (H1) *hold for* $\kappa = \theta$. *Furthermore, the* q*-potential of the entrance law* $(\mathbf{n}_s, s > 0)$, *admits the representation*

$$\int_0^{\infty} \mathrm{d}s\, \mathrm{e}^{-qs}\mathbf{n}_s f = \gamma_{\alpha,\theta}\int_0^{\infty} f(y)\mathbf{E}^{\natural}(\exp\{-qy^{1/\alpha}J\})y^{(1-\alpha-\alpha\theta)/\alpha}\,\mathrm{d}y,$$

where

$$\gamma_{\alpha,\theta} = \left(\alpha\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)\right)^{-1},$$

for every $f \in C_b(\mathbb{R}^+)$.

(ii) *If* $\alpha\theta \geqslant 1$, *then either* (H1a) *or* (H1b) *fails to hold.*

Proof. (i) That (H1a) holds is easily proved. Indeed, since $0 < \alpha\theta < 1$ we have from Bingham *et al*. (1989, Corollary 8.1.7) that the result in Lemma 4 is equivalent to

$$\lim_{x\to 0}\frac{\mathbb{E}_x(1-\mathrm{e}^{-T_0})}{x^\theta}=\lim_{x\to 0}\frac{\mathbf{E}(1-\mathrm{e}^{-x^{1/\alpha}I})}{x^\theta}=\Gamma(1-\alpha\theta)\frac{\alpha\mathbf{E}(I^{-(1-\alpha\theta)})}{m^\natural}. \tag{11}$$

To prove (H1b) we recall the identity,

$$\frac{V_q f(x)}{x^\theta}=V_q^\natural(f/h^*)(x),$$

where $V_q^\natural$ is the resolvent of the semigroup $P_t^\natural$ and $h^*(x)=x^\theta$, $x>0$. As has already been pointed out, the results in Bertoin and Yor (2002a) are applicable in our setting to the self-similar process $X^\natural$. In particular, their formula (4) states that

$$\lim_{x\to 0}V_q^\natural g(x)=\frac{\alpha}{m^\natural}\int_0^\infty g(y^\alpha)\mathbf{E}^\natural(\mathrm{e}^{-qyJ})\mathrm{d}y,$$

for every function $g\in C_b(\mathbb{R}^+)$. Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{V_q f(x)}{x^\theta}&=\lim_{x\to 0}V_q^\natural(f/h^*)(x)\\&=\frac{\alpha}{m^\natural}\int_0^\infty f(y^\alpha)y^{-\alpha\theta}\mathbf{E}^\natural(\mathrm{e}^{-qyJ})\mathrm{d}y,\\&=\frac{1}{m^\natural}\int_0^\infty f(y)\mathbf{E}^\natural(\mathrm{e}^{-qy^{1/\alpha}J})y^{(1-\alpha-\alpha\theta)/\alpha}\mathrm{d}y\end{aligned}\tag{12}$$

for every $f\in C_K]0,\infty[$. Thus we have verified hypotheses (H1) and the expression for the q-resolvent of the entrance law $(\mathbf{n}_s,\ s>0)$ follows from the identity (3) using the calculations in equations (11) and (12).

(ii) If $\alpha\theta\geqslant 1$, Fatou's lemma and the scaling property imply

$$\liminf_{x\to 0}\frac{\mathbb{E}_x(1-\mathrm{e}^{-T_0})}{x^\theta}\geqslant\int_0^\infty \mathrm{e}^{-s}s^{-\alpha\theta}\Big(\liminf_{t\to\infty}t^{\alpha\theta}\mathbf{P}(I>t)\Big)\mathrm{d}s=\infty.$$

But from the proof of (i) we know that the limit

$$\lim_{x\to 0}\frac{V_q f(x)}{x^\theta},\qquad q>0,$$

still exists and is not 0 for every non-negative function $f\in C_K]0,\infty[$ and, indeed, $f>0$ in a set of positive Lebesgue measure. As a consequence, even if there exists $\kappa<\theta$, such that the limit $\lim_{x\to 0}x^{-\kappa}\mathbb{E}_x(1-\mathrm{e}^{-T_0})$, exists and is positive, the limit $\lim_{x\to 0}x^{-\kappa}V_q f(x)$ is equal to zero for every continuous function f with bounded support on $]0,\infty[$. □

Proposition 3 proves that hypotheses (H2) imply (H1). In the next proposition we establish a converse.

Proposition 4. *Assume that there exists a* $\kappa>0$ *such that hypotheses* (H1) *hold. Then*

(i) $0<\alpha\kappa<1$,
(ii) (H2b) *and* (H2c) *are satisfied with* $\theta=\kappa$.

Proof. To prove (i) we recall that under (H1) Vuolle-Apiala (1994, Theorem 2.1) proved that the q-resolvent of the entrance law $(\mathbf{n}_s, s > 0)$ is characterized by equation (3). Next, it is easily verified using the self-similarity of the minimal process (X, T_0) that, for every $q > 0$, $c > 0$,

$$\lim_{x\to 0} \frac{V_q f(x)}{\mathbb{E}_x(1-\mathrm{e}^{-T_0})} = c^{(1-\alpha\kappa)/\alpha} \lim_{x\to 0} \frac{V_{qc^{1/\alpha}} H_c f(x)}{\mathbb{E}_x(1-\mathrm{e}^{-T_0})}.$$

Then the excursion measure $\mathbf{n}$ is such that, for every $c > 0$,

$$\mathbf{n}\left(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s)\mathrm{d}s\right) = c^{(1-\alpha\kappa)/\alpha}\mathbf{n}\left(\int_0^{T_0} \mathrm{e}^{-qc^{1/\alpha}s} H_c f(X_s)\mathrm{d}s\right).$$

The latter fact implies that (ii) in Lemma 2 is satisfied with $\gamma = \alpha\kappa$ and $0 < \alpha\kappa < 1$.

Next we prove (ii). We first prove that under (H1) the process $(X_t^\kappa, t > 0)$ is a martingale for $\mathbb{P}_x$, which implies Cramér's condition (H2b). Indeed, since (H1a) holds we have that

$$\lim_{x\to 0} \frac{\mathbb{E}_x(1-\mathrm{e}^{-T_0})}{x^\kappa} = B \in]0, \infty[,$$

and, given that $0 < \alpha\kappa < 1$, the existence of this limit is equivalent to the existence of the limit

$$\lim_{s\to\infty} s^{\alpha\kappa}\mathbb{P}_x(T_0 > s) = x^\kappa B/\Gamma(1-\alpha\kappa).$$

This fact suffices to prove that, for every $x > 0$ and $t > 0$,

$$\lim_{s\to\infty} \mathbb{P}_x(A | T_0 > s) = x^{-\kappa}\mathbb{P}_x(X_t^\kappa, A \cap \{t < T_0\}),$$

for any $A \in \mathcal{G}_t$. To see this, just repeat the arguments in the proof of (ii) in Proposition 2. In particular, we have that, for every $x > 0$ and $t > 0$, $x^\kappa = \mathbb{E}_x(X_t^\kappa, t < T_0)$. Using the Markov property we obtain that, for every $x > 0$, under $\mathbb{P}_x$ the process X^κ is a martingale and as a consequence Cramér's condition follows. Moreover, the Lévy process ξ associated with X via Lamperti's transformation has a characteristic exponent Ψ that admits an analytic extension to the complex strip $\mathfrak{I}(z) \in [-\kappa, 0[$ defined by $\psi(z) = -\Psi(-\mathrm{i}z)$ (see the survey at the beginning of this subsection). Now to prove that (H2c) is satisfied, we recall that under (H1) we have that

$$\lim_{s\to\infty} s^{\alpha\kappa}\mathbf{P}(I > s) = x^{-\kappa} \lim_{s\to\infty} s^{\alpha\kappa}\mathbb{P}_x(T_0 > s) = B/\Gamma(1-\alpha\kappa),$$

and that $\mathbf{E}(I^{-(1-\alpha\kappa)}) < \infty$, the latter being a consequence of Lemma 3. Repeating the arguments in the calculation of the constant in the proof of Lemma 4, we obtain that

$$\mathbf{E}(I^{-(1-\alpha\kappa)}) = B\psi'(\theta-)/\Gamma(1-\alpha\kappa) < \infty,$$

that is, the exponent ψ of ξ has a left derivative at κ which is equivalent to

$$\mathbf{E}(\xi_1 \mathrm{e}^{\kappa\xi_1}, 1 < \zeta) < \infty.$$

Using the elementary relation

$$0 \leqslant (\xi_1 \exp\{\kappa\xi_1\})^- = \xi_1^- \exp\{\kappa\xi_1\} = \xi_1^- \exp\{-\kappa\xi_1^-\} \leqslant \kappa^{-1}$$

with $a^- = (-a) \vee 0$, we obtain that $0 \leqslant \mathbf{E}((\xi_1 e^{\kappa\xi_1})^-, 1 < \zeta) < 1/\kappa$. Therefore, $\mathbf{E}(\xi_1 e^{\kappa\xi_1}, 1 < \zeta) < \infty$ if and only if $\mathbf{E}(\xi_1^+ e^{\kappa\xi_1}, 1 < \zeta) < \infty$, which concludes the proof. □

Remarks

1. If $0 < \alpha\theta < 1$ we have the equality
$$\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)}).$$
Indeed, straightforward calculations lead to
$$\int e^{-s}\mathbf{n}_s 1\, ds = \gamma_{\alpha,\theta}\int_0^\infty \mathbf{E}^\natural(e^{-y^{1/\alpha}J})y^{(1-\alpha-\alpha\theta)/\alpha}\, dy = \frac{\mathbf{E}^\natural(J^{-(1-\alpha\theta)})}{\mathbf{E}(I^{-(1-\alpha\theta)})},$$
and comparing this with the fact that $\int e^{-s}\mathbf{n}_s 1 ds = 1$ gives the equality.
2. A consequence of Lemma 4 is that
$$\mathbf{E}(I^{\beta\alpha}) < \infty, \qquad \text{for every } 0 < \beta < \theta,$$
and that $\mathbf{E}(I^{\alpha\theta}) = \infty$. Then under (H2) any extension which leaves 0 by jumps a.s. has a jumping-in measure $\eta(dx) = b_{\alpha,\beta}x^{-(1+\beta)}dx$, $x > 0$, with $0 < \beta < \theta \wedge 1/\alpha$ and $b_{\alpha,\beta}$ as defined in Proposition 1.

3. Existence of recurrent extensions that leaves 0 continuously

We next study the excursion measure such that the related extension leaves 0 continuously. To this end, we suppose throughout the rest of this section that hypotheses (H2) hold.

Theorem 1. *There exists a pseudo-excursion measure* $\mathbf{n}'$ *such that* $\mathbf{n}'(X_{0+} > 0) = 0$. *Its associated entrance law* $(\mathbf{n}'_s, s > 0)$ *is given by*

$$\mathbf{n}'_s f = \mathbb{E}^\natural_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0.$$

We have that $\mathbf{n}'$ *is an excursion measure if and only if* $0 < \alpha\theta < 1$. *Assume that this condition holds and let*

$$a_{\alpha,\theta} = \alpha\mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^\natural.$$

Then the measure $(a_{\alpha,\theta})^{-1}\mathbf{n}'$, *is the normalized excursion measure* $\mathbf{n}$.

Proof. We know from Proposition 2 that the function $h(x) = x^{-\theta}$ is excessive for the semigroup $P^\natural_t$ and the corresponding h-transform is P_t. Let $\mathbf{n}'$ be the h-transform of $\mathbb{E}^\natural_{0+}$ via the excessive function $h(x) = x^{-\theta}$, $x > 0$. That is, $\mathbf{n}'$ is the unique measure in $\mathbb{D}^+$ carried by $\{T_0 > 0\}$, such that under $\mathbf{n}'$ the coordinate process is Markovian with semigroup P_t and for every $\mathcal{G}_t$-stopping time T and any $A_T \in \mathcal{G}_T$,

$$\mathbf{n}'(A_T, T < T_0) = \mathbb{E}^\natural_{0+}(A_T, X_T^{-\theta}).$$

Therefore, $\mathbf{n}'$ is a pseudo-excursion measure such that $\mathbf{n}'(X_{0+} > 0) = 0$ and the entrance law associated with $\mathbf{n}'$ is defined by

$$\mathbf{n}'_s f := \mathbf{n}'(f(X_s),\ s < T_0) = \mathbb{E}^{\natural}_{0+}(f(X_s)X_s^{-\theta}), \qquad s > 0, \tag{13}$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable.

To prove the second assertion we have to specify when $\mathbf{n}'(1 - \mathrm{e}^{-T_0})$ is finite. Using standard arguments, we obtain that

$$\begin{aligned}\mathbf{n}'(1 - \mathrm{e}^{-T_0}) &= \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s}\mathbf{n}'(T_0 > s)\\ &= \int_0^\infty \mathrm{d}s\, \mathrm{e}^{-s}\mathbb{E}^{\natural}_{0+}(X_s^{-\theta})\\ &= \begin{cases} \alpha\mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m^{\natural}, & \text{if } \alpha\theta < 1,\\ \infty, & \text{if } \alpha\theta \geqslant 1;\end{cases}\end{aligned}$$

the third equality is obtained from (10). If $0 < \alpha\theta < 1$, then $\mathbf{E}^{\natural}(J^{-(1-\alpha\theta)}) < \infty$ since $\mathbf{E}^{\natural}(J^{-1}) < \infty$. As a consequence, $\mathbf{n}'(1 - \mathrm{e}^{-T_0}) < \infty$ if and only if $0 < \alpha\theta < 1$. If we assume that $0 < \alpha\theta < 1$, it follows that the measure $a_{\alpha,\theta}^{-1}\mathbf{n}'$ is a normalized excursion measure compatible with the semigroup P_t. Furthermore, it is straightforward to check that $a_{\alpha,\theta}^{-1}\mathbf{n}'$ satisfies condition (ii) in Lemma 2 for $\gamma = \alpha\theta$. The normalized excursion measure $a_{\alpha,\theta}^{-1}\mathbf{n}'$ is equal to the measure $\mathbf{n}$ since this is the unique normalized excursion measure having the property $\mathbf{n}(X_{0+} > 0) = 0$. □

A consequence of the Markov property is that under $\mathbf{n}'$ the excursions leave 0 continuously and either hit 0 continuously or by a jump according to whether $\mathbf{k} = 0$ or $\mathbf{k} > 0$, that is,

$$\mathbf{n}'(X_{0+} > 0,\ X_{T_0-} > 0) = 0 \quad \text{or} \quad \mathbf{n}'(X_{0+} > 0,\ X_{T_0-} = 0) = 0,$$

respectively.

In the following theorem we give a simple criterion to determine, in terms of the Lévy process ξ, whether there exists a self-similar recurrent extension of (X, T_0) that leaves 0 continuously. Furthermore, with this result we give a complete solution to the problem posed by Lamperti since we have already established the existence of self-similar recurrent extensions of the minimal process that leave 0 by jumps.

Theorem 2. (i) *Assume* $0 < \alpha\theta < 1$. *The minimal process admits a unique self-similar recurrent extension* $\tilde{X} = (\tilde{X}_t,\ t \geqslant 0)$ *that leaves* 0 *continuously a.s. The resolvent of* $\tilde{X}$ *is determined by*

$$U_q f(0) = \frac{\gamma_{\alpha,\theta}}{q^{\alpha\theta}}\int_0^\infty f(y)\mathbf{E}^{\natural}(\mathrm{e}^{-qy^{1/\alpha}J})y^{(1-\alpha-\alpha\theta)/\alpha}\,\mathrm{d}y,$$

with $\gamma_{\alpha,\theta}$ *as defined in Proposition 3 and*

$$U_q f(x) = V_q f(x) + \mathbb{E}_x(\mathrm{e}^{-qT_0}) U_q f(0), \qquad x > 0,$$

for $f \in C_b(\mathbb{R}^+)$. *The resolvent* U_q *is Fellerian.*

(ii) *If* $\alpha\theta \geqslant 1$, *there does not exist any self-similar recurrent extension that leaves* 0 *continuously.*

Proof. To obtain (i) we use Lemma 1. This enables us to apply the results of Blumenthal (1983) to ensure that associated with the excursion measure $\mathbf{n}$, described in Theorem 1, there exists a Markov process $\tilde{X}$, having a Feller resolvent which is an extension of the minimal process. The self-similarity of $\tilde{X}$ follows from Lemma 2. All that needs justifying is the expression for the q-resolvent of the extension. Using the compensation formula for Poisson point processes, we obtain that

$$U_q f(0) = \mathbf{n}\left(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s) \mathrm{d}s \right) / \mathbf{n}(1 - \mathrm{e}^{-qT_0}),$$

for every $f \in C_b(\mathbb{R}^+)$. From Lemma 2 we deduce that $\mathbf{n}(1 - \mathrm{e}^{-qT_0}) = q^{\alpha\theta}$. The expression of $U_q f(0)$ is then obtained from Proposition 3.

The proof of (ii) is a straightforward consequence of Lemma 5 below. □

The next lemma states that if $\alpha\theta \geqslant 1$, the only excursion measures compatible with (X, T_0) which satisfy (ii) in Lemma 2 are those associated with a jumping-in measure as in (ii) in Proposition 1.

Lemma 5. *Assume that* $\alpha\theta \geqslant 1$. *If there exists a normalized excursion measure* $\mathbf{m}$ *compatible with the minimal process such that conditions (ii) and (iii) in Lemma 2 are satisfied, then* $\mathbf{m}(X_{0+} = 0) = 0$.

Sketch of proof. We recall from the proof of Proposition 3 that if $\alpha\theta \geqslant 1$ then we have that

$$\liminf_{x \to 0} \frac{\mathbb{E}_x(1 - \mathrm{e}^{-T_0})}{x^\theta} = \infty,$$

and that

$$\lim_{x \to 0} \frac{V_q f(x)}{x^\theta}, \qquad q > 0,$$

exists in $\mathbb{R}$ for every function $f \in C_K]0, \infty[$. Therefore,

$$\lim_{x \to 0} \frac{V_q f(x)}{\mathbb{E}_x(1 - \mathrm{e}^{-T_0})} = 0,$$

for every function $f \in C_K]0, \infty[$. Then we may simply repeat the arguments in Vuolle-Apiala (1994, Lemma 1.1) to prove that, for $q > 0$,

$$\mathbf{m}\left(\int_0^{T_0} \mathrm{e}^{-qs} f(X_s) \mathrm{d}s \right) = b \int_0^\infty V_q f(x) x^{-(1+\beta)} \mathrm{d}x,$$

for some $\beta \in]0, 1/\alpha[$ and a constant $b \in]0, \infty[$. The result follows. $\square$

Corollary 1. *Assume* $0 < \alpha\theta < 1$.

(i) *The law of* T_0 *under* $\mathbf{n}$ *is*

$$\mathbf{n}(T_0 \in \mathrm{d}s) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)} s^{-(1+\alpha\theta)}\mathrm{d}s.$$

(ii) *Under* $\mathbf{n}$ *the law of the height of the excursion, say* $H := \sup_{0 \leqslant t \leqslant T_0} X_s$, *is given by*

$$\mathbf{n}(H > z) = p_{\alpha,\theta} z^{-\theta}, \qquad z > 0,$$

with $p_{\alpha,\theta} = p(\alpha\theta \mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta))^{-1}$ *and* $p \in]0, 1]$ *a constant that depends on the law of* ξ.

Proof. The result in (i) follows from the fact that the subordinator σ which is the inverse local time of $\tilde{X}$ is a stable subordinator of parameter $\alpha\theta$; cf. Lemma 2.

The main ingredient in the proof of (ii) is that the tail distribution of the random variable $S_\zeta = \sup_{0<r<\zeta}\xi_r$ is such that

$$\lim_{s\to\infty} \mathrm{e}^{\theta s}\mathbf{P}(S_\zeta > s) = p/m^{\natural}\theta,$$

for a constant $p \in]0, 1]$. This result was obtained by Bertoin and Doney (1994) in the case $\mathbf{k} = 0$, but in fact their proof extends easily to the case $\mathbf{k} > 0$. We deduce from this a tail estimate for the behaviour of the supremum of the minimal process (X, T_0) as the initial point tends to 0. More precisely, defining $S^X_\infty := \sup_{0 \leqslant r < T_0} X_r$,

$$\lim_{x\to 0} x^{-\theta}\mathbb{P}_x(S^X_\infty > z) = z^{-\theta}(p/m^{\natural}\theta), \qquad z > 0.$$

Let $H_t = \sup_{t \leqslant s < T_0} X_s$, $t > 0$. We have that, for any $z > 0$,

$$\lim_{t\to 0+} \mathbf{n}(H_t > z, t < T_0) = \mathbf{n}(H > z),$$

and that for any $\epsilon, \delta > 0$, there exists a $t_0 > 0$ such that

$$\mathbf{n}(X_t \in (\epsilon, \infty), t < T_0) \leqslant \delta, \qquad \forall t < t_0.$$

Therefore,

$$\mathbf{n}(X_t \in]0, \epsilon[, H_t > z, t < T_0) \leqslant \mathbf{n}(H_t > z, t < T_0) \leqslant \delta + \mathbf{n}(X_t \in]0, \epsilon[, H_t > z, t < T_0),$$

and by the Markov property under $\mathbf{n}$, we obtain that

$$\begin{aligned}\mathbf{n}(X_t \in]0, \epsilon[, H_t > z, t < T_0) &= (a_{\alpha,\theta})^{-1}\mathbb{E}^{\natural}_{0+}(X_t \in]0, \epsilon[, X_t^{-\theta}\mathbb{E}_{X_t}(S^X_\infty > z)) \\ &\sim p_{\alpha,\theta} z^{-\theta}\mathbb{E}^{\natural}_{0+}(X_t \in]0, \epsilon[) \\ &\sim p_{\alpha,\theta} z^{-\theta},\end{aligned}$$

for t small enough. Thus,

$$p_{\alpha,\theta}z^{-\theta} \leqslant \mathbf{n}(H > z) \leqslant \delta + p_{\alpha,\theta}z^{-\theta},$$

and the result follows by letting $\delta \to 0$. □

If $0 < \alpha\theta < 1$, it was shown by Vuolle-Apiala that given an excursion measure, the extension $\tilde{X}$ associated with this excursion measure leaves 0 either continuously or by jumps. This fact is natural when we observe that the excursions that leave 0 continuously have different duration than those leaving 0 by jumps. Indeed, the duration of the former has distribution

$$\mathbf{n}(T_0 > t) = t^{-\alpha\theta}(\Gamma(1-\alpha\theta))^{-1},$$

and for the latter

$$n^j(T_0 > t) = t^{-\alpha\beta}(\Gamma(1-\alpha\beta))^{-1}, \qquad 0 < \beta < \theta.$$

In the case where the Lévy process ξ is a Brownian motion with negative drift, the criterion in Theorem 2 coincides with the classification from Feller's diffusion theory for 0 to be a regular or an exit boundary point, as is explained in Example 2 below. By analogy, we can say that 0 is a regular boundary point for $\tilde{X}$ if $0 < \alpha\theta < 1$ and an exit boundary point if $1 \leqslant \alpha\theta$. Even in the case $\alpha\theta < 0$, which is not considered in this paper, it is easy to see that if ξ is a Lévy process with infinite lifetime and such that $\theta < 0$ in Cramér's condition then the Lévy process ξ drifts to ∞. The only way to extend a self-similar Markov process X associated with a Lévy process that drifts to ∞ is by making 0 an entrance boundary point. This possibility is considered by Bertoin and Caballero (2002), Bertoin and Yor (2002a; 2002b) and Caballero and Chaumont (2004).

4. Excursions conditioned by their durations

It is well known that the excursion measure for the Brownian motion can be described using the law of the excursion process conditioned to return to 0 at time 1, that is, the law of a Bessel(3) bridge of length 1 (McKean 1963; Revuz and Yor 1999, Section XII.4). In this section we follow this idea to describe the law under the excursion measure $\mathbf{n}$ defined in Theorem 1 of the excursion process conditioned to return to zero at a given time. We then give an alternative description of the excursion measure $\mathbf{n}$.

4.1. The case $\mathbf{k} = 0$

To deal with this case, we will make the additional hypothesis:

(H2d) $\mathbf{E}(\xi_1) > -\infty$ and the distribution of the Lévy exponential functional I has a continuous density on $[0, \infty[$, say ρ, with respect to Lebesgue measure.

The condition that the law of the exponential functional I has a continuous density is satisfied by a wide variety of Lévy processes (Carmona *et al*. 1997, Proposition 2.1). We next introduce another self-similar process. Denote by $\hat{\xi} = (-\xi_s, s > 0)$ the dual Lévy process, and by $\hat{\mathbf{P}}$ and $\hat{\mathbf{E}}$ its probability and expectation. Then define $(\hat{\mathbb{P}}_x, x > 0)$ to be the distribution on $\mathbb{D}^+$ of the α-self-similar process associated with the Lévy process with law $\hat{\mathbf{P}}$. The process $\hat{X}$ is usually called the dual α-self-similar process; the term 'dual' is justified by the relation

$$\int_0^\infty g(x)V_q f(x)x^{(1-\alpha)/\alpha}\mathrm{d}x = \int_0^\infty f(x)\hat{V}_q g(x)x^{(1-\alpha)/\alpha}\mathrm{d}x, \tag{14}$$

for every $f, g :]0, \infty[\to \mathbb{R}^+$ measurable (Bertoin and Yor 2002a, Lemma 2). By (H2d) we have that $0 < m := |\psi'(0^+)| = \hat{\mathbf{E}}(\xi_1) < \infty$. Let $\hat{\mathbb{P}}_{0^+}$ be the limit in the sense of finite-dimensional marginals of $\hat{\mathbb{P}}_x$ as $x \to 0$, whose existence is ensured by Bertoin and Yor (2002a, Theorem 1). The latter theorem also establishes that for every $t > 0$ and for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable, we have

$$\hat{\mathbb{E}}_{0^+}(f(X_t)) = \frac{\alpha}{m}\mathbf{E}(f((t/I)^\alpha)/I), \tag{15}$$

where I is defined in (2). Hypothesis (H2d) implies that for any $t > 0$ the law of X_t under $\hat{\mathbb{P}}_{0^+}$ has a density with respect to the measure $\upsilon(\mathrm{d}y) = y^{(1-\alpha)/\alpha}\mathrm{d}y$, $y > 0$, given by the formula

$$\frac{\hat{\mathbb{P}}_{0^+}(X_t \in \mathrm{d}y)}{\upsilon(\mathrm{d}y)} = m^{-1}y^{-1/\alpha}\rho(ty^{-1/\alpha}) := \hat{p}_t(y), \qquad y > 0.$$

Let $(\mu_s(\mathrm{d}y) = \hat{\mathbb{P}}_{0^+}(X_s \in \mathrm{d}y), s > 0)$. A consequence of the duality relation (14) is that the relation $\mu_s\hat{P}_{t-s} = \mu_t$ for $s < t$ can be shifted to the semigroup of the minimal process P_t as $\hat{p}_t = P_s\hat{p}_{t-s}$ υ-a.s. It was proved in Rivero (2003, Section 4) that these densities can be used to construct a regular version of the family of probability measures $(\mathbb{P}_x(\cdot|T_0 = r), r > 0)$ when the underlying Lévy process is a subordinator. Moreover, the same argument applies to any Lévy process assuming only (H2d). Here the densities $(\hat{p}_t, t \geqslant 0)$ will be used to construct a bridge for the coordinate process under $\mathbb{E}^\natural_{0^+}$; the techniques here used are reminiscent of those in Fitzsimmons *et al*. (1993).

Recall that the semigroup $(P^\natural_t, t \geqslant 0)$ is the h-transformation of the semigroup $(P_t, t \geqslant 0)$ via the invariant function $h(x) = x^\theta$, $x > 0$. Using the fact that for every $t > s > 0$, the equality $\hat{p}_t = P_s\hat{p}_{t-s}$ υ-a.s. holds, we obtain that, for $r > 0$ arbitrary, the function

$$h^{\natural r}(s, x) = \hat{p}_{r-s}(x)x^{-\theta}1_{\{s<r\}}, \qquad x > 0, s > 0,$$

is excessive for the semigroup $(\pi_t \otimes P^\natural_t, t \geqslant 0)$ of the space-time process. Let $\overline{\Lambda}^r$ be the h-transform of the measure $\mathbb{E}^\natural_{0^+}$ by means of the space–time excessive function $h^{\natural r}(s, x)$. Then under $\overline{\Lambda}^r$ the space process $(X_t, t > 0)$ is an inhomogeneous Markov process with entrance law

$$\overline{\Lambda}^r_s f = \mathbb{E}^\natural_{0^+}(f(X_s)\hat{p}_{r-s}(X_s)X_s^{-\theta}), \qquad 0 < s < r,$$

for $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable, and inhomogeneous semigroup

$$K^r_{t,t+s}(x, \mathrm{d}y) = \frac{P^\natural_s(x, \mathrm{d}y)h^{\natural r}(t+s, y)}{h^{\natural r}(t, x)} = \frac{P_s(x, \mathrm{d}y)\hat{p}_{r-(t+s)}(y)}{\hat{p}_{r-t}(x)}, \qquad y > 0;\ t,\ t+s < r.$$

Observe that the inhomogeneous semigroup $K^r_{t,t+s}$ is that of X conditioned to die at 0 at time r (Rivero 2003, Lemma 7). Moreover, using the fact that $\overline{\Lambda}^r$ is an h-transform of the measure $\mathbb{E}^{\natural}_{0+}$ it is easily verified that the measure $\overline{\Lambda}^r$ has the property

$$\overline{\Lambda}^r(F(X_s, 0 \leqslant s < r)) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(F(r^{\alpha}X_s, 0 \leqslant s < 1)),$$

for every positive measurable F. In particular, the total mass of $\overline{\Lambda}^r$ is determined by

$$b_r := \overline{\Lambda}^r(1) = r^{-(1+\alpha\theta)}\overline{\Lambda}^1(1),$$

and it will be shown below that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}m} < \infty. \tag{16}$$

Therefore, assuming hypotheses (H2a)–(H2d) and $\overline{\Lambda}^1(1) < \infty$, we can define a probability measure on $\mathcal{G}_\infty$ by $\Lambda^r = b_r^{-1}\overline{\Lambda}^r$. The distribution under Λ^r of the lifetime T_0 is the Dirac distribution at r, that is, $\Lambda^r(T_0 = r) = 1$ (Rivero 2003, Lemma 7). We can now state the main result of this section.

Proposition 5 (Itô's description of the measure n). *Assume hypotheses* (H2a)–(H2d) *hold and* $0 < \alpha\theta < 1$. *Then* $\overline{\Lambda}^1(1) < \infty$. *Let* $\mathbf{n}$ *be the unique normalized excursion measure such that* $\mathbf{n}(X_{0+} > 0) = 0$. *For* $F \in \mathcal{G}_\infty$,

$$\mathbf{n}(F) = \frac{\alpha\theta}{\Gamma(1-\alpha\theta)}\int_0^\infty \Lambda^r(F \cap \{T_0 = r\})\frac{\mathrm{d}r}{r^{1+\alpha\theta}}.$$

The proof of this proposition is similar to that given in Revuz and Yor (1999, Theorem XII.4.2) for the analogous result for Brownian excursion measure.

Proof. We first show that

$$\mathbf{n}(F) = \frac{m}{a_{\alpha,\theta}}\int_0^\infty \overline{\Lambda}^r(F \cap \{T_0 = r\})\mathrm{d}r, \tag{17}$$

with $a_{\alpha,\theta}$ as defined in Theorem 3. We will deduce from this that

$$\overline{\Lambda}^1(1) = \frac{\alpha^2\theta\mathbf{E}^{\natural}(J^{-(1-\alpha\theta)})}{m^{\natural}m}.$$

Indeed, by the monotone class theorem it is enough to prove the assertion for sets F of the form

$$F = \bigcap_{i=1}^n \{X(t_i) \in B_i\},$$

with $0 < t_1 < t_2 <\ldots< t_n$ and Borel sets $B_i \subset\,]0, \infty[$, $i \in \{1, \ldots, n\}$. On the one hand, according to Theorem 1, we have

$$\mathbf{n}(F) = \int_{B_1} \mathbf{n}_{t_1}(\mathrm{d}x_1) \int_{B_2} P_{t_2 - t_1}(x_1, \mathrm{d}x_2) \cdots \int_{B_n} P_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n).$$

On the other hand, using that $F \cap \{T_0 < t_n\} = \varnothing$, we have that the right-hand term in (17) can be written as

$$\frac{m}{a_{\alpha,\theta}} \int_{t_n}^{\infty} \mathrm{d}r \int_{B_1} \overline{\Lambda}^r_{t_1}(\mathrm{d}x_1) \int_{B_2} K_{t_1,t_2}(x_1, \mathrm{d}x_2) \cdots \int_{B_n} K_{t_{n-1},t_n}(x_{n-1}, \mathrm{d}x_n). \tag{18}$$

Recall from Theorem 1 that

$$\overline{\Lambda}^r_{t_1}(\mathrm{d}x_1) = \mathbb{P}^{\natural}_{0^+}(X_{t_1} \in \mathrm{d}x_1) \hat{p}_{r-t_1}(x_1) x_1^{-\theta} = a_{\alpha,\theta} \mathbf{n}_{t_1}(\mathrm{d}x_1) \hat{p}_{r-t_1}(x_1).$$

Using this identity and the expression of the transition probabilities $K_{t_i,t_{i+1}}$ we obtain that (18) is equal to

$$m \int_{t_n}^{\infty} \mathrm{d}r \int_{B_1} \mathbf{n}_{t_1}(\mathrm{d}x_1) \int_{B_2} P_{t_2-t_1}(x_1, \mathrm{d}x_2) \cdots \int_{B_n} P_{t_n - t_{n-1}}(x_{n-1}, \mathrm{d}x_n) \hat{p}_{r-t_n}(x_n).$$

Finally, using

$$m \int_s^{\infty} \hat{p}_{r-s}(x) \mathrm{d}r = \int_s^{\infty} \rho((r-s)x^{-1/\alpha}) \frac{\mathrm{d}r}{x^{1/\alpha}} = 1,$$

for all $x > 0$, we conclude that both expressions in (17) for $\mathbf{n}(F)$ coincide. In particular, if $F = 1 - \mathrm{e}^{-T_0}$ we have that

$$1 = \mathbf{n}(1 - \mathrm{e}^{-T_0}) = \frac{m}{a_{\alpha,\theta}} \int_0^{\infty} \overline{\Lambda}^r(1)(1 - \mathrm{e}^{-r}) \mathrm{d}r = \frac{\overline{\Lambda}^1(1) m}{a_{\alpha,\theta}} \left(\frac{\Gamma(1 - \alpha\theta)}{\alpha\theta} \right).$$

The value of $\overline{\Lambda}^1(1)$ in (16) is obtained by using the expression for $a_{\alpha,\theta}$ and we derive from (17) that

$$\mathbf{n}(F) = \frac{m \overline{\Lambda}^1(1)}{a_{\alpha,\theta}} \int_0^{\infty} \Lambda^r(F \cap \{T_0 = r\}) \frac{\mathrm{d}r}{r^{1+\alpha\theta}},$$

and the result follows. □

Remark. A result analogous to that in Proposition 5 can be obtained for the excursion measure n^j obtained via the jumping-in measure $\eta(\mathrm{d}x) = b_{\alpha,\beta} x^{-(1+\beta)} \mathrm{d}x$. The method is similar and we leave the details to the interested reader.

4.2. The case $\mathbf{k} > 0$

In this setting we have noted that the random variable I has the same law as $A_{\mathbb{e}} = \int_0^{\mathbb{e}} \exp\{\check{\xi}_s/\alpha\} \mathrm{d}s$, with $\check{\xi}$ the unkilled Lévy process and $\mathbb{e}$ an exponential random variable of parameter $\mathbf{k}$ and independent of $\check{\xi}$. Then it is easy to prove that, under our assumptions, the law of the random variable $A_{\mathbb{e}}$ has a density ρ with respect to Lebesgue measure (cf. Carmona *et al.* 1994, Proposition 2.3). More precisely,

$$\check{\mathbf{P}}(A_{\mathrm{e}} \in \mathrm{d}t) = \rho(t)\mathrm{d}t = \mathbf{k}\mathbb{E}_1^{\natural}(X_t^{-1/\alpha-\theta})\mathrm{d}t, \qquad t > 0.$$

Furthermore, the Markov property implies that, for any $r > 0$, the function

$$h^{\natural r}(s, x) := \mathbb{E}_x^{\natural}(X_{r-s}^{-1/\alpha-\theta})1_{\{s<r\}}, \qquad x > 0, s \geqslant 0,$$

is excessive for the space-time Markov process $((t, X_t^{\natural}), t \geqslant 0)$. Therefore we can simply repeat the arguments in the previous subsection to construct the law Λ^r of the excursion process conditioned to have a length r and obtain a description of the excursion measure $\mathbf{n}$ similar to that given in Proposition 5.

It was proved by Chaumont (1997, Theorem 3) that in the case where X is a stable process with negative jumps killed at its first entrance into $]-\infty, 0]$, the law of the excursion process conditioned to have a given length is absolutely continuous with respect to the law of the stable meander process. An analogous result still holds in our setting. To give a precise statement, we next recall the definition of the law of the meander process. For any $r > 0$, the probability measure M^r defined over $\mathbb{D}^+([0, r])$ by

$$M^r(\cdot) := \mathbf{n}(\cdot \circ k_r, T_0 > r)/\mathbf{n}(T_0 > r),$$

with k_r the killing operator at time $r > 0$, is called the law of the meander process. This corresponds to the law of the process $(\tilde{X}_{g_t+s}, 0 \leqslant s \leqslant t - g_t)$ conditioned by $t - g_t = r$ for some $t > r$ and g_t the last hitting time of 0 before t, $g_t = \sup\{s \leqslant t\colon \tilde{X}_s = 0\}$ (Getoor 1979).

We can now state the following corollary which is the analogue of (Chaumont 1997, Theorem 3).

Corollary 2. *For any $r > 0$, $t < r$ and $F \in \mathcal{G}_t$, we have that*

$$\Lambda^r(F) = \frac{r\mathbf{k}}{\alpha\theta} M^r(F, X_r^{-1/\alpha}).$$

Proof. On the one hand, by the very definition of the law of the meander and Theorem 1, we have that

$$M^r(F) = \frac{r^{\alpha\theta}\Gamma(1-\alpha\theta)}{a_{\alpha,\theta}} \mathbb{E}_{0+}^{\natural}(F, X_r^{-\theta}).$$

On the other hand, by the construction of Λ^r in Proposition 4.1 and the Markov property, we have that

$$\Lambda^r(F) = (b_r)^{-1}\mathbb{E}_{0+}^{\natural}(F, h^{\natural r}(t, Y_t)) = (b_r)^{-1}\mathbb{E}_{0+}^{\natural}(F, X_r^{-(1/\alpha)-\theta}).$$

Finally, in this case the normalizing constant b_r is given by $b_r = r^{-1-\alpha\theta}b_1$ with $b_1 := \alpha^2\theta\mathbb{E}^{\natural}(J^{-(1-\alpha\theta)})/m^{\natural}\mathbf{k}$. The result follows by identifying the constants. □

5. Duality

In this section we will construct a self-similar Markov process which is in weak duality with the process $\tilde{X}$ and whose excursion measure is the image under time reversal of $\mathbf{n}$. To this end, we first introduce some notation.

Let $\xi^\natural$ be a Lévy process with law $\mathbf{P}^\natural$ and $\widehat{\xi^\natural}$ its dual, that is, $\widehat{\xi^\natural} = -\xi^\natural$. Denote by $\widehat{\mathbf{P}^\natural}$ and $\widehat{\mathbf{E}^\natural}$ the probability and expectation for $\hat{\xi}^\natural$. The process $\hat{\xi}^\natural$ drifts to $-\infty$ since $\xi^\natural$ drifts to ∞. Let $(\hat{\mathbb{P}}^\natural_x, x \geqslant 0)$ be the law on $\mathbb{D}^+$ of the α-self-similar process $\hat{X}^\natural = (\hat{X}^\natural_t, t \geqslant 0)$ associated by Lamperti's transformation with the Lévy process with law $\widehat{\mathbf{P}^\natural}$. The process $\hat{X}^\natural$ has a lifetime $\hat{T}_0 = \inf\{t > 0: \hat{X}^\natural_t = 0\}$ which is finite $\hat{\mathbb{P}}^\natural_x$-a.s. for all $x \geqslant 0$. Denote by $(\hat{P}^\natural_t, t \geqslant 0)$ and $(\hat{V}^\natural_q, q > 0)$ the semigroup and resolvent of the minimal process for $\hat{X}^\natural$:

$$\hat{P}^\natural_t f(x) = \hat{\mathbb{P}}^\natural_x(f(X_t), t < T_0), \qquad t \geqslant 0,$$

and

$$\hat{V}^\natural_q f(x) = \int e^{-qt} \hat{P}^\natural_t f(x) dt, \qquad q > 0.$$

By the duality relation (14), the resolvents $V^\natural_q$ and $\hat{V}^\natural_q$ are in weak duality with respect to the measure $v(dx) = x^{(1-\alpha)/\alpha} dx$, $x > 0$. Furthermore, it follows that the resolvents V_q and $\hat{V}^\natural_q$ are in weak duality with respect to the measure $Q_{\mathbf{n}}(dx) = x^{(1-\alpha-\alpha\theta)/\alpha} dx$, $x > 0$.

In the following lemma we construct a candidate for the process dual to $\tilde{X}$.

Lemma 6. *Assume hypotheses* (H2) *and suppose that* $0 < \alpha\theta < 1$.

(i) *Let* $\mathbf{k} = 0$. *Assume*
(H2e) $\mathbf{E}(\xi_1^-) < \infty$, *with* $a^- = (-a) \vee 0$.
Then the minimal process $(\hat{X}^\natural, \hat{T}_0)$ *admits a unique extension* $(\tilde{Z}_t, t \geqslant 0)$, *which leaves* 0 *continuously a.s. Its resolvent is given by*

$$\hat{U}_q f(0) = \frac{\hat{\gamma}_{\alpha,\theta}}{q^{\alpha\theta}} \int_0^\infty f(y) \mathbf{E}(e^{-qy^{1/\alpha} I}) Q_{\mathbf{n}}(dy), \qquad \hat{U}_q f(x) = \hat{V}^\natural_q f(x) + \mathbb{E}^\natural_x(e^{-qT_0}) \hat{U}_q f(0),$$

for $x > 0$, *with* $\hat{\gamma}_{\alpha,\theta} = (\alpha \mathbf{E}(I^{-(1-\alpha\theta)}) \Gamma(1-\alpha\theta)/m)^{-1}$.

(ii) *Let* $\mathbf{k} > 0$. *The process* $(\hat{X}^\natural, \hat{T}_0)$ *admits a self-similar recurrent extension* $Z_\theta = (Z_{\theta,t}, t \geqslant 0)$ *which leaves* 0 *by a jump according to the jumping-in measure*

$$\eta_\theta(dx) = b_{\alpha,\theta} x^{-(1+\theta)} dx, \qquad x > 0,$$

with $b_{\alpha,\theta} = \theta / \Gamma(1-\alpha\theta) \mathbf{E}^\natural(J^{\alpha\theta})$. *The resolvent of* Z_θ *is given by*

$$\mathcal{U}_q f(0) = b_{\alpha,\theta} q^{-\alpha\theta} \int_0^\infty y^{-(1+\theta)} \hat{V}^\natural_q f(y) dy, \qquad \mathcal{U}_q f(x) = \hat{V}^\natural_q f(x) + \widehat{\mathbb{E}}^\natural_x(e^{-qT_0}) \mathcal{U}_q f(0),$$

for $x > 0$.

Proof. (i) According to Theorem 1, all that we have to do is to verify that $\widehat{\xi^\natural}$ satisfies hypotheses (H2a)–(H2c). Indeed, that (H2b) holds follows from

$$\widehat{\mathbf{E}^{\natural}}(\mathrm{e}^{\theta\xi_1}) = \mathbf{E}^{\natural}(\mathrm{e}^{-\theta\xi_1}) = \mathbf{E}(\mathrm{e}^{-\theta\xi_1}\mathrm{e}^{\theta\xi_1}) = 1,$$

and it is verified in same way that (H2c) holds,

$$\widehat{\mathbf{E}}^{\natural}(\xi_1^+\mathrm{e}^{\theta\xi_1}) = \mathbf{E}^{\natural}((-\xi_1)^+\mathrm{e}^{-\theta\xi_1}) = \mathbf{E}(\xi_1^-) < \infty.$$

The results in Section 3 can be applied to the minimal process $(\hat{X}^{\natural}, \hat{T}_0)$ to ensure that there exists a unique normalized excursion measure $\hat{\mathbf{n}}$ compatible with the semigroup $(\hat{P}_t^{\natural}, t \geqslant 0)$. The entrance law associated with $\hat{\mathbf{n}}$ admits the representation

$$\hat{\mathbf{n}}_s f = (\hat{a}_{\alpha,\theta})^{-1}\hat{\mathbb{E}}_{0^+}(f(X_s)X_s^{-\theta}), \qquad s > 0,$$

where $\hat{a}_{\alpha,\theta} = \alpha\mathbf{E}(I^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta)/m$, for f continuous and bounded. To see this it should be verified that the measure $\widehat{\mathbf{P}^{\natural\natural}}$ obtained by h-transformation of the law $\widehat{\mathbf{P}^{\natural}}$ by means of the function $h(x) = \mathrm{e}^{\theta x}$ is $\hat{P}$. To this end, it suffices to prove that both probability measures have the same one-dimensional marginals. Indeed,

$$\widehat{\mathbf{P}^{\natural\natural}}(f(\xi_s)) = \widehat{\mathbf{P}^{\natural}}(f(\xi_s)\mathrm{e}^{\theta\xi_s}) = \mathbf{P}^{\natural}(f(-\xi_s)\mathrm{e}^{-\theta\xi_s}) = \mathbf{P}(f(-\xi_s)) = \hat{\mathbf{P}}(f(\xi_s)),$$

for every bounded continuous f. Then the α-self-similar Markov process associated with the Lévy process with law $\widehat{\mathbf{P}^{\natural\natural}}$ is equivalent to that associated to the Lévy process with law $\hat{\mathbf{P}}$. Note that the law of J under $\widehat{\mathbf{P}^{\natural\natural}}$ is the same as that of I under $\mathbf{P}$.

(ii) According to Proposition 1, all that we have to verify in order to prove the claimed result is that $\widehat{\mathbf{E}^{\natural}}(I^{\alpha\theta}) < \infty$. Indeed, owing to (H2c) we have that $-\widehat{\mathbf{E}^{\natural}}(\xi_1) = m^{\natural} \in]0, \infty[$ and by the identity (2.7) that $\widehat{\mathbf{E}^{\natural}}(I^{-1}) = m^{\natural}/\alpha < \infty$ (observe that I under $\widehat{\mathbf{P}^{\natural}}$ is equal to J under $\mathbf{P}^{\natural}$). Therefore, we have that $\widehat{\mathbf{E}^{\natural}}(I^{\alpha\theta-1}) < \infty$. The claim follows using the identity

$$\widehat{\mathbf{E}^{\natural}}(I^{\alpha\beta}) = \frac{\alpha\beta}{-\psi^{\natural}(\beta)}\widehat{\mathbf{E}^{\natural}}(I^{\alpha\beta-1}), \qquad \text{for } 0 < \beta \leqslant \theta, \tag{19}$$

with $\psi^{\natural}: [0, \theta] \to \mathbb{R}$ defined by

$$\widehat{\mathbf{E}^{\natural}}(\mathrm{e}^{\lambda\xi_1}) = \mathrm{e}^{\psi^{\natural}(\lambda)}, \qquad 0 \leqslant \lambda \leqslant \theta.$$

The identity (19) is analogous to that in (7) and is proved as in Bertoin and Yor (2002b, Proposition 2). Note that $\psi^{\natural}(\lambda) = \psi(\theta - \lambda)$, for every $0 \leqslant \lambda \leqslant \theta$. □

Because of the weak duality relation between the resolvents V_q and $\hat{V}_q^{\natural}$ it is natural to ask if this property is inherited by the resolvents U_q and $\hat{U}_q$ (or $\mathcal{U}_q$). That is the content of the following result.

We assume throughout this section that the hypothesis (H2) are satisfied and, if $\mathbf{k} = 0$, that $\mathbf{E}(\xi_1^-) < \infty$.

Lemma 7. *If* $\mathbf{k} = 0$*, for any* $q > 0$ *the resolvents* U_q *and* $\hat{U}_q$ *are in weak duality with respect to the measure* $Q_{\mathbf{n}}(\mathrm{d}x) = x^{(1-\alpha-\alpha\theta)/\alpha}\mathrm{d}x$, $x > 0$. *If* $\mathbf{k} > 0$*, the same result holds true with* $\mathcal{U}_q$ *instead of* $\hat{U}_q$.

Proof. We first treat the case $\mathbf{k} = 0$. Using the expression for the resolvents U_q and $\hat{U}_q$ of $\tilde{X}$ and $\tilde{Z}$ respectively, obtained in Theorem 2 and Lemma 6 (i) respectively, it is straightforward that, for any $f,\ g : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$\begin{aligned}
\int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)g(y)U_q f(y) &= \int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)g(y)V_q f(y) + U_q f(0)\int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)g(y)\mathbb{E}_y(\mathrm{e}^{-qT_0}) \\
&= \int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)f(y)\hat{V}^\natural_q g(y) + U_q f(0)\int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)g(y)\mathbf{E}(\mathrm{e}^{-qy^{1/\alpha}I}) \\
&= \int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)f(y)\hat{V}^\natural_q g(y) \\
&\quad + \frac{\hat{a}_{\alpha,\theta}m}{a_{\alpha,\theta}m^\natural}\hat{U}_q g(0)\int_0^\infty Q_{\mathbf{n}}(\mathrm{d}x)f(x)\mathbf{E}^\natural(\mathrm{e}^{-qx^{1/\alpha}J}) \\
&= \int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)f(y)\hat{U}_q g(y),
\end{aligned}$$

where the last equality follows from the fact that the constants $\gamma_{\alpha,\theta}$ and $\hat{\gamma}_{\alpha,\theta}$ are equal. To see this, recall that $\mathbf{E}(I^{-(1-\alpha\theta)}) = \mathbf{E}^\natural(J^{-(1-\alpha\theta)})$, as remarked after Proposition 3.

The case $\mathbf{k} > 0$ follows the same lines but uses the following identity. For every $q > 0$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable,

$$b_{\alpha,\theta}\int_0^\infty y^{-(1+\theta)}\hat{V}^\natural_q f(y)\mathrm{d}y = C_{\alpha,\alpha\theta}\int_0^\infty Q_{\mathbf{n}}(\mathrm{d}y)f(y)\mathbb{E}_y(\mathrm{e}^{-qT_0})\mathrm{d}y,$$

with $C_{\alpha,\alpha\theta} := (\alpha\mathbf{E}^\natural(J^{-(1-\alpha\theta)})\Gamma(1-\alpha\theta))^{-1} = b_{\alpha,\theta}/\mathbf{k}$, and $b_{\alpha,\theta}$ as in Lemma 6. The preceding identity is an easy consequence of the fact that the random variable $A_{\mathbf{e}}$ has density $\rho(t) = \mathbf{k}\mathbb{E}^\natural_1(Y_t^{-(1/\alpha)-\theta})$ for $t > 0$ and that under $\mathbb{P}_y$ the random variable T_0 has the same law as $y^{1/\alpha}A_{\mathbf{e}}$. $\square$

Some results on time reversal can be derived from the preceding facts. To give a precise statement we introduce some notation. Let ϱ denote the operator of time reversal at time T_0, that is

$$(\varrho X(\omega))(t) = \begin{cases} X_{(T_0-t)^-}(\omega), & \text{if } 0 \leqslant t < T_0 < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\varrho\mathbf{n}$ denote the image under time reversal at time T_0 of $\mathbf{n}$. Recall that L is a return time if

$$L \circ \theta_t = (L-t)^+, \qquad \text{a.s. for all } t \geqslant 0.$$

The first part of the following result is an extension for self-similar processes of the celebrated result on time reversal of Williams (1974): a three-dimensional Bessel process starting from 0 and reversed at its last exit time from $x > 0$ is identical in law to a Brownian motion killed at its first hitting time of 0. In the second part we determine $\varrho\mathbf{n}$.

Proposition 6. (i) *If L is a finite return time then under $\mathbb{E}^{\natural}_{0+}$ the reversed process $(X_{(L-t)-},\ 0 \leqslant t < L)$ is Markovian and has semigroup $(\hat{P}^{\natural}_t,\ t \geqslant 0)$.*

(ii) $\mathbf{k} = 0$, *we have that $\varrho\mathbf{n} = \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ the Itô excursion measure of $\tilde{Z}$.*

(iii) *If $\mathbf{k} > 0$, we have that*

$$\varrho\mathbf{n}(\cdot) = b_{\alpha,\theta}\int_0^\infty \mathrm{d}x\, x^{-(1+\theta)}\hat{\mathbb{P}}^{\natural}_x(\cdot).$$

In particular, $\mathbf{n}(X_{T_0-} \in \mathrm{d}x) = b_{\alpha,\theta}x^{-(1+\theta)}\mathrm{d}x$, $x > 0$, *and* $\varrho\mathbf{n}(\cdot|X_{T_0-} = x) = \hat{\mathbb{P}}^{\natural}_x(\cdot)$.

Proof. (i) The potential of the measure $\mathbb{E}^{\natural}_{0+}$ is determined by

$$\begin{aligned}\mathbb{E}^{\natural}_{0+}\left(\int_0^\infty dsf(X_s)\right) &= a_{\alpha,\theta}\int_0^\infty \mathrm{d}s\, \mathbf{n}_s(fh^*)\\ &= a_{\alpha,\theta}\int f(y)y^{(1-\alpha)/\alpha}\mathrm{d}y,\end{aligned}$$

with the notation of Sections 2.3 and 3. Because of the weak duality between the resolvents $V^{\natural}_\lambda$ and $\hat{V}^{\natural}_\lambda$ with respect to the measure $y^{(1-\alpha)/\alpha}\mathrm{d}y$, $y > 0$, the statement in (i) is a direct consequence of a result of Nagasawa (1964) on time reversal. A general version of Nagasawa's result can be found in Dellacherie *et al*. (1992, Section XVIII.46).

(ii) Since $\mathbf{n}(X_{0+} > 0, X_{T_0-} > 0) = 0$, the excursion of $\tilde{X}$ from 0 starts and ends at 0, (Getoor and Sharpe 1982, Section 9). This and the weak duality in Lemma 7 enable us to use a result due to Mitro (1984, Section 4) to deduce that $\varrho\mathbf{n} = \hat{\mathbf{n}}$.

(iii) We first note that an application of Lemma 3 proves that the entrance laws

$$(\mathbf{n}_s(\mathrm{d}y),\ s > 0) \quad \text{and} \quad \left(N^\theta_s f = b_{\alpha,\theta}\int_0^\infty \mathrm{d}x\, x^{-(1+\theta)}\hat{P}^{\natural}_s f(x),\ s > 0\right),$$

for the semigroups $(P_t,\ t \geqslant 0)$ and $(\hat{P}^{\natural}_s,\ s \geqslant 0)$ respectively, have the same potential

$$\int_0^\infty \mathrm{d}s\, \mathbf{n}_s f = C_{\alpha,\alpha\theta}\int_0^\infty f(x)x^{1/\alpha-1-\theta}\mathrm{d}x = \int_0^\infty \mathrm{d}s N^\theta_s f,$$

with $C_{\alpha,\alpha\theta} = (m^{\natural}a_{\alpha,\theta})^{-1}$. This enables us to use a result on time reversal of Kusnetzov measures established in Dellacherie *et al*. 1992 (Section XIX.33) to verify the claimed result. □

Remark. A consequence of Lemma 6 and Getoor and Sharpe (1981, Theorem 4.8) is that the process obtained by time-reversing one by one the excursions of $\tilde{X}$ starting at 0 has the same law as $\tilde{Z}$ (Z_θ) started at 0. Furthermore, it follows from Proposition 6 that the process $\tilde{Z}$ (Z_θ) has the same law as that constructed using Ito's programme and the Poisson point process $\varrho\Delta = (\varrho\Delta_s,\ s \geqslant 0)$ which is the image under ϱ of the Poisson point process of excursions of $\tilde{X}$, $\Delta = (\Delta_s,\ s \geqslant 0)$.

6. Examples

Example 2 *Self-similar diffusions*. Here we consider the case where the Lévy process is a Brownian motion with negative drift. Let $(\xi_t = \varepsilon B_t - \mu t,\ t \geqslant 0)$ with $(B_t,\ t \geqslant 0)$ a Brownian motion and $\varepsilon, \mu > 0$. Hypotheses (H2) are satisfied with $\theta = 2\mu/\varepsilon^2$ and under $\mathbf{P}^\natural$ the law of $\xi^\natural$ is that of $\varepsilon B_t + \mu t$. Then the α-self-similar Markov process X associated with ξ has continuous paths and has an infinitesimal generator of the form

$$Lf(x) = (\varepsilon^2/2 - \mu)x^{1-1/\alpha} f'(x) + \varepsilon^2/2x^{2-1/\alpha} f''(x), \qquad x > 0.$$

Then for $\alpha > 0$ we have that $0 < \alpha\theta < 1$ if and only if $0 < \mu < \varepsilon^2/2\alpha$. This corresponds to the case when the point 0 is a regular boundary point for the self-similar diffusion associated with the infinitesimal generator L just described; in the case $1 \leqslant \alpha\theta$, or equivalently $\varepsilon^2/2\alpha \leqslant \mu$, 0 is an exit boundary point. See Lamperti (1972, Theorem 5.1) and Vuolle-Apiala (1994, Theorem 3.1) for a related discussion. If $0 < \mu < \varepsilon^2/2\alpha$ holds, the process X admits a unique extension that is continuous and is characterized by Theorem 2. Furthermore, using the fact that the law of J under $\mathbf{E}^\natural$ is that of $2\alpha^2/(\varepsilon^2 Z_{\alpha\theta})$, with $Z_{\alpha\theta}$ a random variable of law gamma with parameter $\alpha\theta$, (Dufresne 1990), we deduce that the entrance law in Theorem 1 has a density

$$\frac{\mathbf{n}_s(\mathrm{d}y)}{\mathrm{d}y} = c_{\alpha\theta} s^{-2(1-\alpha\theta)-1} y^{2(1-\alpha\theta)/\alpha-1} \exp\left(-y^{1/\alpha} s^{-1} d_{\varepsilon,\alpha}\right), \qquad y > 0,$$

with respect to Lebesgue measure, with

$$c_{\alpha\theta} = \frac{(1-\alpha\theta)\alpha}{\Gamma(1-\alpha\theta)\mu^2}\left(\frac{\varepsilon^2}{2\alpha^2}\right)^{\alpha\theta} \quad \text{and} \quad d_{\varepsilon,\alpha} = \frac{2\alpha^2}{\varepsilon^2}.$$

Example 3 *Reflected stable processes*. Let Y be a stable process of parameter $a \in\ ]0, 2[$ and $(\underline{\mathbb{P}}_x,\ x \geqslant 0)$ its law. Assume that $|Y|$ is not a subordinator. Define $\rho = \underline{\mathbb{P}}(Y_1 > 0)$ and

$$X'_t = \begin{cases} Y_t - \inf_{0 \leqslant s \leqslant t} Y_s, & \text{if } t \geqslant T_{]-\infty,0]} \\ Y_t & \text{if } t < T_{]-\infty,0]}, \end{cases}$$

with $T_{]-\infty,0]}$ the first hitting time of $]-\infty, 0]$ by Y. Then $\rho \in\ ]0, 1[$ and 0 is a regular recurrent state for X' – we refer to Bertoin (1996, Section VIII) and Chaumont (1997) for background on stable processes and its excursion theory. We denote by (X, T_0) the process X' killed at $T_{]-\infty,0]}$; this process is $1/a$-self-similar. Let ξ be the Lévy process associated with (X, T_0) via Lamperti's transformation – see Caballero and Chaumont (2004) for a precise description of ξ. Observe that in the case where Y has negative jumps ξ is a Lévy process killed at an exponential time, while in the case where Y has no negative jumps ξ has infinite lifetime and drifts to $-\infty$. We claim that hypotheses (H2) are satisfied for $\theta = a(1-\rho)$. This can be verified either by doing the calculations using the results in Caballero and Chaumont (2004) or by the following arguments.

It is known that the function $h(x) = x^{a(1-\rho)}$, $x > 0$ is, up to a multiplicative constant, the only invariant function for the semigroup of the process (X, T_0). Then Cramér's condition

(H2b) for ξ is satisfied with $\theta = a(1-\rho)$. A consequence of this fact and Mejane (2002, Proposition 3.1) is that the Lévy exponential functional $I = \int_0^\infty \exp\{a\xi_s\}\mathrm{d}s$ has finite moments

$$\mathbf{E}(I^{\beta/a}) < \infty \qquad \text{for every } 0 < \beta < a(1-\rho).$$

The excursion measure for X' away from 0, say $\underline{\mathbf{n}}$, is an excursion measure compatible with the minimal process (X, T_0) such that its entrance law satisfies (iii) in Lemma 2 with $\gamma = 1-\rho$, and $\underline{\mathbf{n}}(X_{0+} > 0) = 0$ – see Chaumont (1997) and the reference therein. Thus $\mathbf{E}(I^{-\rho}) < \infty$, by Lemma 3. Therefore, it is easily verified by repeating the arguments in the proof of Proposition 4 that condition (H2c) is satisfied.

The excursion measure $\mathbf{n}$ defined in Theorem 1 is equal to $\underline{\mathbf{n}}$ and the recurrent extension $\hat{X}$ in Theorem 2 associated with $\mathbf{n}$ is equivalent to X'. Finally, it can be shown that the process dual of $\hat{X}$ constructed in Section 5 has the same law as the process $-Y$ conditioned to stay positive and reflected at its future infimum. We omit the details.

Example 4 Let ξ be a non-arithmetic Lévy process with no positive jumps such that ξ drifts to $-\infty$. We assume that ξ is neither the negative of a subordinator nor a deterministic drift. The case of the negative of a subordinator was discussed in Example 1 and the case of a deterministic drift can be treated in the same way. From the theory of Lévy processes with no positive jumps we know that $\mathbf{E}(\mathrm{e}^{\lambda\xi_1}) < \infty$, for all $\lambda > 0$. Then the convex function $\psi(\lambda) : \mathbb{R}^+ \to \mathbb{R}$ defined by $\mathbf{E}(\mathrm{e}^{\lambda\xi_1}) = \mathrm{e}^{\psi(\lambda)}$ is such that $\psi(0) = 0$, and $\lim_{\lambda\to\infty}\psi(\lambda) = \infty$. Since ξ drifts to $-\infty$ there exists a unique $\theta > 0$ such that $\psi(\theta) = 0$. It follows that ξ satisfies (H2). Let $0 < \alpha < 1/\theta$, and let (X, T_0) be the α-self-similar minimal process associated with ξ. Owing to the absence of positive jumps, we have that $X_{T_{[z,\infty[}} = z$ whenever $T_{[z,\infty[} < T_0$, with $T_{[z,\infty[} = \inf\{t > 0\colon X_t \geqslant z\}$. The excursion measure $\mathbf{n}$ compatible with the process (X, T_0) defined in Theorem 1 has the property that under the probability measure on $\mathbb{D}^+$, $\mathbf{n}|(T_{[z,\infty[} < T_0)$, the processes $(X_t, t \leqslant T_{[z,\infty[})$ and $(X_{T_z+t}, t \leqslant T_0 - T_{[z,\infty[})$, are independent. The law of the former is $\mathbb{E}^{\natural}_{0+}$ killed at $T_{[z,\infty[}$ and of the latter is that of (X, T_0) started at z. Here $\mathbf{n}|(T_{[z,\infty[} < T_0)$ means $\mathbf{n}(A \cap \{T_{[z,\infty[} < T_0\})/\mathbf{n}(\{T_{[z,\infty[} < T_0\})$ for $A \in \mathcal{G}_\infty$. This claim is easily verified using the fact that the measure $\mathbf{n}$ is a multiple of the h– transform of $\mathbb{E}^{\natural}_{0+}$ via the excessive function $h^*(x) = x^{-\theta}$, $x > 0$. Moreover, the law of the Lévy exponential functional $I = \int_0^\infty \exp\{\xi_s/\alpha\}\mathrm{d}s$, associated with ξ is self-decomposable and as a consequence the law of I has a continuous density (Rivero 2003, Proposition 4). Therefore, to apply the results in Sections 4 & 5, the only hypothesis that should be made on ξ is that $\mathbb{E}(\xi_1) > -\infty$.

Appendix: On dual extensions

This section is motivated by Section 5, where we proved that given two minimal processes X and $\hat{X}$ which are self-similar and in weak duality, there exist Markov processes $\tilde{X}$ and $\tilde{Z}$ extending (X, T_0) and $(\hat{X}, \hat{T}_0)$ respectively, which are still in weak duality. The purpose of this section is to give a generalization of this fact under the hypotheses of Blumenthal. The result given here is of independent interest, and to make the section self-contained we next introduce some notation. Let $(Y_t,\ t \geqslant 0)$ and $(\hat{Y}_t,\ t \geqslant 0)$ be Markov processes having 0 as a trap. Denote by P, E, ($\hat{\mathrm{P}}$, $\hat{\mathrm{E}}$) the probability and expectation for Y ($\hat{Y}$), and by T_0 ($\hat{T}_0$) the first hitting time of 0 for Y ($\hat{Y}$), that is, $T_0 = \inf\{t > 0 : Y_t = 0\}$. Assume $\mathrm{P}_x(T_0 < \infty) = \hat{\mathrm{P}}x(T_0 < \infty) = 1$ for any $x > 0$. Let Q^0_t, W^0_λ ($\hat{Q}^0_t$, $\hat{W}^0_\lambda$) denote the semigroup and λ-resolvent for Y ($\hat{Y}$) killed at 0. For $\lambda > 0$, define the functions $\varphi_\lambda, \hat{\varphi}_\lambda : \mathbb{R}^+ \to [0, 1]$, by

$$\varphi_\lambda(x) = \mathrm{E_x}(\mathrm{e}^{-\lambda \mathrm{T_0}}), \quad \hat{\varphi}_\lambda(x) = \hat{\mathrm{E}}x(\mathrm{e}^{-\lambda \mathrm{T_0}}), \qquad x > 0.$$

The main assumptions of this section are as follows:

(H3a) Y, $\hat{Y}$, both satisfy the basic hypotheses in Blumenthal (1983).
(H3b) The resolvents W^0_λ and $\hat{W}^0_\lambda$ are in weak duality with respect to a σ-finite measure $Q(\mathrm{d}x)$ on $]0, \infty[$.
(H3c) We have

$$\int_{]0,\infty[} Q(\mathrm{d}x)\varphi_\lambda(x) < \infty, \quad \int_{]0,\infty[} Q(\mathrm{d}x)\hat{\varphi}_\lambda(x) < \infty, \qquad \text{for all } \lambda > 0.$$

Theorem 3. *Assume hypotheses* (H3). *Then there exist excursion measures* m *and* $\hat{m}$ *compatible with the semigroups* $(Q^0_t,\ t \geqslant 0)$ *and* $(\hat{Q}^0_t,\ t \geqslant 0)$, *respectively. The Laplace transforms of the entrance laws* $(m_s,\ s > 0)$ *and* $(\hat{m}_s,\ s > 0)$ *associated with* m and $\hat{m}$, *respectively, are determined by*

$$\int_0^\infty \mathrm{e}^{-\lambda s} m_s f\,\mathrm{d}s = \int_{]0,\infty[} Q(\mathrm{d}x) f(x)\hat{\varphi}_\lambda(x); \quad \int_0^\infty \mathrm{e}^{-\lambda s} \hat{m}_s f\,\mathrm{d}s = \int_{]0,\infty[} Q(\mathrm{d}x) f(x)\varphi_\lambda(x),$$

for $\lambda > 0$, *and* f *continuous and bounded. Furthermore, associated wtih these excursion measures there exist Markov processes* Y^* *and* $\hat{Y}^*$ *which are extensions for* Y *and* $\hat{Y}$ *respectively and which are still in weak duality with respect to the measure* $Q(\mathrm{d}x)$.

The proof of this theorem will be given via three lemmas. The first ensures the existence of the excursion measures.

Lemma 8. *The family of finite measures* $M_\lambda f = \int_{]0,\infty[} Q(\mathrm{d}x) f(x)\hat{\varphi}_\lambda(x)$, $\lambda > 0$, *is such that the following hold:*

(i) $\lim_{\lambda\to\infty} M_\lambda 1 = 0$,
(ii) For $\mu, \lambda > 0$, $\mu \neq \lambda$,

$$(\mu - \lambda) M_\lambda W^0_\mu f = M_\lambda f - M_\mu f,$$

for f *continuous and bounded.*

Proof. That $M_\lambda \to 0$ as $\lambda \to \infty$ follows from the monotone convergence theorem. Using the weak duality for the resolvents W^0_λ and $\hat{W}^0_\lambda$, we obtain

$$M_\lambda W^0_\mu f = \int_{]0,\infty[} Q(\mathrm{d}x) W^0_\mu f(x) \hat{\varphi}_\lambda(x)$$
$$= \int_{]0,\infty[} Q(\mathrm{d}x) f(x) \hat{W}^0_\mu \hat{\varphi}_\lambda(x).$$

The result is then obtained from the elementary identity

$$\hat{W}^0_\mu \hat{\varphi}_\lambda(x) = \frac{\hat{\mathrm{E}}_x(\mathrm{e}^{-\lambda T_0} - \mathrm{e}^{-\mu T_0})}{\mu - \lambda}.$$

□

From Lemma 8 and Getoor and Sharpe (1973, Theorem 6.9), there exists a unique entrance law $(m_t,\ t > 0)$, for the semigroup $(Q_t,\ t \geqslant 0)$, such that, for each $\lambda > 0$,

$$M_\lambda f = \int_0^\infty \mathrm{e}^{-\lambda t} m_t f \,\mathrm{d}t,$$

for f measurable and bounded, and

$$\int_0^1 m_t 1 \,\mathrm{d}t < \infty.$$

According to Blumenthal (1983), for an entrance law $(m_s,\ s > 0)$ there exists a unique excursion measure m having this entrance law. The same method ensures the existence of an excursion measure $\hat{m}$ and an entrance law $(\hat{m}_t,\ t > 0)$ for the semigroup $(\hat{Q}_t,\ t \geqslant 0)$.

Using the results in Blumenthal (1983), we obtain that associated with the excursion measure m ($\hat{m}$) there exists a unique Markov process Y^* ($\hat{Y}^*$ extending Y ($\hat{Y}$) and the λ-resolvent of Y^* is determined by

$$W_\lambda f(0) = \frac{M_\lambda f}{\lambda M_\lambda 1}, \qquad W_\lambda f(x) = W^0_\lambda f(x) + \varphi_\lambda(x) W_\lambda f(0), \quad x > 0,$$

for f measurable and bounded; the λ-resolvent for $\hat{Y}^*$, say $\hat{W}_\lambda$, is defined in a similar way. To establish weak duality with respect to the σ-finite measure $Q(\mathrm{d}x)$ for the resolvents W_λ and $\hat{W}_\lambda$ we will need the following technical result.

Lemma 9. *For every $\lambda > 0$, we have that $\lambda M_\lambda 1 = \lambda \hat{M}_\lambda 1$.*

Proof. Since $m_s 1$ is a decreasing function of s and $\int_0^1 m_s 1 \,\mathrm{d}s < \infty$, we have that

$$\mu M_\mu 1 = \mu \int_0^\infty \mathrm{e}^{-\mu t} m_t 1 \,\mathrm{d}t = \lim_{s\to\infty} m_s 1 + \int_0^\infty (1 - \mathrm{e}^{-\mu t}) \nu(\mathrm{d}t),$$

where $\nu(\mathrm{d}t) = -\mathrm{d}m_t 1$. Analogously,

$$\mu \hat{M}_\mu 1 = \mu \int_0^\infty \mathrm{e}^{-\mu t} \hat{m}_t 1 \,\mathrm{d}t = \lim_{s\to\infty} \hat{m}_s 1 + \int_0^\infty (1 - \mathrm{e}^{-\mu t}) \hat{\nu}(\mathrm{d}t).$$

Therefore, to establish the lemma we will prove that, for $\lambda > 0$,

$$\int_0^\infty (1-\mathrm{e}^{-\lambda s})\upsilon(\mathrm{d}s) = \int_0^\infty (1-\mathrm{e}^{-\lambda s})\hat{\upsilon}(\mathrm{d}s), \tag{20}$$

and

$$\lim_{s\to\infty} m_s = 0 = \lim_{s\to\infty} \hat{m}_s.$$

To this end we will use the following elementary identities: for $\lambda, \mu > 0$,

$$(\lambda-\mu)M_\lambda\varphi_\mu = \lambda M_\lambda 1 - \mu M_\mu 1$$

and

$$(\lambda-\mu)\hat{M}_\lambda\hat{\varphi}_\mu = \lambda\hat{M}_\lambda 1 - \mu\hat{M}_\mu 1.$$

Next, since

$$M_\lambda\varphi_\mu = \int_{]0,\infty[} Q(\mathrm{d}x)\hat{\varphi}_\lambda(x)\varphi_\mu(x) = \hat{M}_\mu\hat{\varphi}_\lambda,$$

we have

$$\lambda M_\lambda 1 - \mu M_\mu 1 = \lambda\hat{M}_\lambda 1 - \mu\hat{M}_\mu 1.$$

Letting $\mu \to 0$, we obtain that

$$\lambda M_\lambda 1 - \lim_{s\to\infty} m_s 1 = \lambda\hat{M}_\lambda 1 - \lim_{s\to\infty} \hat{m}_s 1.$$

This proves the equality (20). To prove that $\lim_{s\to\infty} m_s 1 = 0$, we use the fact that m is the excursion measure associated to the entrance law $(m_s, s > 0)$. Indeed,

$$m(1-\mathrm{e}^{-\lambda T_0}) = \lambda M_\lambda 1 = \lim_{s\to\infty} m_s 1 + \int_0^\infty (1-\mathrm{e}^{-\lambda t})\upsilon(\mathrm{d}t).$$

Letting $\lambda \to 0$, in this equation we obtain, thanks to the monotone convergence theorem, that $\lim_{s\to\infty} m_s 1 = 0$. In the same way it is proved that $\lim_{s\to\infty} \hat{m}_s 1 = 0$. □

Finally, the following lemma establishes weak duality for the resolvents W_λ and $\hat{W}_\lambda$.

Lemma 10. *For every $\lambda > 0$ and every measurable function $f, g : [0,\infty[\to \mathbb{R}^+$, we have*

$$\int_{]0,\infty[} Q(\mathrm{d}y)g(y)W_\lambda f(y) = \int_{]0,\infty[} Q(\mathrm{d}y)f(y)\hat{W}_\lambda g(y).$$

The proof of this lemma is a straightforward consequence of Lemma 9 and the construction of W_λ and $\hat{W}_\lambda$; see the proof of Lemma 7.

Remarks.

1. Observe that

$$\lim_{\lambda\to 0}\int_0^\infty \mathrm{d}s\mathrm{e}^{-\lambda s}m_s f = \int_0^\infty \mathrm{d}s m_s f = \int_{]0,\infty[} Q(\mathrm{d}y)f(y).$$

   By the weak duality relation in Lemma 10 we have that $Q(\mathrm{d}y)$ is invariant for the semigroup of Y^* and since 0 is a recurrent state for Y^* then $Q(\mathrm{d}y)$ is in fact the unique (up to a multiplicative constant) excessive measure for this semigroup, (Dellacherie *et al*. 1992, XIX.46).
2. We have not considered here the possibility of a *stickiness* parameter in the construction of the processes Y^* and $\hat{Y}^*$; that is, constructing Y^* and $\hat{Y}^*$ via the subordinators

$$\sigma_t = dt + \sum_{\mathrm{s}\leqslant \mathrm{t}} \mathrm{T}_0(\Delta_{\mathrm{s}}), \quad \hat{\sigma}_{\mathrm{t}} = \hat{d}t + \sum_{s\leqslant t} \hat{T}_0(\Delta_s), \qquad t > 0,$$

   for some d, $\hat{d} > 0$ – see Section 2.1 or Blumenthal (1992, Section 5) for an account. In such a case, the λ-resolvent for Y^* ($\hat{Y}^*$) at 0 is given by

$$W_\lambda f(0) = \frac{df(0) + M_\lambda f}{\lambda d + \lambda M_\lambda 1};\ W_\lambda f(0) = \frac{\hat{d}f(0) + \hat{M}_\lambda f}{\lambda \hat{d} + \lambda \hat{M}_\lambda 1},$$

   for f continuous and bounded and, if $d = \hat{d}$ then the resolvents W_λ and $\hat{W}_\lambda$ are still in weak duality but this time with respect to the measure $Q^d(\mathrm{d}x) = d\delta_0(\mathrm{d}x) + Q(\mathrm{d}x)$.
3. Assume, moreover, that for every $x > 0$, $\hat{\mathrm{P}}_x(T_0 \in \mathrm{d}t)$ is absolutely continuous with respect to Lebesgue measure, having a density

$$a(x,\ t) = \frac{\hat{\mathrm{P}}_x(T_0 \in \mathrm{d}t)}{\mathrm{d}t}, \qquad x,\ t > 0,$$

   which is jointly Borel measurable. Then, for $\lambda > 0$,

$$\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\lambda s} m_s f = \int_{]0,\infty[} Q(\mathrm{d}x)\hat{\varphi}_\lambda(x)f(x) = \int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\lambda s}\int_{]0,\infty[} Q(\mathrm{d}x)a(x,\ s)f(x),$$

   for f continuous and bounded. The second equality is a consequence of Fubini's theorem. By inverting the Laplace transform we obtain that, for $s > 0$,

$$m_s f = \int_{]0,\infty[} Q(\mathrm{d}x)a(x,\ s)f(x).$$

   A similar result was obtained in (Getoor 1979, Proposition 10.10) in a different setting.

Acknowledgements

I am grateful to Jean Bertoin for suggesting the problem to me and for very helpful comments. I am indebted to Olivier Mejane for access to his work (Mejane 2002). The research for this paper was supported by a grant from CONACYT (National Council of Science and Technology Mexico).

References

Bertoin, J. (1996) *Lévy Processes*. Cambridge: Cambridge University Press.

Bertoin, J. and Caballero, M.-E. (2002) Entrance from 0+ for increasing semi-stable Markov processes. *Bernoulli*, **8**, 195–205.

Bertoin, J. and Doney, R.A. (1994) Cramér's estimate for Lévy processes. *Statist. Probab. Lett.*, **21**, 363–365.

Bertoin, J. and Yor, M. (2002a) The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. *Potential Anal.*, **17**(4), 389–400.

Bertoin, J. and Yor, M. (2002b) On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes. *Ann. Fac. Sci. Toulouse, VI. Sér., Math.*, **11**, 33–45.

Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1989) *Regular Variation*. Cambridge: Cambridge University Press.

Blumenthal, R.M. (1983) On construction of Markov processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **63**(4), 433–444.

Blumenthal, R.M. (1992) *Excursions of Markov Processes*. Boston: Birkhäuser.

Caballero, M. and Chaumont, L. (2004) Weak convergence of positive self similar Markov processes and overshoots of Lévy processes. Technical report, available at http://www.proba.jussieu.fr/mathdoc/textes/PMA-899.pdf

Carmona, P., Petit, F. and Yor, M. (1994) Sur les fonctionnelles exponentielles de certains processus de Lévy. *Stochastics Stochastics Rep.*, **47**(1–2), 71–101.

Carmona, P., Petit, F. and Yor, M. (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. In M. Yor (ed.), *Exponential Functionals and Principal Values Related to Brownian Motion*, Bibl. Rev. Mat. Iberoamericana, pp. 73–130. Madrid: Revista Matemática Iberoamericana.

Chaumont, L. (1997) Excursion normalisée, méandre et pont pour les processus de Lévy stables. *Bull. Sci. Math.*, **121**(5), 377–403.

Dellacherie, C., Maisonneuve, B. and Meyer, P.A. (1992) *Probabilités et Potentiel: Processus de Markov (fin). Compléments du Calcul Stochastique*, Vol. V. Paris: Hermann.

Dufresne, D. (1990) The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, 39–79.

Embrechts, P. and Maejima, M. (2002) *Self-similar Processes*. Princeton, NJ: Princeton University Press.

Fitzsimmons, P., Pitman, J. and Yor, M. (1993) Markovian bridges: construction, Palm interpretation, and splicing. In E. Çinlar, K.L. Chung and M.J. Sharpe (eds) *Seminar on Stochastic Processes, 1992*, Progr. Probab. 33, pp. 101–134. Boston: Birkhäuser.

Getoor, R.K. (1979) Excursions of a Markov process. *Ann. Probab.*, **7**(2), 244–266.

Getoor, R.K. and Sharpe, M.J. (1973) Last exit times and additive functionals. *Ann. Probab.*, **1**, 550–569.

Getoor, R.K. and Sharpe, M.J. (1981) Two results on dual excursions, In E. Çinlar, K.L. Chung and R.K. Getoor (eds), *Seminar on Stochastic Processes, 1981*, Progr. Probab. Statist. 1, pp. 31–52. Boston: Birkhäuser.

Getoor, R.K. and Sharpe, M.J. (1982) Excursions of dual processes. *Adv. in Math.*, **45**(3), 259–309.

Goldie, C.M. (1991) Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.*, **1**(1), 126–166.

Imhof, J.-P. (1984) Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, **21**(3), 500–510.

Kesten, H. (1973) Random difference equations and renewal theory for products of random matrices. *Acta Math.*, **131**, 207–248.

Lamperti, J. (1972) Semi-stable Markov processes. I. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **22**, 205–225.

McKean, Jr., H.P. (1963) Excursions of a non-singular diffusion. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **1**, 230–239.

Mejane, O. (2002) Cramer's estimate for the exponential functional of a Lévy process. Laboratoire de Statistiques et Probabilités, Université Paul Sabatier. Available at http://front.math.ucdavis.edu/math.PR/0211409

Meyer, P.A. (1971) Processus de Poisson ponctuels, d'après K. Ito. In *Séminaire de Probabilités V*, Lecture Notes in Math. 191, pp. 177–190. Berlin: Springer-Verlag.

Mitro, J.B. (1984) Exit systems for dual Markov processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **66**(2), 259–267.

Nagasawa, M. (1964) Time reversion of Markov processes. *Nagoya Math. J.*, **24**, 177–204.

Revuz, D. and Yor, M. (1999) *Continuous Martingales and Brownian Motion*, 3rd edition. Berlin: Springer-Verlag.

Rivero, V.M. (2003) A law of iterated logarithm for increasing self-similar Markov processes. *Stochastics Stochastics Rep.*, **75**(6), 443–472.

Rogers, L.C.G. (1983) Itô excursion theory via resolvents. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **63**(2), 237–255.

Salisbury, T.S. (1986a) Construction of right processes from excursions. *Probab. Theory Related Fields*, **73**(3), 351–367.

Salisbury, T.S. (1986b) On the Itô excursion process. *Probab. Theory Related Fields*, **73**(3), 319–350.

Sato, K.-I. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Sharpe, M. (1988) *General Theory of Markov Processes*. Boston: Academic Press.

Vuolle-Apiala, J. (1994) Itô excursion theory for self-similar Markov processes. *Ann. Probab.*, **22**, 546–565.

Williams, D. (1974) Path decomposition and continuity of local time for one-dimensional diffusions. I. *Proc. London Math. Soc. (3)*, **28**, 738–768.

Received August 2003 and revised April 2004