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Stability of the tail Markov chain and the evaluation of improper priors for an exponential rate parameter

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Let Z be a continuous random variable with a lower semicontinuous density f that is positive on $(0, \infty)$ and 0 elsewhere. Put $G(x) = \int_x^\infty f(z)\,\mathrm{d}z$. We study the tail Markov chain generated by Z, defined as the Markov chain $\Phi = (\Phi_n)_{n=0}^\infty$ with state space $[0, \infty)$ and Markov transition density $k(y|x) = f(y + x)/G(x)$. This chain is irreducible, aperiodic and reversible with respect to G. It follows that Φ is positive recurrent if and only if Z has a finite expectation. We prove (under regularity conditions) that if $\mathrm{E}Z = \infty$, then Φ is null recurrent if and only if $\int_1^\infty 1/[z^3 f(z)]\mathrm{d}z = \infty$. Furthermore, we describe an interesting decision-theoretic application of this result. Specifically, suppose that X is an $\mathrm{Exp}(\theta)$ random variable; that is, X has density $\theta\,\mathrm{e}^{-\theta x}$ for $x > 0$. Let ν be an improper prior density for θ that is positive on $(0, \infty)$. Assume that $\int_0^\infty \theta\,\nu(\theta)\,\mathrm{d}\theta < \infty$, which implies that the posterior density induced by ν is proper. Let m_ν denote the marginal density of X induced by ν; that is, $m_\nu(x) = \int_0^\infty \theta\,\mathrm{e}^{-\theta x}\,\nu(\theta)\,\mathrm{d}\theta$. We use our results, together with those of Eaton and of Hobert and Robert, to prove that ν is a $\mathcal{P}$-admissible prior if $\int_1^\infty 1/[x^2\, m_\nu(x)]\mathrm{d}x = \infty$.

Keywords: admissibility; coupling; hazard rate; null recurrence; reversibility; stochastic comparison; stochastically monotone Markov chain; transience

1. Introduction

1.1. Tail Markov chains and the main result

Let Z be a random variable whose density (with respect to Lebesgue measure) is a lower semicontinuous function $f : \mathbb{R} \to [0, \infty)$ that is positive on $\mathbb{R}^+ := (0, \infty)$ and 0 on $(-\infty, 0]$. We will say that such a random variable (and its density) satisfies assumption $\mathcal{A}$. Let G and q denote the *survival function* and *hazard rate*, respectively; that is, $G(x) = \int_x^\infty f(z)\,\mathrm{d}z$ and $q(z) = f(z)/G(z)$.

With each such Z we associate a Markov chain $\Phi = (\Phi_n)_{n=0}^\infty$ with state space $[0, \infty)$ and Markov transition density $k(y|x) = f(y + x)/G(x)$. Thus, for any $n \in \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, any $x \geqslant 0$, and any Borel measurable set $A \subset [0, \infty)$,

$$P(x, A) := \Pr(\Phi_{n+1} \in A | \Phi_n = x) = \int_A k(y|x)\, \mathrm{d}y = \int_A \frac{f(y+x)}{G(x)}\, \mathrm{d}y. \tag{1}$$

One can think of the chain evolving as follows. Suppose that the current state is $\Phi_n = x$ and let Z_x denote a random variable with density proportional to $f(z)\, I(z > x)$. Then Φ_{n+1} is set equal to a realization of $Z_x - x$ whose support is $[0, \infty)$. We call this chain the *tail Markov chain generated by* Z (or *by the density* f).

Since f is positive on $\mathbb{R}^+$, the probability in (1) is positive for any x as long as $\lambda(A) > 0$, where λ denotes Lebesgue measure. Thus, Φ is λ-irreducible and aperiodic; see Meyn and Tweedie (1993) for definitions. Moreover, Φ is a *Feller* Markov chain; that is, for each fixed open set $A \subset [0, \infty)$, $P(x, A)$ is a lower semicontinuous function of x. To see this, let $(x_n)_{n=1}^{\infty}$ be a sequence of positive real numbers such that $x_n \neq x_0$ and $x_n \to x_0 \geqslant 0$ as $n \to \infty$. Now using Fatou's lemma and the fact that products of positive, lower semicontinuous functions are lower semicontinuous, we have

$$\liminf_{n\to\infty} P(x_n, A) \geqslant \int_A \liminf_{n\to\infty} \frac{f(y+x_n)}{G(x_n)}\, \mathrm{d}y \geqslant \int_A \frac{f(y+x_0)}{G(x_0)}\, \mathrm{d}y = P(x_0, A),$$

which implies the desired lower semicontinuity. Because Φ is a Feller chain, every compact set in the state space is a *petite set* (Meyn and Tweedie, 1993, Chapters 5–6). This facilitates several technical arguments later in the paper.

The chain is *reversible* with respect to the function G; that is,

$$k(y|x)\, G(x) = k(x|y)\, G(y), \qquad \forall x,\ y \in [0, \infty).$$

Hence, $\int_0^\infty k(y|x)\, G(x)\, \mathrm{d}x = G(y)$, which means that $G(y)\, \mathrm{d}y$ is an invariant measure for Φ. Since $\int_0^\infty G(y)\, \mathrm{d}y = \mathrm{E}Z$, it follows that the tail Markov chain generated by Z is positive recurrent if $\mathrm{E}Z < \infty$ and is either null recurrent or transient if $\mathrm{E}Z = \infty$. In this paper, we concentrate on differentiating between null recurrence and transience when Z has an infinite mean. The following theorem, which is proved in Section 4, is our main result.

Theorem 1. *Assume that Z satisfies assumption $\mathcal{A}$ and that $\mathrm{E}Z = \infty$ so that the tail Markov chain generated by Z is either null recurrent or transient. Assume that there exists an $M > 0$ such that $q(z)$ is non-increasing for $z > M$. Then Φ is null recurrent if*

$$\int_1^\infty \frac{1}{z^2 G(z)}\, \mathrm{d}z = \infty, \tag{2}$$

and transient if

$$\int_1^\infty \frac{1}{z^3 f(z)}\, \mathrm{d}z < \infty. \tag{3}$$

Remark 1. It is shown in Barlow *et al.* (1963) that if $\mathrm{E}Z^r = \infty$ for $r > 0$, then $\liminf_{z\to\infty} z\, q(z) \leqslant r$. Thus, if $\mathrm{E}Z = \infty$ and $\lim_{z\to\infty} q(z)$ exists, the limit must be 0. Hence, our assumption regarding q is not as restrictive as it may at first seem.

Remark 2. In Section 4, we prove that under the additional condition $\liminf_{z\to\infty} z\, q(z) > 0$, one of (2) or (3) must be true. Thus, under this extra condition, the conclusion of Theorem 1 can be stated as: Then Φ is null recurrent if and only if

$$\int_1^\infty \frac{1}{z^3 f(z)} \mathrm{d}z = \infty.$$

Example 1. Consider the tail Markov chains generated by the (centred) Pareto distributions with densities

$$f(z;\, \alpha,\, \beta) = \frac{\beta\, \alpha^\beta}{(z+\alpha)^{\beta+1}}\, I(z > 0),$$

where $\alpha, \beta > 0$. We restrict attention to the case in which $\beta \leqslant 1$ since otherwise the mean is finite and the chain is positive recurrent. Note that $q(z) = \beta/(z + \alpha)$, which is clearly decreasing. Moreover, $\lim_{z\to\infty} z\, q(z) = \beta > 0$. Now

$$\int_1^\infty \frac{1}{z^3 f(z)} \mathrm{d}z \propto \int_1^\infty \frac{(z+\alpha)^{\beta+1}}{z^3} \mathrm{d}z.$$

This integral diverges if $\beta = 1$ and converges if $\beta \in (0, 1)$. Hence, by Remark 2 above, Φ is null recurrent when $\beta = 1$ and is transient when $\beta \in (0, 1)$.

If G is an intractable integral, it may be difficult to analyse q directly. This makes it difficult to decide if Theorem 1 is applicable. We now prove a result providing a simple sufficient condition (involving only f) for q to be eventually non-increasing.

Lemma 1. *Suppose Z satisfies assumption $\mathcal{A}$ and that there exists an $M > 0$ such that $\log f(z)$ is convex for $z > M$. Then $q(z)$ is non-increasing for $z > M$.*

Proof. We use the following property of convex functions (see, for example, Pečarić *et al.*, 1992, p. 2). Let g be a convex function on an interval I. If $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, $x_1 \neq x_2$, and $y_1 \neq y_2$, then

$$\frac{g(x_2) - g(x_1)}{x_2 - x_1} \leqslant \frac{g(y_2) - g(y_1)}{y_2 - y_1}. \tag{4}$$

Now let $M < z < z'$, and let $x > 0$. Applying (4) with $x_1 = z$, $x_2 = z + x$, $y_1 = z'$ and $y_2 = z' + x$, we obtain

$$\frac{\log f(z+x) - \log f(z)}{x} \leqslant \frac{\log f(z'+x) - \log f(z')}{x}.$$

It follows that $f(z'+x)f(z) \geqslant f(z')f(z+x)$ for all $x > 0$. Thus,

$$f(z)\int_0^\infty f(z'+x)\,\mathrm{d}x \geqslant f(z')\int_0^\infty f(z+x)\,\mathrm{d}x,$$

and hence $q(z) \geqslant q(z')$. □

Example 2. Consider the tail Markov chains generated by the inverse gamma (IG) distributions with densities

$$f(z;\,\alpha,\,\beta) = \frac{\mathrm{e}^{-1/z\beta}}{\Gamma(\alpha)\,\beta^{\alpha}\,z^{\alpha+1}}\;I(z > 0),$$

where $\alpha, \beta > 0$. We restrict attention to the case in which $\alpha \leqslant 1$ since otherwise the mean is finite. We have $(\partial^2/\partial z^2)\log f(z;\,\alpha,\,\beta) > 0$ as long as $z > 2/[\beta(\alpha+1)]$. Thus, by Lemma 1, Theorem 1 is applicable. If $\alpha \in (0, 1)$,

$$\int_1^{\infty} \frac{1}{z^3 f(z)}\,\mathrm{d}z \propto \int_1^{\infty} \frac{\mathrm{e}^{1/z\beta}}{z^{2-\alpha}}\,\mathrm{d}z < \infty,$$

which implies that Φ is transient. If $\alpha = 1$, it is easy to show that $G(z) < c/z$, where c is a constant, and it follows from (2) that Φ is null recurrent in this case.

Example 3. Consider the tail Markov chains generated by the F distributions. The F densities are given by

$$f(z;\,\alpha,\,\beta) = \frac{\Gamma((\alpha+\beta)/2)}{\Gamma(\alpha/2)\Gamma(\beta/2)}\left(\frac{\alpha}{\beta}\right)^{\alpha/2}\frac{z^{(\alpha-2)/2}}{[1+(\alpha/\beta)\,z]^{(\alpha+\beta)/2}}\;I(z > 0),$$

where $\alpha, \beta > 0$. We restrict attention to the case in which $\beta \leqslant 2$ since otherwise the mean is finite. First,

$$\frac{\partial^2}{\partial z^2}\log f(z;\,\alpha,\,\beta) = -\left(\frac{\alpha-2}{2}\right)\frac{1}{z^2} + \left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha}{\beta}\right)^2\left[1+\frac{\alpha}{\beta}\,z\right]^{-2}.$$

If $\alpha \leqslant 2$, then $\log f(z)$ is clearly convex on all of $\mathbb{R}^+$. Now suppose that $\alpha > 2$. A straightforward calculation shows that $(\partial^2/\partial z^2)\log f(z;\,\alpha,\,\beta) > 0$ as long as

$$z > \frac{\beta(\alpha-2) + \beta\{(\alpha-2)(\alpha+\beta)\}^{1/2}}{\alpha(\beta+2)} > 0.$$

Thus, by Lemma 1, Theorem 1 is applicable. If $\beta \in (0, 2)$, then

$$\int_1^{\infty} \frac{1}{z^3 f(z)}\,\mathrm{d}z = c\int_1^{\infty} \frac{[1+(\alpha/\beta)\,z]^{(\alpha+\beta)/2}}{z^{\alpha/2+2}}\,\mathrm{d}z < c'\int_1^{\infty} \frac{1}{z^{2-\beta/2}}\,\mathrm{d}z < \infty,$$

and hence Φ is transient. Now, if $\beta = 2$, it's easy to show that $G(z) < c/z$ and it follows from (2) that Φ is null recurrent in this case.

In the next subsection, we describe a connection between null recurrent tail Markov chains and good prior distributions for an exponential rate parameter.

1.2. Evaluating improper priors for an exponential rate parameter

Suppose that X is an $\mathrm{Exp}(\theta)$ random variable; that is, the conditional density of X given θ

is $h(x|\theta) = \theta \exp\{-x\theta\} I(x > 0)$, where $\theta > 0$. Let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\int_{\mathbb{R}^+} \nu(\theta)\,\mathrm{d}\theta = \infty$ and $\int_{\mathbb{R}^+} \theta\,\nu(\theta)\,\mathrm{d}\theta < \infty$. Note that

$$\int_{\mathbb{R}^+} \theta\,\nu(\theta)\,\mathrm{d}\theta < \infty \Rightarrow \int_{\mathbb{R}^+} \theta^{k+1} \exp\{-x\theta\}\,\nu(\theta)\,\mathrm{d}\theta < \infty$$

whenever $x > 0$ and $k \geqslant 0$. Thus, $\nu(\theta)$ can be viewed as an *improper* prior density that yields a *proper* posterior density given by

$$\pi(\theta|x) = \frac{\theta \exp\{-x\theta\}\,\nu(\theta)\,I(\theta > 0)}{m_\nu(x)},$$

where, of course,

$$m_\nu(x) := \int_{\mathbb{R}^+} \theta \exp\{-x\theta\}\,\nu(\theta)\,\mathrm{d}\theta.$$

An example of a prior satisfying these conditions is $\nu(\theta; p) = \theta^{-1} I(0 < \theta < 1) + \theta^{-p} I(\theta > 1)$ for any $p > 2$.

Priors satisfying these conditions are 'proper at ∞' in the sense that $\int_1^\infty \nu(\theta)\,\mathrm{d}\theta < \infty$ but 'improper at 0' in the sense that $\int_0^1 \nu(\theta)\,\mathrm{d}\theta = \infty$. The exponential scale family can easily be transformed into a location family by taking logs. If τ is the corresponding prior density for the location parameter $\lambda = -\log\theta$, then $\int_{-\infty}^0 \tau(\lambda)\,\mathrm{d}\lambda < \infty$ and $\int_0^\infty \tau(\lambda)\,\mathrm{d}\lambda = \infty$, so the prior is proper in one tail but improper in the other.

Consider a statistical decision problem where $R(\delta, \theta)$ is the risk function for the decision rule δ. If ν is an improper prior, a decision rule δ_0 is said to be *almost-ν-admissible* if, for any decision rule δ_1 which satisfies $R(\delta_1, \theta) \leqslant R(\delta_0, \theta)$ for all θ, we have $\nu(\{\theta : R(\delta_1, \theta) < R(\delta_0, \theta)\}) = 0$. The prior ν is called *$\mathcal{P}$-admissible* if the generalized Bayes estimator of every bounded function of θ is almost-ν-admissible under squared error loss (Eaton, 1992; Hobert and Robert, 1999). (Such improper priors have also been called *strongly admissible*.)

With each prior ν satisfying $\int_{\mathbb{R}^+} \nu(\theta)\,\mathrm{d}\theta = \infty$ and $\int_{\mathbb{R}^+} \theta\,\nu(\theta)\,\mathrm{d}\theta < \infty$, we associate a Markov chain Φ^ν with state space $[0, \infty)$ and Markov transition density

$$k_\nu(y|x) = \int_{\mathbb{R}^+} h(y|\theta)\,\pi(\theta|x)\,\mathrm{d}\theta = \frac{\displaystyle\int_{\mathbb{R}^+} \theta^2 \exp\{-(x+y)\theta\}\,\nu(\theta)\,\mathrm{d}\theta}{\displaystyle\int_{\mathbb{R}^+} \theta \exp\{-x\theta\}\,\nu(\theta)\,\mathrm{d}\theta}$$

for $x, y \in [0, \infty)$. It follows from results of Eaton (1992) and Hobert and Robert (1999) that if Φ^ν is (null) recurrent, then the prior ν is $\mathcal{P}$-admissible. See Eaton (1997) for a detailed introduction to these ideas. Other key papers in which connections between admissibility and recurrence are established include Brown (1971), Johnstone (1984; 1986), Lai (1996) and Eaton (2001).

The Markov chain Φ^ν is actually the tail Markov chain generated by the density

$$f_\nu(z) = \frac{\displaystyle\int_{\mathbb{R}^+} \theta^2 \exp\{-z\,\theta\}\, \nu(\theta)\,\mathrm{d}\theta}{\displaystyle\int_{\mathbb{R}^+} \theta\, \nu(\theta)\,\mathrm{d}\theta}\, I(z > 0),$$

which is clearly lower semicontinuous and hence satisfies assumption $\mathcal{A}$. Note also that $\int_{\mathbb{R}^+} z\, f_\nu(z)\,\mathrm{d}z \propto \int_{\mathbb{R}^+} \nu(\theta)\,\mathrm{d}\theta = \infty$. Hence, Φ^ν is never positive recurrent. The hazard rate is given by

$$q_\nu(z) = \frac{\displaystyle\int_{\mathbb{R}^+} \theta^2 \exp\{-z\,\theta\}\, \nu(\theta)\,\mathrm{d}\theta}{\displaystyle\int_{\mathbb{R}^+} \theta \exp\{-z\,\theta\}\, \nu(\theta)\,\mathrm{d}\theta}.$$

We now show that q_ν is non-increasing, which means that Theorem 1 is applicable. Consider the exponential family of probability densities given by

$$g(w;\, \eta) = w\, \nu(w) \exp\{w\eta - \psi(\eta)\}\, I(w > 0),$$

where $\eta < 0$ and $\psi(\eta) = \log \int_{\mathbb{R}^+} w\, \nu(w)\, \mathrm{e}^{w\eta}\,\mathrm{d}w$. Brown (1986) shows that the derivatives of ψ exist and can be computed by differentiating under the integral sign. Moreover, $\psi''(\eta) = \mathrm{var}_\eta(W)$, where W is a random variable with density $g(w;\, \eta)$. Now, for $z > 0$, $q_\nu(z) = \psi'(-z)$ and hence $(\mathrm{d}/\mathrm{d}z) q_\nu(z) = -\psi''(-z) \leqslant 0$. Thus, $q_\nu(z)$ is non-increasing. Applying Theorem 1 in this context leads to a simple sufficient condition for the $\mathcal{P}$-admissibility of ν.

Theorem 2. *Suppose that $X \sim \mathrm{Exp}(\theta)$ and let $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ be an improper prior for θ such that $\int_{\mathbb{R}^+} \theta\, \nu(\theta)\,\mathrm{d}\theta < \infty$. Then ν is $\mathcal{P}$-admissible if*

$$\int_1^\infty \frac{1}{x^2\, m_\nu(x)}\,\mathrm{d}x = \infty. \tag{5}$$

Example 4. Let $\nu(\theta;\, p) = \theta^{-1}\, I(0 < \theta < 1) + \theta^{-p}\, I(\theta > 1)$, where $p > 2$. Then

$$m_\nu(x) = \int_0^1 \mathrm{e}^{-x\,\theta}\,\mathrm{d}\theta + \int_1^\infty \theta^{1-p}\, \mathrm{e}^{-x\,\theta}\,\mathrm{d}\theta < \int_0^1 \mathrm{e}^{-x\,\theta}\,\mathrm{d}\theta + \int_1^\infty \mathrm{e}^{-x\,\theta}\,\mathrm{d}\theta = \frac{1}{x}.$$

Thus, by Theorem 2, all the priors in this class are $\mathcal{P}$-admissible.

Example 5. Consider the improper conjugate priors

$$\nu(\theta;\, \alpha, \beta) = \theta^{\alpha-1} \exp\{-\beta\theta\}\, I(\theta > 0),$$

where $\alpha \in (-1, 0]$ and $\beta > 0$. The marginal density is given by $m_\nu(x) = \Gamma(\alpha+1)(\beta+x)^{-\alpha-1}$, and hence

$$\int_1^\infty \frac{1}{x^2\, m_\nu(x)}\,\mathrm{d}x = \frac{1}{\Gamma(\alpha+1)} \int_1^\infty \frac{(\beta+x)^{\alpha+1}}{x^2}\,\mathrm{d}x,$$

which diverges if $\alpha = 0$. Thus, by Theorem 2, all priors of the form $\theta^{-1} \exp\{-\beta\theta\} I(\theta > 0)$ with $\beta > 0$ are $\mathcal{P}$-admissible. Hobert and Robert (1999) arrived at this conclusion through a completely different argument.

The results of Hobert and Robert (1999) also imply that the prior $\theta^{-1} I(\theta > 0)$ is $\mathcal{P}$-admissible. Alternatively, the fact that $\theta^{-1} I(\theta > 0)$ is $\mathcal{P}$-admissible can be deduced from Example 3.1 of Eaton (1992). The fact that this prior is $\mathcal{P}$-admissible does not, however, follow from the results of the present paper, because the condition $\int_{\mathbb{R}^+} \theta\, \nu(\theta)\, \mathrm{d}\theta < \infty$ is needed to define the density $f_\nu(z)$. This is also the reason why we needed to assume $p > 2$ in Example 4.

The rest of this paper is organized as follows. Section 2 contains two results that are used in the proof of Theorem 1. We first prove that the tail Markov chain generated by Z is *stochastically monotone* if $q(z)$ is non-increasing on $\mathbb{R}^+$. We then prove that given a density, $f(z)$, whose hazard rate is eventually non-increasing, there exists another density that is both equal to $f(z)$ for all large z and has a hazard rate that is non-increasing on $\mathbb{R}^+$. In Section 3, we describe a discrete analogue of Φ and state a result of Hobert and Schweinsberg (2002) that is also used in the proof of Theorem 1. Section 4 contains the proof of Theorem 1 as well as a lemma connecting the limiting behaviour of $z\, q(z)$ with the integrals in (2) and (3).

2. Stochastic monotonicity and monotone hazard rate

Define

$$K(y|x) := \Pr(\Phi_{n+1} \leqslant y | \Phi_n = x) = P(x, [0, y]) = \int_0^y \frac{f(t+x)}{G(x)}\, \mathrm{d}t.$$

The Markov chain Φ is called *stochastically monotone* (Daley, 1968) if, for every pair $0 \leqslant x_1 < x_2$ and every $y > 0$, $K(y|x_1) \geqslant K(y|x_2)$. Note that $K(y|x)$ is the distribution function of the random variable $Z_x - x$. Hence, stochastic monotonicity of Φ is equivalent to saying that $Z_{x_2} - x_2$ is stochastically larger than $Z_{x_1} - x_1$ whenever $0 \leqslant x_1 < x_2$. The following result gives a direct connection between the stochastic monotonicity of Φ and the behaviour of q.

Lemma 2. *Suppose Z satisfies assumption $\mathcal{A}$. If Z has a non-increasing hazard rate, then the tail Markov chain generated by Z is stochastically monotone.*

Proof. First, it is simple to verify that

$$G(x) = \exp\left\{-\int_0^x q(t)\, \mathrm{d}t\right\}.$$

Thus, we can write

$$K(y|x) = 1 - \frac{G(x+y)}{G(x)} = 1 - \exp\left\{-\int_x^{x+y} q(t)\,\mathrm{d}t\right\}.$$

Now fix x_1, x_2 and y such that $0 \leqslant x_1 < x_2$ and $y > 0$. Clearly, $\int_{x_1}^{x_1+y} q(t)\,\mathrm{d}t \geqslant \int_{x_2}^{x_2+y} q(t)\,\mathrm{d}t$, and hence $K(y|x_1) \geqslant K(y|x_2)$. Thus, $K(y|x)$ is non-increasing in x for each fixed y. □

Remark 3. If we assume that f is continuous, the conclusion of Lemma 1 can be written: The tail Markov chain generated by Z is stochastically monotone if and only if Z has a non-increasing hazard rate. Indeed, $K(y|x)$ is non-increasing in x for each fixed y if and only if $\int_x^{x+y} q(t)\,\mathrm{d}t$ is non-increasing in x for each fixed y. Taking a derivative (q is continuous), we find that $K(y|x)$ is non-increasing in x for each fixed y if and only if $q(x+y) \leqslant q(x)$ for all $x > 0$ for each fixed y.

Now suppose all we can say regarding the monotonicity of q is that there exists an $M > 0$ such that $q(z)$ is non-increasing for all $z > M$. We now consider whether it is possible to find a $z^* \geqslant M$ and a density f^* such that the following four conditions hold:

1. f^* satisfies assumption $\mathcal{A}$.
2. f^* has non-increasing hazard rate.
3. $f(z^*) = f^*(z^*)$.
4. $\int_{z^*}^{\infty} f(z)\,\mathrm{d}z = \int_{z^*}^{\infty} f^*(z)\,\mathrm{d}z$.

If such an f^* exists, then the density

$$\tilde{f}(z) = \begin{cases} f^*(z) & \text{if } z < z^*, \\ f(z) & \text{if } z \geqslant z^*, \end{cases} \tag{6}$$

satisfies assumption $\mathcal{A}$, has non-decreasing hazard rate, *and* has exactly the same tail as f. We will now prove that the answer to the question above is 'yes' (as long as there exists an $r > 0$ such that $\mathrm{E}Z^r = \infty$). In fact, one can always find a Weibull density that does the job. Write the Weibull density as $w(z;\lambda,\alpha) = \lambda\,\alpha\,z^{\alpha-1}\exp\{-\lambda\,z^{\alpha}\}I(z>0)$, where λ, $\alpha > 0$. The hazard rate of the Weibull density is non-increasing whenever $\alpha \leqslant 1$.

Lemma 3. *Assume that Z satisfies assumption $\mathcal{A}$, $\mathrm{E}Z^r = \infty$ for some $r > 0$, and that there exists an $M > 0$ such that $q(z)$ is non-increasing for all $z > M$. Then there exists a $z^* \geqslant M$ and a density f^* such that* (1), (2), (3) *and* (4) *all hold.*

Proof. We simply demonstrate the existence of a Weibull density satisfying all the conditions. First, Barlow *et al.* (1963) show that

$$\mathrm{E}Z^r = \infty \Rightarrow \liminf_{z\to\infty} z\,q(z) \leqslant r.$$

Therefore,

$$\liminf_{z\to\infty} \frac{z\,q(z)}{-\log G(z)} = 0.$$

Thus, there exists a $z^* \geqslant M$ such that

$$\frac{z^* \, q(z^*)}{-\log G(z^*)} < 1.$$

Now fix z^* as above, and consider the following system of two equations and two unknowns:

$$w(z^*; \lambda, \alpha) = f(z^*),$$

$$\int_{z^*}^{\infty} w(z; \lambda, \alpha) \mathrm{d}z = G(z^*).$$

Solving for α and λ yields

$$\hat{\alpha} = \frac{z^* \, q(z^*)}{-\log G(z^*)} \quad \text{and} \quad \hat{\lambda} = [-\log G(z^*)](z^*)^{-[z^* \, q(z^*)/\{-\log G(z^*)\}]}.$$

Since $\hat{\alpha} < 1$ by construction, the Weibull density that is the solution has non-increasing hazard rate. $\square$

Example 2 (continued). Consider the IG(1, 1) density; that is, $f(z) = z^{-2} \exp\{-1/z\} I(z > 0)$. We know that $\mathrm{E}Z = \infty$. It is easy to show that the hazard rate, $q(z)$, is increasing for small z and decreasing for $z > 1$. Taking $z^* = 2$, the Weibull solution has $\lambda \doteq 0.526$ and $\alpha \doteq 0.826$. Figure 1 shows f and f^*.

3. The discrete analogue of Φ

Hobert and Schweinsberg (2002) studied a discrete analogue of Φ and one of their results will be used in the proof of Theorem 1. Suppose W is a discrete random variable with support $\mathbb{Z}^+$. Let $\Psi = (\Psi_n)_{n=0}^{\infty}$ be a Markov chain with state space $\mathbb{Z}^+$ and transition probabilities given by

$$p_{ij} := \Pr(\Psi_{n+1} = j | \Psi_n = i) = \frac{P(W = i + j)}{P(W \geqslant i)} \tag{7}$$

for all $i, j \in \mathbb{Z}^+$. The fact that $P(W = i + j) > 0$ for all $i, j \in \mathbb{Z}^+$ implies that Ψ is irreducible and aperiodic. Let $\pi_i = P(W \geqslant i)$ and note that $\pi_i \, p_{ij} = \pi_j \, p_{ji}$ for all $i, j \in \mathbb{Z}^+$. Thus, Ψ is *reversible* and the sequence $(\pi_i)_{i=0}^{\infty}$ is an invariant sequence for Ψ since

$$\sum_{i=0}^{\infty} \pi_i \, p_{ij} = \sum_{i=0}^{\infty} \pi_j \, p_{ji} = \pi_j$$

for all $j \in \mathbb{Z}^+$. It follows that if $\sum_{i=0}^{\infty} \pi_i < \infty$, then the chain is positive recurrent, and if $\sum_{i=0}^{\infty} \pi_i = \infty$, then the chain is either null recurrent or transient. Moreover, since $\sum_{i=0}^{\infty} \pi_i = 1 + \mathrm{E}W$, the Markov chain Ψ is positive recurrent if and only if $\mathrm{E}W < \infty$. The following result is due to Hobert and Schweinsberg (2002).

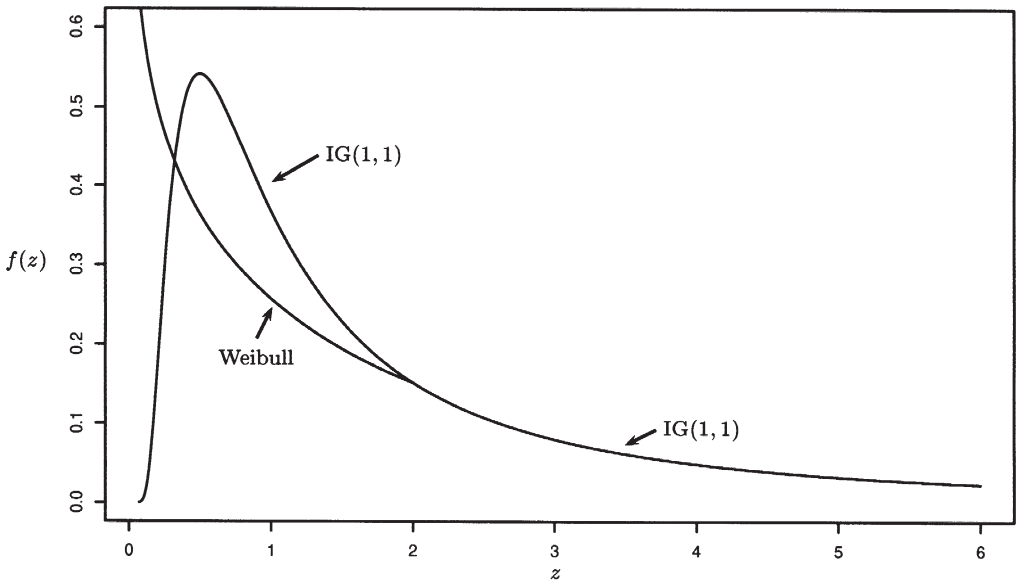

Figure 1. The IG(1,1) density between 0 and 6 and the Weibull density with $\lambda \doteq 0.526$ and $\alpha \doteq 0.826$ between 0 and 2. The densities are equal at the point 2 and the area under the curve between 0 and 2 is the same for the two densities.

Theorem 3. *If* $\sum_{i=1}^{\infty}[i^3 P(W=i)]^{-1} < \infty$, *then the Markov chain* Ψ *is transient. If* $\sum_{i=1}^{\infty}[i^2 P(W \geqslant i)]^{-1} = \infty$, *then* Ψ *is recurrent.*

Theorem 1 is the continuous analogue of Theorem 3. It is important to note, however, that the techniques used to prove Theorem 3 are based on connections between reversible Markov chains and electrical networks and consequently are specific to Markov chains on countable state spaces. Thus, while Φ and Ψ are quite similar in structure, the methods used to prove Hobert and Schweinsberg's (2002) result *cannot* be applied to Φ.

4. The main result

This section contains the proof of Theorem 1. The proof has two parts. The first part is a coupling argument that requires Z to be *stochastically monotone*. In this part of the argument, we assume that q is non-increasing on all of $\mathbb{R}^+$. The second part involves relaxing the assumption that q is non-increasing on all of $\mathbb{R}^+$ and is based on a stochastic comparison technique (Meyn and Tweedie, 1993, p. 220).

Proof of Theorem 1. We first show that the result is true under the more restrictive assumption that q is non-increasing on all of $\mathbb{R}^+$. Define a $\mathbb{Z}^+$-valued random variable W such that

$$P(W = i) = P(i < Z \leqslant i + 1)$$

for all $i \in \mathbb{Z}^+$. Define another $\mathbb{Z}^+$-valued random variable W' by

$$P(W' = i) = \frac{P(i + 1 < Z \leqslant i + 2)}{P(Z > 1)}$$

for all $i \in \mathbb{Z}^+$. Now, for fixed $i \in \mathbb{Z}^+$, let W_i be a random variable with support $\{i, i + 1, \ldots\}$ and probabilities proportional to those of W. Define W'_i similarly.

We now construct three coupled Markov chains, which we denote by Ψ, Φ and Ψ'. Let $U_0, U_1, U_2, \ldots$ be a sequence of independent and identically distributed Uniform(0, 1) random variables. Fix a real number $s \geqslant 0$. Let $\Phi_0 = s$, and then let Ψ_0 and Ψ'_0 be non-negative integers such that $\Psi_0 \leqslant \Phi_0 \leqslant \Psi'_0 + 1$. Given $\Psi_n = i$, $\Phi_n = x$ and $\Psi'_n = i'$, we define

$$\Psi_{n+1} = \inf\{j \in \mathbb{Z}^+ : P(W_i - i \leqslant j) \geqslant U_n\},$$

$$\Phi_{n+1} = \inf\{y \in [0, \infty) : P(Z_x - x \leqslant y) \geqslant U_n\},$$

$$\Psi'_{n+1} = \inf\{j \in \mathbb{Z}^+ : P(W'_{i'} - i' \leqslant j) \geqslant U_n\}.$$

Note that Ψ is a Markov chain with transition probabilities given by (7), and Ψ' is a Markov chain with transition probabilities given by (7) with W' in place of W. Also, Φ is a Markov chain whose transition densities are given by (1).

We now prove by induction that $\Psi_n \leqslant \Phi_n \leqslant \Psi'_n + 1$ for all $n \in \mathbb{Z}^+$. Suppose we have $\Psi_n \leqslant \Phi_n \leqslant \Psi'_n + 1$ for some n. If $i, j \in \mathbb{Z}^+$ and $j \geqslant 1$, then

$$P(Z_i - i \leqslant j) = \frac{P(i < Z \leqslant i + j)}{P(Z > i)} = \frac{P(i \leqslant W \leqslant i + j - 1)}{P(W \geqslant i)} = P(W_i - i \leqslant j - 1)$$

and

$$P(Z_{i+1} - (i + 1) \leqslant j + 1) = \frac{P(i + 1 < Z \leqslant i + j + 2)}{P(Z > i + 1)} = \frac{P(i \leqslant W' \leqslant i + j)}{P(W' \geqslant i)} = P(W'_i - i \leqslant j).$$

If $\Psi_n = i$ and $\Psi_{n+1} = j \geqslant 1$, then $P(W_i - i \leqslant j - 1) < U_n$. If we also have $\Phi_n = x$, then our assumption about the hazard rate of Z implies that $Z_x - x$ is stochastically larger than $Z_i - i$ and hence

$$P(Z_x - x \leqslant j) \leqslant P(Z_i - i \leqslant j) = P(W_i - i \leqslant j - 1) < U_n,$$

which means $\Phi_{n+1} \geqslant j = \Psi_{n+1}$. Likewise, if $\Psi'_n = i'$ and $\Psi'_{n+1} = j'$, then $P(W'_{i'} - i' \leqslant j') \geqslant U_n$. Therefore, if we also have $\Phi_n = x$, then

$$P(Z_x - x \leqslant j' + 1) \geqslant P(Z_{i'+1} - (i' + 1) \leqslant j' + 1) = P(W'_{i'} - i' \leqslant j') \geqslant U_n,$$

which means $\Phi_{n+1} \leqslant j' + 1 = \Psi'_{n+1} + 1$. Thus, by induction, $\Psi_n \leqslant \Phi_n \leqslant \Psi'_n + 1$ for all $n \in \mathbb{Z}^+$, as claimed.

Suppose (3) holds. Then, using Jensen's inequality, we have

$$\sum_{i=1}^{\infty} \frac{1}{i^3 P(W=i)} = \sum_{i=1}^{\infty} \frac{1}{i^3 \displaystyle\int_i^{i+1} f(z)\mathrm{d}z} \leqslant \sum_{i=1}^{\infty} \frac{1}{i^3} \int_i^{i+1} \frac{1}{f(z)}\mathrm{d}z$$

$$\leqslant 8 \sum_{i=1}^{\infty} \int_i^{i+1} \frac{1}{z^3 f(z)}\mathrm{d}z = 8\int_1^{\infty} \frac{1}{z^3 f(z)}\mathrm{d}z < \infty.$$

Thus, by Theorem 3, the chain Ψ is transient. Fix a positive real number K, and define $U(s, K) = \sum_{n=0}^{\infty} P(\Phi_n \leqslant K)$. (Recall that $\Phi_0 = s$.) Since $\Psi_n \leqslant \Phi_n$ for all n and Ψ is transient, we have $U(s, K) \leqslant \sum_{n=0}^{\infty} P(\Psi_n \leqslant K) < \infty$. It follows that Φ is transient.

Now suppose (2) holds. Then

$$\sum_{i=1}^{\infty} \frac{1}{i^2 P(W' \geqslant i)} = \sum_{i=1}^{\infty} \frac{G(1)}{i^2 G(i+1)} \geqslant G(1)\int_1^{\infty} \frac{1}{z^2 G(z)}\mathrm{d}z = \infty.$$

Therefore, Theorem 3 implies that Ψ' is recurrent, which means $\Psi'_n = 0$ infinitely often. Thus, $\Phi_n \in [0, 1]$ infinitely often, and it follows from Theorem 8.3.5 of Meyn and Tweedie (1993, p. 187) that Φ is null recurrent. (We are using the fact that $[0, 1]$ is a petite set. Since $[0, 1]$ is compact, this follows from the fact that Φ is a Feller chain.)

We have so far shown that the result holds under the assumption that q is non-increasing on all of $\mathbb{R}^+$. We now relax this assumption and suppose only that there exists an $M > 0$ such that $q(z)$ is non-increasing for $z > M$. Lemma 3 implies the existence of $\tilde{f}$ defined in (6). Note that $\tilde{f}$ satisfies assumption $\mathcal{A}$, has non-increasing hazard rate on all of $\mathbb{R}^+$ and is identical to f on $[z^*, \infty)$. Let $\tilde{G}(z) = \int_z^{\infty} \tilde{f}(t)\,\mathrm{d}t$. Define $\tilde{\Phi}$ to be the tail Markov chain generated by $\tilde{f}$ and let $\tilde{k}(y|x)$ be the corresponding Markov transition density; that is, $\tilde{k}(y|x) = \tilde{f}(y+x)/\tilde{G}(x)$ for $x, y \in [0, \infty)$. By construction, $k(y|x) = \tilde{k}(y|x)$ for all $y \geqslant 0$ whenever $x \geqslant z^*$. Put $C = [0, z^*]$ and define

$$\tau_C = \min\{n \geqslant 1 : \Phi_n \in C\} \quad \text{and} \quad \tilde{\tau}_C = \min\{n \geqslant 1 : \tilde{\Phi}_n \in C\}.$$

Then for any $x \in C^c$ and any $n \in \{2, 3, \ldots\}$,

$$\begin{aligned}\Pr(\tau_C \geqslant n|\Phi_0 = x) &= \int_{C^c} \cdots \int_{C^c} k(t_{n-1}|t_{n-2}) \cdots k(t_1|x)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \\ &= \int_{C^c} \cdots \int_{C^c} \tilde{k}(t_{n-1}|t_{n-2}) \cdots \tilde{k}(t_1|x)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} \qquad (8) \\ &= \Pr(\tilde{\tau}_C \geqslant n|\tilde{\Phi}_0 = x).\end{aligned}$$

Meyn and Tweedie (1993, p. 220) show that from (8) we may conclude that Φ is null recurrent if and only if $\tilde{\Phi}$ is null recurrent. (Here again we are using the fact that C is a petite set.)

Assume (2) holds. Clearly, (2) implies that $\int_1^{\infty}[z^2\,\tilde{G}(z)]^{-1}\mathrm{d}z = \infty$. Now since the hazard rate of $\tilde{f}$ is non-increasing on all of $\mathbb{R}^+$, we may conclude that $\tilde{\Phi}$ is null recurrent, and this in turn implies that Φ is null recurrent. A similar argument works for the transient case.□

An obvious question regarding Theorem 1 is whether it is possible to find a Z such that neither (2) nor (3) holds. We will now show that either (2) or (3) must hold when $\liminf_{z\to\infty} z\, q(z) > 0$. We will then give an example in which $\liminf_{z\to\infty} z\, q(z) = 0$ and neither (2) nor (3) holds. Define $\liminf_{z\to\infty} z\, q(z) = \underline{L}$ and $\limsup_{z\to\infty} z\, q(z) = \bar{L}$. The next result gives two relationships between these limits and the integrals in (2) and (3).

Lemma 4. *Assume that Z satisfies assumption $\mathcal{A}$.*

(i) *If $\underline{L} > 0$, then*

$$\int_1^\infty \frac{1}{z^2 G(z)} \mathrm{d}z < \infty \Rightarrow \int_1^\infty \frac{1}{z^3 f(z)} \mathrm{d}z < \infty.$$

(ii) *If $\bar{L} < 1$, then*

$$\int_1^\infty \frac{1}{z^2 G(z)} \mathrm{d}z < \infty.$$

Proof. (i) Let $0 < L < \underline{L}$. There exists $0 < A < \infty$ such that $z\, q(z) > L$ for all $z > A$. Thus,

$$\int_A^\infty \frac{1}{z^3 f(z)} \mathrm{d}z = \int_A^\infty \frac{1}{z^2 G(z)\, z\, q(z)} \mathrm{d}z < \frac{1}{L} \int_A^\infty \frac{1}{z^2 G(z)} \mathrm{d}z < \infty.$$

(ii) Let $\bar{L} < L < 1$. There exists $0 < B < \infty$ such that $z\, q(z) < L$ for all $z > B$. Thus, $q(z) < L/z$ for all $z > B$. Integration of both sides yields

$$\int_B^z q(t)\, \mathrm{d}t < L \log\left(\frac{z}{B}\right)$$

for all $z > B$. Exponentiating and rearranging yields

$$\exp\left\{\int_0^z q(t)\, \mathrm{d}t\right\} < \left(\frac{z}{B}\right)^L \exp\left\{\int_0^B q(t)\, \mathrm{d}t\right\}$$

for all $z > B$. Thus, for all $z > B$, we have

$$\frac{1}{z^2\, G(z)} < c\, \frac{1}{z^{2-L}},$$

where c is a constant that does not depend on z. Finally, since $2 - L > 1$,

$$\int_B^\infty \frac{1}{z^2\, G(z)}\, \mathrm{d}z < c \int_B^\infty \frac{1}{z^{2-L}}\, \mathrm{d}z < \infty.$$

□

Remark 4. Part (i) shows that if $\liminf_{z\to\infty} z\, q(z) > 0$, then one of (2) or (3) must hold.

Example 6. This example shows that it is possible that neither (2) nor (3) holds, even if the other conditions of Theorem 1 are satisfied. For all non-negative integers n, let $a_n = 2^{2^n}$. Note that $a_{n+1} = a_n^2$. For positive integers n, let $r_n = (2^{n-3} \log 2)/(a_n - a_{n-1})$. Next, define the function q by setting $q(z) = r_n$ for $z \in [a_{n-1}, a_n)$ and $q(z) = r_1$ for $z \in (0, a_0)$. Since the sequence $(r_n)_{n=1}^{\infty}$ is decreasing, $q(z)$ is a non-increasing function of z on $(0, \infty)$. Define $G(z) = \exp\{-\int_0^z q(x)\,\mathrm{d}x\}$ and $f(z) = q(z)G(z)$ for $z > 0$. Since $\int_0^\infty q(z)\,\mathrm{d}z = \infty$, the function f is a density function. Since f is lower semicontinuous and positive on $(0, \infty)$, we see that f satisfies assumption $\mathcal{A}$. Also, note that if Z is a random variable with density f, then G is the survival function of Z and q is the hazard rate.

For $n \geqslant 2$,

$$G(a_n) = \exp\left\{-a_1 r_1 - \sum_{i=2}^{n}(a_i - a_{i-1})r_i\right\} = \exp\left\{-a_1 r_1 - \sum_{i=2}^{n} 2^{i-3}\log 2\right\}$$

$$= \exp\left\{-\frac{1}{2}\log 2 - (\log 2)\left(2^{n-2} - \frac{1}{2}\right)\right\} = 2^{-2^{n-2}} = a_{n-2}^{-1}.$$

Since $G(z)$ is a decreasing function of z, we have

$$\mathrm{E}[Z] \geqslant \sum_{n=2}^{\infty}\int_{a_{n-1}}^{a_n} G(z)\,\mathrm{d}z \geqslant \sum_{n=2}^{\infty} a_{n-2}^{-1}(a_n - a_{n-1}) = \sum_{n=2}^{\infty} a_{n-2}(a_{n-2}^2 - 1) \geqslant \sum_{n=2}^{\infty} a_{n-2} = \infty.$$

Thus, all of the hypotheses of Theorem 1 are satisfied. Now since $f(z) \geqslant G(a_n)r_n = a_{n-2}^{-1}r_n$ for all z such that $a_{n-1} \leqslant z < a_n$, we have

$$\int_1^\infty \frac{1}{z^3 f(z)}\,\mathrm{d}z \geqslant \sum_{n=2}^{\infty}\int_{a_{n-1}}^{a_n}\frac{1}{z^3 f(z)}\,\mathrm{d}z \geqslant \sum_{n=2}^{\infty}\frac{1}{G(a_{n-1})r_n}\int_{a_{n-1}}^{a_n}\frac{1}{z^3}\,\mathrm{d}z$$

$$= \sum_{n=2}^{\infty}\frac{a_{n-3}(a_n - a_{n-1})}{2^{n-3}\log 2}\left[\frac{1}{2a_{n-1}^2} - \frac{1}{2a_n^2}\right]$$

$$= \sum_{n=2}^{\infty}\frac{4a_n^{1/8}(a_n - a_n^{1/2})}{\log a_n}\left[\frac{1}{a_n} - \frac{1}{a_n^2}\right] = \infty,$$

so (3) does not hold. Furthermore, letting $c = \int_1^4 [z^2 G(z)]^{-1}\,\mathrm{d}z$, we obtain

$$\int_1^\infty \frac{1}{z^2 G(z)}\,\mathrm{d}z = c + \sum_{n=2}^{\infty}\int_{a_{n-1}}^{a_n}\frac{1}{z^2 G(z)}\,\mathrm{d}z \leqslant c + \sum_{n=2}^{\infty}\frac{1}{G(a_n)}\int_{a_{n-1}}^{a_n}\frac{1}{z^2}\,\mathrm{d}z$$

$$\leqslant c + \sum_{n=2}^{\infty}\frac{a_{n-2}}{a_{n-1}} = c + \sum_{n=2}^{\infty}\frac{1}{a_{n-2}} < \infty.$$

Thus, (2) also fails to hold. Note that

$$\lim_{n\to\infty} a_{n-1}q(a_{n-1}) = \lim_{n\to\infty} a_{n-1}r_n = \lim_{n\to\infty}\frac{a_{n-1}(2^{n-3}\log 2)}{a_n - a_{n-1}} = 0,$$

so $\liminf_{z\to\infty} z\,q(z) = 0$, as it must.

Remark 5. If $\underline{L} > 0$ and $\bar{L} < 1$, then Lemma 4 implies that Φ is transient. Furthermore, if

$\underline{L} > 1$, then $\mathrm{E}Z < \infty$ (Barlow *et al.*, 1963), which means that Φ is positive recurrent. It is tempting to conjecture that if $L = \lim_{z\to\infty} z\,q(z)$ exists, then Φ is positive recurrent, null recurrent or transient as L is greater than 1, equal to 1, or less than 1. However, the next example shows that Φ can be transient when $L = 1$.

Example 7. Consider the density

$$f(z) = \frac{C[\log(z+1)]^2}{(z+1)^2} I(z > 0),$$

where C is a constant. Note that f is lower semicontinuous and positive on $(0, \infty)$, and thus satisfies assumption $\mathcal{A}$. Also, one can check that

$$\frac{\partial^2}{\partial z^2} \log f(z) = \frac{2}{(z+1)^2}\left(1 - \frac{1}{\log(z+1)} - \frac{1}{[\log(z+1)]^2}\right),$$

which is positive for sufficiently large z. Therefore, by Lemma 1, the function $q(z)$ is non-increasing for sufficiently large z. Note that

$$\int_0^\infty z f(z)\mathrm{d}z = \int_0^\infty \frac{Cz[\log(z+1)]^2}{(z+1)^2}\mathrm{d}z = \infty,$$

and

$$\int_1^\infty \frac{1}{z^3 f(z)}\mathrm{d}z = \int_1^\infty \frac{(z+1)^2}{Cz^3[\log(z+1)]^2}\mathrm{d}z < \infty.$$

Therefore, by Theorem 1, the tail Markov chain generated by f is transient. It remains to show that $L = 1$. Changing variables from x to $u = 1/(x+1)$, we have

$$G(z) = C\int_z^\infty \frac{[\log(x+1)]^2}{(x+1)^2}\mathrm{d}x = \frac{C}{z+1}\{[\log(z+1)]^2 + 2\log(z+1) + 2\}.$$

Therefore,

$$z\,q(z) = \frac{zf(z)}{G(z)} = \frac{z[\log(z+1)]^2}{(z+1)\{[\log(z+1)]^2 + 2\log(z+1) + 2\}}$$

and hence $\lim_{z\to\infty} z\,q(z) = 1$, as claimed.

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