Bernoulli **24**(4B), 2018, 3751–3790
https://doi.org/10.3150/17-BEJ977

The class of multivariate max-id copulas with ℓ_1-norm symmetric exponent measure

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Members of the well-known family of bivariate Galambos copulas can be expressed in a closed form in terms of the univariate Fréchet distribution. This formula extends to any dimension and can be used to define a whole new class of tractable multivariate copulas that are generated by suitable univariate distributions. This paper gives necessary and sufficient conditions on the underlying univariate distribution which ensure that the resulting copula exists. It is also shown that these new copulas are in fact dependence structures of certain max-id distributions with ℓ_1-norm symmetric exponent measure. The basic dependence properties of this new class of multivariate exchangeable copulas is investigated, and an efficient algorithm is provided for generating observations from distributions in this class.

Keywords: Clayton copula; completely monotone function; exponent measure; Galambos copula; Laplace transform; ℓ_1-norm symmetric max-id distributions

1. Introduction

Copula models are ubiquitous in multivariate data analysis and the topic has been the object of several books and innumerable articles in the past 30 years; see, for example, [13] for a review. The popularity of copula models stems from the flexibility they provide in capturing the behavior of any random vector $(X_1, \dots, X_d) \in \mathbb{R}^d$ by coupling parametric distributions $F_1, \dots, F_d$ for its marginals with a copula C, so that, for all $x_1, \dots, x_d \in \mathbb{R}$,

$$\Pr(X_1 \le x_1, \dots, X_d \le x_d) = C\{F_1(x_1), \dots, F_d(x_d)\}.$$

Common choices of parametric families for C include Archimedean, Archimax, elliptical, and extreme-value copulas, as described, for example, in [3,7,25]. Even greater flexibility can be achieved by resorting to vine, hierarchical, and factor copula constructions; see, for example, [15, 16,20]. Nevertheless, there is still a need to expand the supply of multivariate copula structures, particularly with models that are tractable, interpretable, and relatively easy to simulate.

The purpose of this paper is to propose, and explore the properties of, a new class of copulas having these desirable features. Each member of this new class is generated by a continuous univariate distribution function F with support $[0, x_F]$ for some $x_F \in (0, \infty]$. In the bivariate case, the copula C_F generated by F is defined, for all $u_1, u_2 \in (0,1)$, by

$$C_F(u_1,u_2) = \frac{u_1 u_2}{F\{F^{-1}(u_1) + F^{-1}(u_2)\}}, \tag{1}$$

where F^{-1} is the generalized inverse of F defined, at all $u \in (0,1]$, by $F^{-1}(u) = \inf\{t \in \mathbb{R} : F(t) \ge u\}$, and at $u = 0$ by $F^{-1}(0) = \sup\{t \in \mathbb{R} : F(t) = 0\}$. We propose to call members of this class *reciprocal Archimedean copulas* because in dimension $d = 2$, their form is reminiscent of Archimedean copulas [6,24] defined, for all $u_1, u_2 \in (0,1)$, by

$$C_\psi(u_1,u_2) = \psi\{\psi^{-1}(u_1) + \psi^{-1}(u_2)\} \tag{2}$$

for some suitable choice of generator $\psi : [0,\infty) \to [0,1]$. Bivariate reciprocal Archimedean copulas were first explored briefly by Khoudraji [14] who observed, among other things, that the Galambos and Clayton copulas with positive dependence are included in this class.

To define a reciprocal Archimedean copula with generator F in any dimension $d \ge 2$, let

$$\mathcal{P}_{d,o} = \big\{A \subseteq \{1,\dots,d\} : A \neq \varnothing \text{ and } |A| \text{ is odd}\big\},$$
$$\mathcal{P}_{d,e} = \big\{A \subseteq \{1,\dots,d\} : A \neq \varnothing \text{ and } |A| \text{ is even}\big\},$$

and set, for all $u_1, \dots, u_d \in (0,1)$,

$$C_F(u_1,\dots,u_d) = \prod_{A \in \mathcal{P}_{d,o}} F\bigg\{\sum_{k \in A} F^{-1}(u_k)\bigg\} \Big/ \prod_{A \in \mathcal{P}_{d,e}} F\bigg\{\sum_{k \in A} F^{-1}(u_k)\bigg\}. \tag{3}$$

As the numerator and denominator of (3) have 2^{d-1} and $2^{d-1} - 1$ factors respectively, the expression for C_F is more involved than that of a multivariate Archimedean copula with generator ψ, viz.

$$C_\psi(u_1,\dots,u_d) = \psi\{\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)\}. \tag{4}$$

Nonetheless there turn out to be surprising parallels between these two classes of copulas.

First, it will be shown here that a continuous distribution function F with support $[0, x_F]$ generates a reciprocal Archimedean copula in dimension d if and only if $\Lambda = -\ln(F)$ is d-monotone. This resembles the result that C_ψ given in (4) is a copula if and only if ψ is a d-monotone Archimedean generator [21,24].

Second, and more importantly, it will be seen that reciprocal Archimedean copulas have a stochastic representation that parallels that of Archimedean copulas. As shown in [24], the function given, for all $x_1, \dots, x_d \in [0,\infty)$, by

$$C_\psi\{\psi(x_1),\dots,\psi(x_d)\} = \psi(x_1 + \cdots + x_d)$$

is the survival function of a d-variate ℓ_1-norm symmetric distribution, whose radial variable is in one-to-one relationship with ψ. The surprising result established here is that the function defined, for all $x_1, \dots, x_d \in [0, \infty)$, by

$$H_F(x_1, \dots, x_d) = C_F\{F(x_1), \dots, F(x_d)\} = \prod_{A \in \mathcal{P}_{d,o}} F\Big(\sum_{i \in A} x_i\Big) \Big/ \prod_{A \in \mathcal{P}_{d,e}} F\Big(\sum_{i \in A} x_i\Big) \tag{5}$$

is the cumulative distribution function of a *max infinitely divisible* (max-id) distribution whose exponent measure μ has an ℓ_1-norm symmetric structure. The radial measure ν of μ, which plays an analogous role to the radial variable of an ℓ_1-norm symmetric distribution, is in one-to-one relationship with the reciprocal Archimedean generator F. Furthermore, in analogy with the Archimedean case, ν can be computed explicitly using the inverse Williamson d-transform of Λ.

Third, just as ψ must be completely monotone for (4) to be an Archimedean copula in every dimension, $\Lambda = -\ln(F)$ must be completely monotone for F to generate a reciprocal Archimedean copula for all $d \geq 2$. In this special case, which was briefly investigated in [19], additional parallels can be drawn between reciprocal Archimedean copulas and the dependence structures of multiplicative hazard models [22].

The layout of this article is as follows. Preliminary observations are given in Section 2, consolidating the initial investigations in [14,19]. The new, key insight offered in Section 3 is that reciprocal Archimedean copulas are precisely the dependence structures of max-id distributions with ℓ_1-norm symmetric exponent measure. It follows that C_F is a copula in a fixed dimension d if and only if $\Lambda = -\ln(F)$ is d-monotone on $(0, \infty)$. A geometric insight into the structure of ℓ_1-norm symmetric exponent measures is provided in Section 4, which leads, in Section 5, to a stochastic representation for reciprocal Archimedean copulas in terms of Poisson point processes, and to an effective simulation algorithm. Section 6 presents a number of new examples and illustrations. The case of completely monotone Λ is briefly discussed in Section 7, and Section 8 sketches a roadmap of the case where $F^{-1}(0) > 0$, which includes, among others, Ali–Mikhail–Haq copulas with positive dependence. Section 9 contains brief concluding remarks. Lengthy proofs are relegated to a series of Appendices.

In what follows, $\boldsymbol{x} = (x_1, \dots, x_d)$ and $\boldsymbol{y} = (y_1, \dots, y_d)$ represent vectors in $\mathbb{R}^d$; furthermore, $\boldsymbol{0}$ stands for the origin in $\mathbb{R}^d$ and $\boldsymbol{\infty} = (\infty, \dots, \infty)$. Unless stated otherwise, all expressions such as $\boldsymbol{x} + \boldsymbol{y}$, $\max(\boldsymbol{x}, \boldsymbol{y})$ or $\boldsymbol{x} \leq \boldsymbol{y}$ are understood as component-wise operations. The set $(\boldsymbol{x}, \boldsymbol{y}]$ refers to $(x_1, y_1] \times \cdots \times (x_d, y_d]$ and similarly for $[-\boldsymbol{\infty}, \boldsymbol{x})$, $[\boldsymbol{0}, \boldsymbol{\infty}]$, etc. Furthermore, $\|\boldsymbol{x}\|_1 = |x_1| + \cdots + |x_d|$ denotes the ℓ_1-norm of $\boldsymbol{x}$. Finally, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{R}^d_+ = [0, \infty)^d$.

2. Elementary properties

This section surveys some of the basic properties of the class of reciprocal Archimedean copulas. For any given integer $d \geq 2$, let $\mathcal{F}_d$ be the set of continuous distribution functions F with support $[0, x_F]$ for some $x_F \in (0, \infty]$ such that C_F defined in (3) is a *bona fide* d-dimensional copula. Let also $\mathcal{C}_d = \{C_F : F \in \mathcal{F}_d\}$ be the set of all reciprocal Archimedean copulas in dimension d.

First, note that it is immediate from (3) that any reciprocal Archimedean copula C_F is exchangeable, that is, for any permutation π of the integers $1, \dots, d$ and any $u_1, \dots, u_d \in [0, 1]$,

$C_F(u_1,\ldots,u_d) = C_F(u_{\pi(1)},\ldots,u_{\pi(d)})$. Furthermore, all lower-dimensional margins of $C_F \in \mathcal{C}_d$ are again reciprocal Archimedean with the same generator F. Consequently, $\mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots$.

Observe that the generator of a reciprocal Archimedean copula is not unique. However, as stated below and proved in Appendix A, all generators must be members of the same scale family. Reciprocal Archimedean generators could thus be made unique by imposing an additional scaling condition, such as $F(1) = 1/2$; this option is not considered here, however.

Lemma 1. *Two distributions $F, G \in \mathcal{F}_d$ generate the same reciprocal Archimedean copula $C_F = C_G$ if and only if there exists $\kappa \in (0,\infty)$ such that, for all $t \in \mathbb{R}$, $G(t) = F(\kappa t)$.*

Distributions $F \in \mathcal{F}_d$ are most easily characterized in terms of higher-order monotonicity of the map $\Lambda : (0,\infty) \to [0,\infty)$ defined, for all $t \in (0,\infty)$, by

$$\Lambda(t) = -\ln\{F(t)\}.$$

Clearly, Λ is non-negative and non-increasing; moreover, $\Lambda(t) \to \infty$ as $t \to 0$ and $\Lambda(t) \to 0$ as $t \to \infty$. It was further observed in [14] that if $F \in \mathcal{F}_2$ has derivative f on $(0, x_F)$, then the reverse hazard rate $r = f/F$ must be non-increasing, which implies that Λ must be convex. More generally, it can be shown from first principles that if $F \in \mathcal{F}_d$, then Λ is d-monotone. This is stated formally below and proved in Appendix A, where a formal definition of d-monotonicity is also recalled.

Proposition 1. *If $F \in \mathcal{F}_d$, then $\Lambda = -\ln(F)$ is d-monotone. Furthermore, $\Lambda(t) \to \infty$ as $t \to 0$ and $\Lambda(t) \to 0$ as $t \to \infty$.*

An easy but important consequence of this result is that in the bivariate case, the distribution H_F given by (5) is TP_2, that is, totally positive of order 2, as proved in Appendix A.

Corollary 1. *When $d = 2$, H_F is TP_2, that is, for all $x_1, x_2, y_1, y_2 \in (0,\infty)$,*

$$x_1 < y_1 \quad \text{and} \quad x_2 < y_2 \quad \Rightarrow \quad H_F(x_1,x_2)H_F(y_1,y_2) \geq H_F(x_1,y_2)H_F(y_1,x_2). \tag{6}$$

Accordingly, all bivariate reciprocal Archimedean copulas are TP_2.

The TP_2 concept is a very strong notion of positive dependence originally introduced in [18]. In particular, it implies that any measure of concordance in the sense of [27] is non-negative. For completeness, we record below simple formulas given in [19] for three such classical dependence measures.

Lemma 2. *Let $F \in \mathcal{F}_2$ with density f and reverse hazard rate $r = f/F$. If C_F is defined by (1), then Blomqvist's beta and Spearman's rho are respectively given by*

$$\beta(C_F) = -1 + 1/F\{2F^{-1}(1/2)\},$$
$$\rho(C_F) = 3\int_0^\infty \int_0^\infty H^2(x_1,x_2)\{r(x_1)+r(x_2)\}f(x_1+x_2)\,dx_2\,dx_1.$$

Furthermore, if f is differentiable, then Kendall's tau can be expressed as

$$\tau(C_F) = -2\int_0^\infty \int_0^\infty H^2(x_1, x_2) r'(x_1 + x_2)\, dx_2\, dx_1.$$

Next, the tail dependence coefficients [12] of bivariate reciprocal Archimedean copulas are reported here for the first time. To this end, recall that a function $g : (0, \infty) \to (0, \infty)$ is regularly varying at infinity of degree $\eta \in \mathbb{R}$, denoted $g \in RV_\eta$, if and only if

$$\forall_{x\in(0,\infty)} \qquad \lim_{t\to\infty} \frac{1 - g(xt)}{1 - g(t)} = x^\eta.$$

Recall also that $g : (0, \infty) \to (0, \infty)$ is called rapidly varying at infinity if

$$\lim_{t\to\infty} \frac{1 - g(xt)}{1 - g(t)} = \begin{cases} 0 & \text{if } x \in (1, \infty), \\ \infty & \text{if } x \in (0, 1). \end{cases}$$

In what follows, $\bar{F} = 1 - F$ denotes the survival function corresponding to F, and F^* refers to the survival function defined, for all $t \in (0, \infty)$, by $F^*(t) = F(1/t)$.

Lemma 3. *Let C_F be a bivariate reciprocal Archimedean copula with generator $F \in \mathcal{F}_2$.*

(i) *If $\bar{F} \in RV_{-\alpha}$ for some $\alpha \in (0, \infty)$, then the upper tail dependence coefficient of C_F is*

$$\lambda_U(C_F) = 2 - \lim_{u\uparrow 1} \frac{1 - C_F(u, u)}{1 - u} = 2^{-\alpha}$$

while $\lambda_U(C_F) = 0$ either when $x_F < \infty$ or when $x_F = \infty$ and $\bar{F}$ is rapidly varying.

(ii) *If $F^* \in RV_{-\beta}$ for some $\beta \in (0, \infty)$, then the lower tail dependence coefficient of C_F is*

$$\lambda_L(C_F) = \lim_{u\downarrow 0} \frac{C_F(u, u)}{u} = 2^{-\beta},$$

while $\lambda_L(C_F) = 0$ whenever F^* is rapidly varying.

Remark 1. The results in Lemma 3 are reminiscent of the fact that the coefficients of lower and upper tail dependence of a bivariate Archimedean copula are related to the regular variation of its generator ψ and of the function ψ^* defined, for all $t \in (0, \infty)$, by $\psi^*(t) = 1 - \psi(1/t)$; see, for example, [4,17]. It is further interesting to note that if Z is a random variable with distribution F, Lemma 3 and classical univariate extreme-value theory [26] imply that $\lambda_U(C_F) = 2^{-\alpha}$ holds if Z is in the maximum domain of attraction (MDA) of the Fréchet distribution Φ_α with parameter $\alpha \in (0, \infty)$, while if Z is in the Weibull or the Gumbel domain of attraction, $\lambda_U(C_F) = 0$. Similarly, $\lambda_L(C_F) = 2^{-\beta}$ if $1/Z$ is in the maximum domain of attraction of the Fréchet distribution Φ_β. If $1/Z$ is in the Gumbel domain, $\lambda_L(C_F) = 0$. Note that the conditions on the support of F imply that the upper endpoint of the distribution of $1/Z$ is infinity, and hence $1/Z$ cannot be in the Weibull domain.

Using Lemma 3, one can easily recover the well-known expressions for the tail dependence coefficients of the classical bivariate Galambos and Clayton copulas with positive dependence, both of which belong to the class $\mathcal{C}_2$ [14].

Example 1. The reciprocal Archimedean generator of the bivariate Galambos copula with parameter $\theta \in (0, \infty)$ is given, for all $t \in (0, \infty)$, by

$$F_G(t) = \Phi_{1/\theta}(t) = e^{-t^{-1/\theta}},$$

that is, the Fréchet distribution $\Phi_{1/\theta}$ with parameter $1/\theta$. It is easily seen that $\bar{F}_G \in RV_{-1/\theta}$ and that F_G^* is rapidly varying, implying that $\lambda_L = 0$ and $\lambda_U = 2^{-1/\theta}$.

Example 2. The reciprocal Archimedean generator of the bivariate Clayton copula with parameter $\theta \in (0, \infty)$ is given, for all $t \in (0, \infty)$, by

$$F_C(t) = (1 - e^{-t})^{1/\theta},$$

that is, a Lehmann alternative to the unit exponential distribution. Using L'Hospital's rule, one finds $F_C^* \in RV_{-1/\theta}$ and that $\bar{F}_C$ is rapidly varying. Hence, $\lambda_L = 2^{-1/\theta}$ and $\lambda_U = 0$.

As a final general comment, note that the independence (or product) copula defined, for all $u_1, \ldots, u_d \in [0, 1]$, by $\Pi(u_1, \ldots, u_d) = u_1 \times \cdots \times u_d$, is not a member of the class $\mathcal{C}_d$. It may represent a limiting case, however, as when $\theta \to 0$ in the bivariate Galambos or Clayton families discussed above. The proof that $\Pi \notin \mathcal{C}_F$ is by contradiction. If $C_F = \Pi$ held for some $F \in \mathcal{F}_d$, it would follow from (5) that, for all $x_1, x_2 \in (0, \infty)$, $F(x_1)F(x_2) = F(x_1)F(x_2)/F(x_1 + x_2)$ and consequently $F(x_1 + x_2) = 1$. This is clearly impossible, given that F is nondegenerate. If the endpoint x_F of F is finite, however, then for all $u_1, \ldots, u_d \in [0, 1]$ such that for some $j < k \in \{1, \ldots, d\}$, $F^{-1}(u_j) + F^{-1}(u_k) \geq x_F$, one has

$$C_F(u_1, \ldots, u_d) = u_j u_k C_F(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{k-1}, u_{k+1}, \ldots, u_d).$$

This means that reciprocal Archimedean copulas can coincide with the independence copula on a subset of $[0, 1]^d$. This is illustrated in Example 9; see Section 6.

Remark 2. Note that for any $F \in \mathcal{F}_d$, it necessarily holds that $F^{-1}(0) = \sup\{t \in \mathbb{R} : F(t) = 0\} = 0$. This constraint could be relaxed, in principle. The first observation, made in [14], is that $F^{-1}(0) < 0$ is not possible. To see this, it suffices to consider the case $d = 2$. If $F^{-1}(0)$ were negative but finite, one could find $x_1, x_2 \in (F^{-1}(0), \infty)$ with $x_1 + x_2 < F^{-1}(0)$, so that $F(x_1 + x_2) = 0$ and $H_F(x_1, x_2) = F(x_1)F(x_2)/F(x_1 + x_2)$ is infinite, which is a contradiction. If $F^{-1}(0) = -\infty$, fix $x_1 < 0$ and note that $F(x_2)/F(x_1 + x_2) \geq 1$ for all $x_2 \in \mathbb{R}$ because F is non-decreasing. Thus, $H_F(x_1, x_2) = F(x_1)F(x_2)/F(x_1 + x_2)$ does not tend to 0 as $x_2 \to -\infty$, which is also a contradiction.

In contrast, $F^{-1}(0) > 0$ can happen. For example, setting, for an arbitrary $\theta \in (0, 1)$, $F(t) = 1 - e^{-t}/\theta$ whenever $t \geq -\ln(\theta)$ and $F(t) = 0$ otherwise, gives $F^{-1}(0) = -\ln(\theta) > 0$. It can be verified that this F indeed generates a *bona fide* copula in $d = 2$ through (1); in fact, C_F coincides with the Ali–Mikhail–Haq Archimedean copula with parameter θ. However, Proposition 1 does not hold when $F^{-1}(0) > 0$ and a stochastic representation of the corresponding reciprocal Archimedean copula seems rather cumbersome. For this reason, only generators with support of the form $[0, x_F]$ are considered in this paper; partial results for generators with $F^{-1}(0) > 0$ are provided in Section 8.

3. Connection with max-id distributions

There is more to the TP_2 property of bivariate reciprocal Archimedean copulas established in Corollary 1. As shown in [23], any bivariate distribution with this property is in fact max-id, and vice versa. Consequently, when $d = 2$, the distribution function H_F given by (5) is max-id. As will be shown in this section, the concept of max infinite divisibility is central to reciprocal Archimedean copulas in any dimension.

For any given integer $d \geq 2$, a d-variate continuous distribution function G is said to be max-id if and only if G^p is a distribution function for any $p \in (0, \infty)$; see [26]. This is equivalent to the condition that the copula of G is max-id, in the sense defined below.

Definition 1. A copula C is said to be *max-id* if and only if for all $p \in (0, \infty)$, there exists a copula C_p such that for all $u_1, \ldots, u_d \in (0, 1)$, $C^p(u_1^{1/p}, \ldots, u_d^{1/p}) = C_p(u_1, \ldots, u_d)$.

We will establish that a reciprocal Archimedean copula C_F with $F \in \mathcal{F}_d$ is max-id by showing that the distribution function H_F given in (5) has this property. This requires a deeper insight into the structure of max-id distributions. The concepts needed for the study of H_F are recalled next; for a detailed treatment, see, for example, [26].

Let $E_d = [\mathbf{0}, \infty] \setminus \{\mathbf{0}\} \subset [0, \infty]^d$ and observe that relatively compact subsets of E_d are sets that are bounded away from the origin. An exponent measure on E_d is defined as follows.

Definition 2. A measure μ on E_d is said to be an exponent measure if μ is Radon, that is, $\mu(B) < \infty$ for any relatively compact set $B \subseteq E_d$, and if μ is such that

$$\mu\left(\bigcup_{k=1}^{d} \{\boldsymbol{x} \in E_d : x_k = \infty\}\right) = 0.$$

As explained in the discussion surrounding Example 5.7 in Section 5.3 of [26], an exponent measure μ can be conveniently characterized as follows.

Lemma 4. *A measure μ on E_d is an exponent measure if $\mu[-\infty, \boldsymbol{x}]^{\complement} < \infty$ for all $\boldsymbol{x} > 0$ and if $\mu[-\infty, \boldsymbol{x}]^{\complement} \to 0$ as $\boldsymbol{x} \to \infty$, where $^{\complement}$ denotes set complement within the space E_d.*

If μ is an exponent measure on E_d, it follows from Proposition 5.8 in [26] that the function H_μ defined, for all $\boldsymbol{x} \geq \mathbf{0}$, by

$$H_\mu(\boldsymbol{x}) = e^{-\mu[-\infty, \boldsymbol{x}]^{\complement}} \tag{7}$$

is a max-id distribution. In the present context, it suffices to focus on max-id distributions whose exponent measure has the additional property that, for all $k \in \{1, \dots, d\}$,

$$\mu\{\boldsymbol{x} \in E_d : x_k = 0\} = 0, \tag{8}$$

that is, μ places no mass on $E_d \setminus (\mathbf{0}, \boldsymbol{\infty}) = \bigcup_{k=1}^d \{\boldsymbol{x} \in E_d : x_k = 0 \text{ or } x_k = \infty\}$ because

$$\mu\big(E_d \setminus (\mathbf{0}, \boldsymbol{\infty})\big) \le \sum_{k=1}^d \mu\{\boldsymbol{x} \in E_d : x_k = 0\} + \mu\left(\bigcup_{k=1}^d \{\boldsymbol{x} \in E_d : x_k = \infty\}\right) = 0.$$

Let $\mathcal{M}_d$ denote the class of exponent measures on E_d with property (8). As stated in Lemma 5 below and proved in Appendix B, members of this class are conveniently characterized through the following concept, which extends the notion of a survival function.

Definition 3. The generalized survival function $S_\mu : E_d \to [0, \infty)$ of an exponent measure $\mu \in \mathcal{M}_d$ is defined, for all $\boldsymbol{x} \in E_d$, by $S_\mu(\boldsymbol{x}) = \mu(\boldsymbol{x}, \boldsymbol{\infty}]$.

Note that S_μ is well defined as for all $\boldsymbol{x} \in E_d$, the set $(\boldsymbol{x}, \boldsymbol{\infty}]$ is relatively compact and hence $\mu(\boldsymbol{x}, \boldsymbol{\infty}] < \infty$. In what follows, let also $\mathcal{S}_d = \{S_\mu : \mu \in \mathcal{M}_d\}$.

Lemma 5. *If $S_\mu : E_d \to [0, \infty)$ for some $\mu \in \mathcal{M}_d$, then S_μ has the following properties*:

(i) $S_\mu(\boldsymbol{x}) \to 0$ *when* $x_k \to \infty$ *for at least one* $k \in \{1, \dots, d\}$.
(ii) S_μ *is right-continuous, that is, for all* $\boldsymbol{x} \in E_d$,

$$\forall_{\varepsilon \in (0,\infty)} \exists_{\delta \in (0,\infty)} \forall_{\boldsymbol{y} \ge \boldsymbol{x}} \qquad \|\boldsymbol{y} - \boldsymbol{x}\|_1 < \delta \quad \Rightarrow \quad \big|S_\mu(\boldsymbol{y}) - S_\mu(\boldsymbol{x})\big| < \varepsilon.$$

(iii) *For all* $\boldsymbol{x}, \boldsymbol{y} \in E_d$ *such that* $\boldsymbol{x} \le \boldsymbol{y}$, *the* S_μ-*volume of* $[\boldsymbol{x}, \boldsymbol{y}]$ *is non-negative, that is,*

$$\sum_{\boldsymbol{c} \in S_{\boldsymbol{x},\boldsymbol{y}}} \operatorname{sign}(\boldsymbol{c}) S_\mu(\boldsymbol{c}) \ge 0,$$

where $S_{\boldsymbol{x},\boldsymbol{y}} = \{\boldsymbol{c} \in E_d : \forall_{k \in \{1,\dots,d\}} c_k \in \{x_k, y_k\}\}$ *and*

$$\operatorname{sign}(\boldsymbol{c}) = \begin{cases} 1 & \text{if } c_k = y_k \text{ for an even number of integers } k \in \{1, \dots, d\}, \\ -1 & \text{if } c_k = y_k \text{ for an odd number of integers } k \in \{1, \dots, d\}. \end{cases}$$

Conversely, if a function $S : E_d \to [0, \infty)$ satisfies the above three conditions, it is the generalized survival function of a unique exponent measure $\mu \in \mathcal{M}_d$.

Finally, we introduce the notion of ℓ_1-*norm symmetric exponent measure*, which is key to the study of distributions of the form (5) and which parallels the notion of an ℓ_1-norm symmetric distribution [5,24].

Definition 4. An exponent measure $\mu \in \mathcal{M}_d$ is called ℓ_1-norm symmetric if there exists a map $\Lambda_\mu : (0,\infty) \to [0,\infty)$ such that $\Lambda_\mu(t) \to 0 \equiv \Lambda_\mu(\infty)$ as $t \to \infty$ and, for all $\boldsymbol{x} \in E_d$,

$$S_\mu(x_1,\dots,x_d) = \Lambda_\mu(x_1 + \cdots + x_d).$$

The map Λ_μ, which is unique, is called the generator of μ.

To generate an ℓ_1-norm symmetric exponent measure μ, the map Λ_μ must be such that S_μ satisfies conditions (i)–(iii) of Lemma 5. The following result, proved in Appendix B, characterizes the class $\mathcal{L}_d$ of generators of ℓ_1-norm symmetric exponent measures $\mu \in \mathcal{M}_d$.

Proposition 2. *A map* $\Lambda : (0,\infty) \to [0,\infty)$ *belongs to* $\mathcal{L}_d$ *if and only if* Λ *is* d*-monotone and* $\Lambda(t) \to 0$ *as* $t \to \infty$.

Now suppose that μ is an ℓ_1-norm symmetric exponent measure with generator Λ_μ. It then emerges from (B.3) in the proof of Lemma 5 that, for any $\boldsymbol{x} > \boldsymbol{0}$,

$$\mu[-\infty, \boldsymbol{x}]^{\complement} = \sum_{k=1}^{d} (-1)^{k+1} \sum_{A \subseteq \{1,\dots,d\}, |A|=k} \Lambda_\mu\Bigl(\sum_{j\in A} x_j\Bigr).$$

The max-id distribution H_μ with exponent measure μ defined in (7) is thus precisely of the form (5) with $F = F_\mu$, where

$$F_\mu(t) = \begin{cases} e^{-\Lambda_\mu(t)} & \text{if } t \in (0,\infty), \\ 0 & \text{otherwise.} \end{cases}$$

If in addition $\Lambda_\mu(t) \to \infty$ as $t \to 0$, then F_μ is continuous, and so is H_μ. The unique underlying copula of H_μ is then reciprocal Archimedean with generator F_μ. Furthermore, the generator of μ satisfies $\Lambda_\mu = -\ln(F_\mu)$. This insight allows for the complete characterization of the class $\mathcal{F}_d$ of reciprocal Archimedean generators in any given dimension $d \geq 2$.

Theorem 1. *A continuous distribution function* F *having support* $[0, x_F]$ *for some* $0 < x_F \leq \infty$ *belongs to* $\mathcal{F}_d$ *if and only if the map* $\Lambda = -\ln(F)$ *on* $(0,\infty)$ *is* d*-monotone and satisfies* $\Lambda(t) \to 0$ *as* $t \to \infty$, *and* $\Lambda(t) \to \infty$ *as* $t \to 0$, *that is,* $\Lambda \in \mathcal{L}_d$ *and* $\Lambda(t) \to \infty$ *as* $t \to 0$. *The class of all such functions* Λ *is denoted* $\mathcal{L}_d^0$.

Proof. The necessity of the conditions on Λ was established in Proposition 1. Sufficiency follows from the discussion preceding Theorem 1 given that $\Lambda \in \mathcal{L}_d$ by Proposition 2, that is, $\Lambda = \Lambda_\mu$ is the generator of an ℓ_1-norm symmetric exponent measure $\mu \in \mathcal{M}_d$. □

Corollary 2. *A distribution function* H_F *of the form* (5) *for some* $F \in \mathcal{F}_d$ *is a continuous max-id distribution with* ℓ_1*-norm symmetric exponent measure* μ_F *generated by* $\Lambda = -\ln(F)$ *with* $\Lambda \in \mathcal{L}_d^0$, *and vice versa.*

Thus, for any $F \in \mathcal{F}_d$, the reciprocal Archimedean copula C_F is precisely the dependence structure of a continuous max-id distribution whose exponent measure is ℓ_1-norm symmetric and generated by $\Lambda = -\ln(F)$. In particular, therefore, reciprocal Archimedean copulas are multivariate totally positive in the sense of [1]. As they are also associated, they are both positive lower and upper orthant dependent; see, for example, Theorem 8.6 in [13].

Remark 3. Theorem 1 and its corollary resemble the characterization of d-variate Archimedean copulas. As shown in [21,24], a function $\psi : [0,\infty) \to [0,\infty)$ generates an Archimedean copula in dimension d if and only if ψ is d-monotone and such that $\psi(0) = 1$ and $\psi(t) \to 0$ as $t \to \infty$. Furthermore, it was established in [24] that Archimedean copulas are precisely the dependence structures of continuous ℓ_1-norm symmetric distributions, that is, distributions whose survival function is of the form $S(\boldsymbol{x}) = \psi(x_1 + \cdots + x_d)$ for all $\boldsymbol{x} \in \mathbb{R}_+^d$.

For later use, illustrations of the above notions are presented below. They pertain to the bivariate Galambos and Clayton copulas with positive dependence.

Example 3. For the generator F_G with parameter $\theta \in (0,\infty)$ given in Example 1, one has $\Lambda_G(t) = -\ln\{F_G(t)\} = t^{-1/\theta}$ for all $t \in (0,\infty)$. Clearly, Λ_G satisfies the limit conditions at 0 and ∞. For any $k \in \mathbb{N}$, the kth derivative of Λ_G is given, for all $t \in (0,\infty)$, by

$$\Lambda_G^{(k)}(t) = (-1)^k t^{-k-1/\theta} \prod_{\ell=0}^{k-1} (\ell + 1/\theta).$$

In particular, therefore, Λ_G is d-monotone for any $d \geq 2$ and F_G can generate a reciprocal Archimedean copula in any dimension. This d-variate copula is the so-called multivariate Galambos extreme-value copula [11].

Example 4. For the generator F_C with parameter $\theta \in (0,\infty)$ given in Example 2, one has $\Lambda_C(t) = -\ln\{F_C(t)\} = -\ln(1 - e^{-t})/\theta$ for all $t \in (0,\infty)$. Obviously, Λ_C satisfies the limit conditions at 0 and ∞. For all $t \in (0,\infty)$, one has $\Lambda_C^{(1)}(t) = -e^{-t}/\{\theta(1 - e^{-t})\}$ and it can further be shown by induction that, for any integer $k \geq 2$,

$$\Lambda_C^{(k)}(t) = (-1)^k \frac{e^{-t}}{\theta(1 - e^{-t})^k} \sum_{\ell=0}^{k-2} A_{k-1,\ell} e^{-\ell t},$$

where, for each $k \in \mathbb{N}$ and $\ell \in \{0, \dots, k-1\}$, $A_{k,\ell}$ is the Eulerian number, that is, the number of permutations of $\{1, \dots, k\}$ having ℓ permutation ascents [10], Section 6.2. These coefficients are such that $A_{k,0} = A_{k,k-1} = 1$ and for all $\ell \in \{1, \dots, k-2\}$,

$$A_{k,\ell} = (\ell + 1)A_{k-1,\ell} + (k - \ell)A_{k-1,\ell-1}.$$

Writing $A_k(t) = A_{k,0} + A_{k,1}t + \cdots + A_{k,k-1}t^{k-1}$ for the kth Eulerian polynomial with the convention that $A_0(t) = 1$, one also has, for all $k \in \mathbb{N}$,

$$\Lambda_C^{(k)}(t) = (-1)^k \frac{e^{-t}}{\theta(1 - e^{-t})^k} A_{k-1}\big(e^{-t}\big).$$

Because the Eulerian numbers are all strictly positive, Λ_C is d-monotone for any $d \geq 2$ and hence F_C can generate a reciprocal Archimedean copula in any dimension. As mentioned in Example 1 and first observed in [14], F_C is a Clayton copula in dimension $d = 2$. Interestingly, however, C_F, called here the *reciprocal Clayton* copula, is *not* Clayton when $d \geq 3$. A numerical verification of this fact is given in [19].

4. Geometric structure of ℓ_1-norm exponent measures

We next show that ℓ_1-norm symmetric exponent measures have a geometric structure that parallels the stochastic representation of ℓ_1-norm symmetric distributions. From [24], the latter are known to be the distributions of random vectors $R \times \boldsymbol{S}$, where R is a strictly positive random variable, independent of the d-variate random vector $\boldsymbol{S}$ uniformly distributed on the simplex

$$\mathbb{S}_d = \{\boldsymbol{s} \in [0,1]^d : \|\boldsymbol{s}\|_1 = 1\}.$$

The following result provides an analogous representation of ℓ_1-norm symmetric exponent measures. Its proof can be found in Appendix C, along with the characterization of d-monotone functions due to Williamson [29] on which it relies.

Theorem 2. *A measure μ on E_d is an ℓ_1-norm symmetric exponent measure if and only if the image measure of μ by the polar coordinate transformation*

$$\mathcal{T} : E_d \to (0,\infty] \times \mathbb{S}_d : (x_1,\dots,x_d) \mapsto \left(\|\boldsymbol{x}\|_1, \frac{x_1}{\|\boldsymbol{x}\|_1}, \dots, \frac{x_d}{\|\boldsymbol{x}\|_1}\right)$$

is of the form $\nu_\mu \times \sigma_d$, where σ_d is the uniform probability distribution on $\mathbb{S}_d$ and ν_μ is a Radon measure on $(0,\infty]$ such that $\nu_\mu\{\infty\} = 0$, termed the radial measure. *Furthermore, the generator Λ_μ of μ is the Williamson d-transform of ν_μ, that is, for any $t \in (0,\infty)$,*

$$\Lambda_\mu(t) = \mathfrak{W}_d(\nu_\mu)(t) \equiv \int_t^\infty (1 - t/r)^{d-1}\,\mathrm{d}\nu_\mu(r), \tag{9}$$

and conversely ν_μ is determined through the inverse Williamson d-transform of Λ_μ, that is, for any $r \in (0,\infty)$, the generalized survival function $S_{\nu_\mu}(r) = \nu_\mu(r,\infty]$ of ν_μ satisfies

$$S_{\nu_\mu}(r) = \mathfrak{W}_d^{-1}(\Lambda_\mu)(r) \equiv \sum_{k=0}^{d-2} \frac{(-1)^k \Lambda_\mu^{(k)}(r)}{k!} r^k + \frac{(-1)^{d-1} \Lambda_{\mu+}^{(d-1)}(r)}{(d-1)!} r^{d-1}, \tag{10}$$

where $\Lambda_{\mu+}^{(d-1)}$ denotes the right-hand derivative of $\Lambda_\mu^{(d-2)}$.

The above result is an analogue of Theorem 3.1 in [24] and justifies the term "ℓ_1-norm symmetric exponent measure." The radial measure ν_μ parallels the notion of the radial distribution of an ℓ_1-norm symmetric distribution. Before proceeding, we compute the latter for the Galambos and the reciprocal Clayton copulas.

Example 5. Consider the generator F_G of the Galambos copula given in Example 1 and let μ_G denote the ℓ_1-norm symmetric exponent measure generated by $\Lambda_G = -\ln(F_G)$. The generalized survival function S_{ν_G} of the radial measure ν_G of μ_G is easily computed using the derivatives of Λ_G given in Example 3. For all $r \in (0, \infty)$,

$$S_{\nu_G}(r) = r^{-1/\theta} \sum_{k=0}^{d-1} \frac{1}{k!} \prod_{\ell=0}^{k-1} (\ell + 1/\theta) = \frac{\theta}{\mathrm{B}(d, 1/\theta)} r^{-1/\theta},$$

where $\mathrm{B}(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ denotes the Beta function at any $\alpha, \beta \in (0, \infty)$.

Example 6. Consider the generator F_C of the reciprocal Clayton copula given in Example 2 and let μ_C denote the ℓ_1-norm symmetric exponent measure generated by $\Lambda_C = -\ln(F_C)$. Calculations in Example 4 imply that the generalized survival function S_{ν_C} of the radial measure ν_C of μ_C satisfies, for all $r \in (0, \infty)$,

$$S_{\nu_C}(r) = \frac{1}{\theta} \left\{ -\ln\bigl(1 - e^{-r}\bigr) + \sum_{k=1}^{d-1} \frac{e^{-r} r^k}{k!(1 - e^{-r})^k} A_{k-1}\bigl(e^{-r}\bigr) \right\},$$

where A_k is the Eulerian polynomial defined in Example 4.

The geometric structure of ℓ_1-norm symmetric exponent measures resembles that of exponent measures corresponding to multivariate extreme-value distributions with unit Fréchet margins. Indeed, as detailed, for example, in Section 5.4.1 in [26], a distribution function G is multivariate extreme-value with Φ_1 margins if and only if it is a max-id distribution of the form (7), whose exponent measure $\mu \equiv \mu_G$ is such that for any Borel set $B \subset E_d$ and $t \in (0, \infty)$, $\mu_G(B) = t\mu_G(tB)$. Thus, the image measure of μ_G by $\mathcal{T}$ is $\nu_G \times \varsigma_d$, where for any $r \in (0, \infty)$, $\nu_G(r, \infty] = d/r$ and ς_d is a probability distribution on $\mathbb{S}_d$ with mean $\mathbf{1}/d$.

While, for max-id distributions with ℓ_1-norm symmetric exponent measure, the measure on $\mathbb{S}_d$ is fixed and the radial measure can vary, for multivariate extreme-value distributions with Φ_1 margins we have the exact opposite, that is, the radial measure is fixed while the measure on $\mathbb{S}_d$ can vary. Because the mean of the uniform distribution σ_d is $\mathbf{1}/d$, the only distribution which belongs to both classes is the max-id distribution H_F with $F = \Phi_1$. Indeed, as can be deduced from Example 5, $\nu_F(r, \infty] = d/r$.

This observation is not in contradiction with the fact that all Galambos copulas are both reciprocal Archimedean and extreme-value as seen in Example 3. This is because the above discussion was limited to extreme-value distributions with unit Fréchet margins. However, it begs the question of whether any other extreme-value copula can be expressed in the form (3) for an appropriate choice of $F \in \mathcal{F}_d$. The answer is negative, as stated below and proved in Appendix C.

Proposition 3. *A copula $C_F \in \mathcal{C}_d$ for some $F \in \mathcal{F}_d$ is extreme-value if and only if F is a Fréchet distribution, that is, there exists $\theta \in (0, \infty)$ such that, for all $t \in (0, \infty)$, $F(t) = \Phi_{1/\theta}(t) = e^{-t^{-1/\theta}}$. The Galambos copulas are thus the only extreme-value copulas in the class $\mathcal{C}_d$.*

The result in Proposition 3 is thus parallel to the characterization of Gumbel copulas as the only Archimedean copulas that are extreme-value [8].

5. Stochastic representation and simulation

It will now be shown that any max-id distribution H_F with $F \in \mathcal{F}_d$ has a convenient stochastic representation that leads to an algorithm for generating observations from a reciprocal Archimedean copula. Recall that H_F has an ℓ_1-norm symmetric exponent measure μ_F generated by $\Lambda = -\ln(F)$, as established in Corollary 2. The stochastic representation of H_F can then be obtained by combining Theorem 2 with results from [2,9] described, for example, in Example 5.7 in [26]. In what follows, ε denotes the Dirac measure and $a \vee b = \max(a, b)$.

Proposition 4. *Given any $F \in \mathcal{F}_d$, let H_F be of the form (5) and let ν_F be the radial measure corresponding to $\Lambda = -\ln(F)$ through (10). Further, let $\zeta = \sum_k \varepsilon_{(r_k, s_k)}$ be a Poisson random measure (PRM) on $(0, \infty] \times \mathbb{S}_d$ with mean measure $\nu_F \times \sigma_d$. Then H_F is the distribution function of the vector*

$$\boldsymbol{Y} = (Y_1, \ldots, Y_d) = \max_k (r_k \times \boldsymbol{s}_k) \vee \boldsymbol{0}, \tag{11}$$

where the maximum is understood as a component-wise operation.

Proof. By Theorem 2, the image measure of $\nu_F \times \sigma_d$ by $\mathcal{T}^{-1}$ is the ℓ_1-norm symmetric exponent measure μ_F generated by Λ. Thus, for any $\boldsymbol{y} \in E_d$,

$$\Pr(\boldsymbol{Y} \le \boldsymbol{y}) = \Pr\bigl[\zeta\bigl\{\mathcal{T}\bigl([-\infty, \boldsymbol{y}]^{\complement}\bigr)\bigr\} = 0\bigr] = \exp\bigl\{-\mu_F[-\infty, \boldsymbol{y}]^{\complement}\bigr\} = H_F(\boldsymbol{y}),$$

while $\Pr(\boldsymbol{Y} \le \boldsymbol{0}) = \Pr[\zeta\{\mathcal{T}(E_d)\} = 0] = 0$ because $\mu_F(E_d) = \infty$. $\square$

Now suppose that C_F is a d-variate reciprocal Archimedean copula generated by $F \in \mathcal{F}_d$. From Sklar's representation and Corollary 2, we know that C_F is the copula of the max-id distribution H_F whose exponent measure μ_F is ℓ_1-norm symmetric with radial measure ν_F. This also means that C_F is the copula of the random vector $\boldsymbol{Y}$ in (11). In principle, a random sample from C_F can thus be obtained by generating observations from $\boldsymbol{Y}$.

The difficulty with this approach is that $\nu_F \times \sigma_d\{(0, \infty] \times \mathbb{S}_d\} = \mu_F(E_d) = \infty$, which means that realizations of the PRM ζ cannot be generated explicitly. To circumvent this problem, consider a decreasing sequence of constants $\infty = \Delta_0 > \Delta_1 > \cdots > 0$ such that $\Delta_k \to 0$ as $k \to \infty$ and for every integer $k \in \mathbb{N}$, set $D_k = (\Delta_k, \Delta_{k-1}] \times \mathbb{S}_d$ so that $\zeta = \sum_{k=1}^{\infty} \zeta_k$, where ζ_k is a PRM on D_k with mean measure $\nu_F \times \sigma_d$ restricted to D_k. Because $\nu_F \times \sigma_d(D_k) = \nu_F(\Delta_k, \Delta_{k-1}] < \infty$, one has that, for every integer $k \in \mathbb{N}$,

$$\zeta_k = \sum_{i=1}^{N_k} \varepsilon_{(R_{ki}, \boldsymbol{S}_{ki})},$$

where N_k is a Poisson random variable with parameter $\lambda_k = \nu_F(\Delta_k, \Delta_{k-1}]$ which is independent of two independent and i.i.d. sequences $R_{k1}, R_{k2}, \ldots$ and $\boldsymbol{S}_{k1}, \boldsymbol{S}_{k2}, \ldots$. Furthermore, for every $i \in \mathbb{N}$, $\boldsymbol{S}_{ki}$ is uniformly distributed on the unit simplex $\mathbb{S}_d$ and R_{ki} is a positive random variable with

$$\Pr(R_{ki} > r) = \begin{cases} \dfrac{S_{\nu_F}(r \vee \Delta_k) - S_{\nu_F}(\Delta_{k-1})}{S_{\nu_F}(\Delta_k) - S_{\nu_F}(\Delta_{k-1})} & \text{if } r < \Delta_{k-1}, \\ 0 & \text{if } r \geq \Delta_{k-1}, \end{cases} \tag{12}$$

where S_{ν_F} is the generalized survival function of ν_F. As a consequence, one can write $\boldsymbol{Y} = \max(\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots)$, where

$$\boldsymbol{Y}_k = (Y_{k1}, \ldots, Y_{kd}) = \max(R_{k1} \times \boldsymbol{S}_{k1}, \ldots, R_{kN_k} \times \boldsymbol{S}_{kN_k}) \vee \mathbf{0}.$$

Note that by construction, the vectors $\boldsymbol{Y}_k$ are mutually independent. Now consider the sequence of cumulative maxima $\boldsymbol{W}_k$ where, for every $k \in \mathbb{N}$,

$$\boldsymbol{W}_k = (W_{k1}, \ldots, W_{kd}) = \max(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_k)$$

and fix an arbitrary element ω of the underlying probability space Ω. Clearly, the sequence $\boldsymbol{W}_1(\omega), \boldsymbol{W}_2(\omega), \ldots$ is increasing. By construction, one also has $Y_{\ell j}(\omega) \leq \Delta_{\ell-1}$ for all $j \in \{1, \ldots, d\}$ and $\ell \in \mathbb{N}$. Thus for any $k \in \mathbb{N}$, the realizations $Y_j(\omega)$ and $W_{kj}(\omega)$ are equal as soon as $W_{kj}(\omega) > \Delta_k$. If $W_{kj}(\omega) \leq \Delta_k$, then $Y_j(\omega) \leq \Delta_k$, so that $|Y_j(\omega) - W_{kj}(\omega)| \leq \Delta_k$. This implies that for every $\omega \in \Omega$, $\boldsymbol{W}_k(\omega) \uparrow \boldsymbol{Y}(\omega)$ as $k \to \infty$.

Thus we propose to approximate $\boldsymbol{Y}$ by $\boldsymbol{W}_K$ for some suitable $K \in \mathbb{N}$. This leads to the following algorithm for generating samples from the reciprocal Archimedean copula C_F.

Algorithm 1. Let C_F be a d-variate reciprocal Archimedean copula with generator $F \in \mathcal{F}_d$. Let $\Lambda = -\ln(F)$ and ν_F be the radial measure of the ℓ_1-norm symmetric exponent measure μ_F of H_F, as given in (10). Let also $\infty = \Delta_0 > \Delta_1 > \cdots$ be a decreasing sequence such that $\Delta_k \to 0$ as $k \to \infty$ and let $K \in \mathbb{N}$ be a fixed integer. To generate an observation $\boldsymbol{U}$ from C_F, proceed as follows.

Step 1. (Initialization) Set $k = 0$ and $\boldsymbol{W}_0 = \mathbf{0}$.
Step 2. (Iteration)

2a. Set $k = k + 1$ and compute $\lambda_k = \nu_F(\Delta_k, \Delta_{k-1}] = S_{\nu_F}(\Delta_k) - S_{\nu_F}(\Delta_{k-1})$.
2b. Generate N_k from the Poisson distribution with parameter λ_k.
2c. Draw N_k independent observations $R_{k1}, \ldots, R_{kN_k}$ from the distribution with survival function (12).
2d. Draw N_k independent d-variate random vectors $\boldsymbol{E}_{k1}, \ldots, \boldsymbol{E}_{kN_k}$ whose components are independent unit exponential variables and for each $\ell \in \{1, \ldots, N_k\}$, set $\boldsymbol{S}_{k\ell} = \boldsymbol{E}_{k\ell}/(E_{k\ell,1} + \cdots + E_{k\ell,d})$.
2e. For each $\ell \in \{1, \ldots, N_k\}$, set $\boldsymbol{X}_{k\ell} = R_{k\ell} \times \boldsymbol{S}_{k\ell}$.
2f. Set $\boldsymbol{Y}_k = \max(\boldsymbol{X}_{k1}, \ldots, \boldsymbol{X}_{kN_k})$ if $N_k > 0$ and $\boldsymbol{Y}_k = \mathbf{0}$ otherwise.
2g. Set $\boldsymbol{W}_k = \max(\boldsymbol{Y}_k, \boldsymbol{W}_{k-1})$.

Step 3. (Stopping criterion). If $k = K$ or $W_{kj} > \Delta_k$ for each $j \in \{1, \dots, d\}$, then set $\boldsymbol{Y} = \boldsymbol{W}_k$ and go to Step 4. Otherwise, go to Step 2.

Step 4. (Output) Set $\boldsymbol{U} = (F(Y_1), \dots, F(Y_d))$.

The performance of Algorithm 1 clearly depends on the choice of the sequence (Δ_k) and the integer K that bounds the number of iterations. From our experiments and the simulations used in this paper, it suffices to set $K = 2$ provided that Δ_1 and Δ_2 are well chosen. When selecting Δ_1, a trade-off between speed and accuracy arises. When Δ_1 is small, chances are high that all components of $\boldsymbol{W}_1$ will be above Δ_1, in which case $\boldsymbol{W}_1$ is a realization of the vector $\boldsymbol{Y}$ and there is no need for further iterations. However, a small value of Δ_1 also means that the Poisson parameter λ_1 is large. High values of N_1 are then likely, which in turn may result in unnecessarily large samples in Steps 2c–d. This is wasteful and can slow down the algorithm considerably.

The choice of Δ_1 is further complicated by the fact that the meaning of "small" depends on the generator F and the dimension d. To see how to proceed, we first derive a lower bound on the probability that Algorithm 1 with $K > 1$ requires only one iteration.

Lemma 6. *For arbitrary $F \in \mathcal{F}_d$ and $\Delta \in (0, \infty)$, let $\zeta^* = \sum_k \varepsilon_{(r_k, s_k)}$ be a PRM on $(\Delta, \infty] \times \mathbb{S}_d$ with mean measure $\nu_F \times \sigma_d$. Let also $\boldsymbol{Y}^* = \max_k(r_k \times \boldsymbol{s}_k) \vee \boldsymbol{0}$. Then*

$$\Pr\big(Y_1^* > \Delta, \dots, Y_d^* > \Delta\big) = \bar{H}_F(\Delta, \dots, \Delta) \geq \big\{1 - F(\Delta)\big\}^d,$$

where $\bar{H}_F$ denotes the survival function of H_F.

This result suggests the following procedure for selecting Δ_1 in order to generate a random sample of size n from a reciprocal Archimedean copula C_F.

Rule of Thumb 1. Choose $p \in (0, 1)$ and let Δ_1 be the largest Δ for which the probability of n consecutive runs of Algorithm 1 with $K > 1$ requiring one iteration each is at least p.

Because $\bar{H}_F(\Delta, \dots, \Delta)$ as a function of Δ may be cumbersome to invert, especially in high dimension d, we suggest to use the inequality in Lemma 6, viz.

$$p = \{\Pr(Y_{11} > \Delta, \dots, Y_{1d} > \Delta)\}^n \geq \{1 - F(\Delta)\}^{nd} \tag{13}$$

so that Δ_1 is given by

$$\Delta_1 = F^{-1}\{1 - p^{1/(nd)}\}. \tag{14}$$

In what follows, Eq. (14) is used with $p = 0.99$ and it seemed enough to take $\Delta_2 = \Delta_1 \times 10^{-4}$ to catch the cases where Algorithm 1 requires more than one iteration.

As an illustration of Algorithm 1 and the implementation of Rule of Thumb 1, its performance is explored below for the Galambos and the reciprocal Clayton family.

Example 7. To generate a random sample of size n from the d-variate Galambos copula with parameter $\theta \in (0, \infty)$, Algorithm 1 must be implemented with the measure ν_G defined in Example 5. In the first iteration, one must draw $N_1 \sim \mathcal{P}(\lambda_1)$ with $\lambda_1 = \theta \Delta_1^{-1/\theta} / \mathrm{B}(d, 1/\theta)$ and

R_1 from a Pareto distribution with shape parameter $1/\theta$ and scale parameter Δ_1, that is, for all $r > \Delta_1$, $\Pr(R_1 > r) = (\Delta_1/r)^{1/\theta}$. Based on Eq. (14), $\Delta_1 = [-\ln\{1 - p^{1/(nd)}\}]^{-\theta}$. The left panel of Figure 1 shows how Δ_1 varies as a function of Kendall's τ (which is the same for all pairs of components) when $p = 0.99$, $n = 1000$ and $d \in \{2, 5, 10\}$. The top panel of Figure 2 displays samples of this size when $\tau = 0.5$.

Example 8. To generate a random sample from the d-variate reciprocal Clayton copula with parameter $\theta \in (0, \infty)$, Algorithm 1 must be implemented with the measure ν_C defined in Example 6. This requires numerical inversion for the generation of R_1. Based on Eq. (14), $\Delta_1 = -\ln[1 - \{1 - p^{1/(nd)}\}^\theta]$. The right panel of Figure 1 shows how Δ_1 varies as a function of Kendall's τ (which is the same for all pairs of components) when $p = 0.99$, $n = 1000$ and $d \in \{2, 5, 10\}$. The bottom panel of Figure 2 displays samples resulting from Algorithm 1 when $\tau = 0.5$ and $d \in \{2, 3\}$, and compares the trivariate sample to a sample of the same size from the trivariate Clayton Archimedean copula with $\tau = 0.5$.

Additional examples of reciprocal Archimedean copulas are discussed in Section 5. In our experience, the values of Δ_1 given by Rule of Thumb 1 tend to be too conservative. As evidenced by Figure 1, the value of Δ_1 decreases as the dimension of the vector and the dependence between its components increase. This suggests that to improve the performance of Algorithm 1, the lower bound given in (13) would need to be refined by taking into account this dependence. We leave this issue aside for the time being.

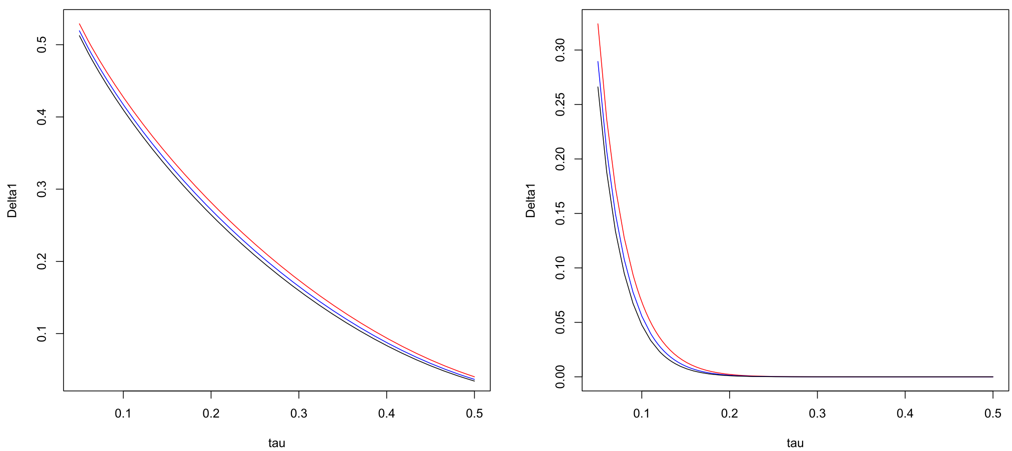

Figure 1. Variation, as a function of bivariate Kendall's τ, of Δ_1 based on Eq. (14) when $p = 0.99$ and $n = 1000$ for the d-variate Galambos copula (left) and reciprocal Clayton copula (right) in dimension $d = 2$ (red), 5 (blue), and 10 (black).

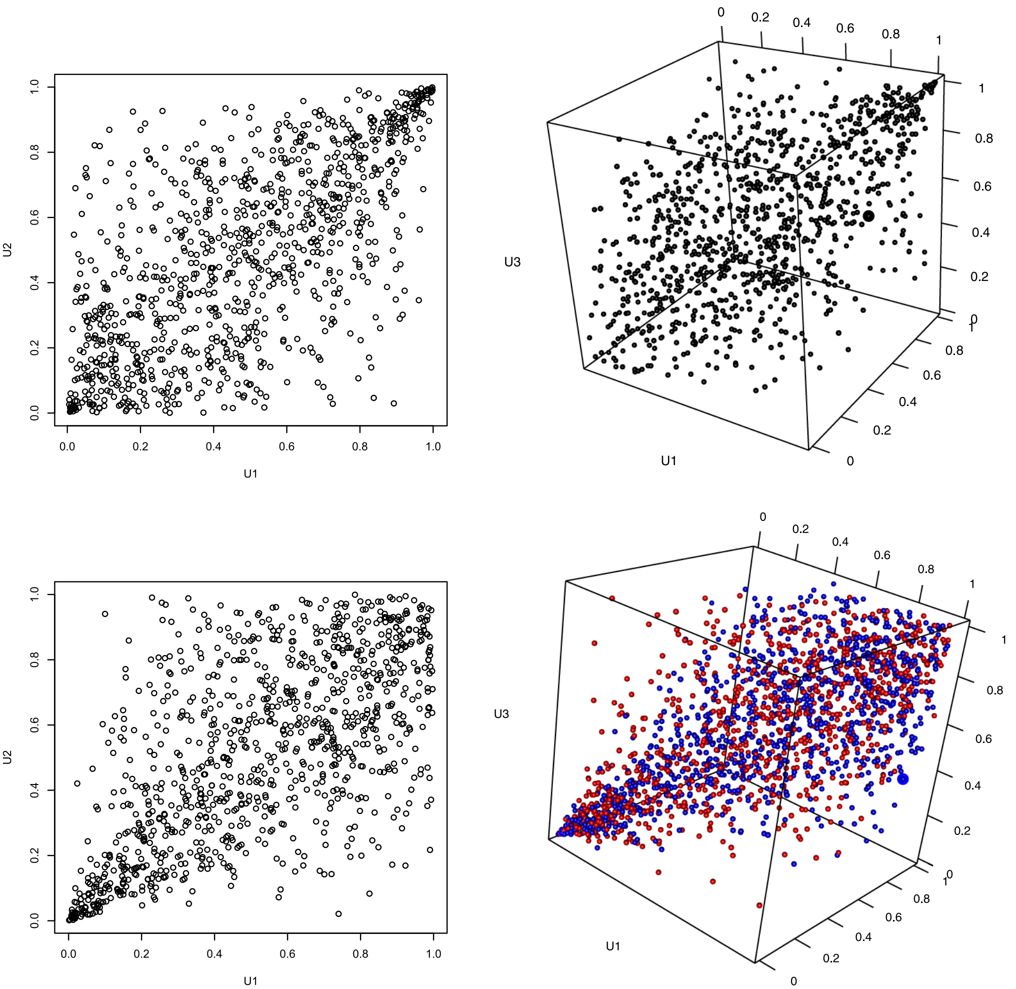

Figure 2. Upper panel: Random samples of size 1000 from the bivariate (upper left) and trivariate (upper right) Galambos copula with dependence $\tau = 0.5$ between all pairs. Lower panel: Random samples of size 1000 from the bivariate Clayton (lower left) and trivariate (upper right) Clayton copula (red) and reciprocal Clayton (blue) copula with dependence $\tau = 0.5$ between all pairs.

6. Illustrations

This section explores mechanisms by which new classes of multivariate reciprocal Archimedean copulas can be constructed. Two approaches are considered, depending on whether it is preferred to specify a parametric class of reciprocal Archimedean generators directly or through the corresponding radial measure. Before proceeding, recall that:

(i) By construction, all reciprocal Archimedean copulas are exchangeable and max-id. Therefore, these copulas are multivariate totally positive and associated [1,13] and capable of modeling positive association only.
(ii) As already mentioned at the end of Section 2, the independence copula Π does not belong to the class $\mathcal{C}_d$ in any dimension $d \geq 2$. This is also easily seen from Condition (8), which prevents the ℓ_1-norm symmetric exponent measure μ of a reciprocal Archimedean copula from placing any mass on the axes going through $\mathbf{0}$. In contrast, the exponent measure of the max-id distribution with independent, identical margins F places all its mass on the axes going through the origin.

6.1. Specifying the radial measure

The easiest way to construct the reciprocal Archimedean generator $F \in \mathcal{F}_d$ is through the radial measure ν. To accomplish this, one must first choose $S \in \mathcal{S}_1$, that is, a non-increasing, right-continuous function $S : (0, \infty) \to [0, \infty)$ such that $S(t) \to 0$ as $t \to \infty$ and $S(t) \to \infty$ as $t \to 0$. Then one sets $\nu(t, \infty] = S(t)$ for all $t \in (0, \infty)$. By Theorem 2, the generator of the corresponding ℓ_1-norm symmetric exponent measure $\mu \in \mathcal{M}_d$ is given by the Williamson d-transform of ν, that is, for any $t \in (0, \infty)$,

$$\Lambda(t) = \int_t^\infty (1 - t/r)^{d-1} \, \mathrm{d}\nu(r).$$

By construction, we then have that $\Lambda \in \mathcal{L}_d$, as well as $\Lambda(t) \to 0$ as $t \to \infty$ and $\Lambda(t) \to \infty$ as $t \to 0$, so that, by Theorem 1, the distribution function given by

$$F(t) = \begin{cases} e^{-\Lambda(t)} & \text{if } t \in (0, \infty), \\ 0 & \text{otherwise} \end{cases}$$

is a valid reciprocal Archimedean generator in dimension d, that is, $F \in \mathcal{F}_d$.

This approach is not only advantageous because the conditions on S are easily met but also because S is explicit, which is useful in Step 2c of Algorithm 1. However, this often comes at the cost of an intractable formula for Λ and F, because the Williamson d-transform of ν may not be explicit. Nonetheless, the integral it involves is one-dimensional, so Step 4 of Algorithm 1 can in principle be carried out using numerical integration.

Perhaps the most natural (though already known) example of such a construction is the Galambos copula, for which $S(t) \propto t^{-1/\theta}$ for all $t \in (0, \infty)$, as computed in Example 5. Below, we expand on it and explore what happens when S vanishes beyond some point $\kappa \in (0, \infty)$.

Example 9. Consider the "truncated" version of the generalized survival function corresponding to the d-variate Galambos copula, given, for $\theta \in (0, \infty)$ and $\kappa \in (0, \infty)$, by

$$S_{\theta,\kappa}(r) = \begin{cases} \dfrac{\theta}{\mathrm{B}(d, 1/\theta)} r^{-1/\theta} & \text{if } r \in (0, \kappa], \\ 0 & \text{otherwise.} \end{cases}$$

Note that $S_{\theta,\kappa}$ is discontinuous at $t=\kappa$. For any $t \in (0,\kappa)$, we have

$$\Lambda_{\theta,\kappa}(t) = \frac{1}{\mathrm{B}(d,1/\theta)}\left\{\theta(1-t/\kappa)^{d-1}\kappa^{-1/\theta} + \int_t^\kappa (1-t/r)^{d-1} r^{-1/\theta-1}\,dr\right\}$$

while $\Lambda_{\theta,\kappa}(t) \equiv 0$ for all $t \in [\kappa,\infty)$. Elementary calculations lead to the alternative expression

$$\Lambda_{\theta,\kappa}(t) = t^{-1/\theta} + \frac{\kappa^{-1/\theta}}{\mathrm{B}(d,1/\theta)} \sum_{\ell=1}^{d-1} \frac{\theta}{\ell+1/\theta} \frac{1}{\mathrm{B}(\ell,d-\ell)} \left(-\frac{t}{\kappa}\right)^{\ell},$$

valid for all $t \in (0,\kappa)$. An explicit expression for the reciprocal Archimedean generator $F_{\theta,\kappa}$ then follows from the fact that, for all $t \in (0,\infty)$, $F_{\theta,\kappa}(t) = e^{-\Lambda_{\theta,\kappa}(t)}$.

When $d=2$, one has, for all $t \in (0,\kappa)$, $\Lambda_{\theta,\kappa}(t) = t^{-1/\theta} - t\kappa^{-1/\theta-1}$ and hence

$$F_{\theta,\kappa}(t) = \begin{cases} \exp\left(-t^{-1/\theta} + t\kappa^{-1/\theta-1}\right) & \text{if } t \in (0,\kappa], \\ 1 & \text{otherwise.} \end{cases}$$

The fact that $F_{\theta,\kappa}$ has a finite upper endpoint $x_F = \kappa$ implies that when $x_1 + x_2 \geq \kappa$, the bivariate distribution H_F reduces to independence, viz. $H_F(x_1,x_2) = F(x_1)F(x_2)$. Consequently, at any $u_1, u_2 \in [0,1]$ such that $F^{-1}(u_1) + F^{-1}(u_2) \geq \kappa$, the reciprocal Archimedean copula C_F generated by F reduces to the independence copula, that is, $C_F(u_1,u_2) = u_1 u_2$. This is illustrated in Figure 3, which shows samples of size $n = 1000$ for $\theta = 1$ and various values of κ. The plots also suggest that there is a singularity along the curve $F^{-1}(u_1) + F^{-1}(u_2) = \kappa$, drawn in red. This is indeed the case and due to the fact that $S_{\theta,\kappa}$ is discontinuous at κ, as explained in Remark 4 below. Finally, note that C_F converges to the independence copula when $\kappa \to 0$ while C_F converges to the Galambos copula with parameter θ when $\kappa \to \infty$.

Remark 4. Given $F \in \mathcal{F}_d$, let H_F be of the form (5) and ν_F be the radial measure corresponding to $\Lambda = -\ln(F)$ through (10). Let also $\boldsymbol{U} = (U_1,\ldots,U_d)$ be distributed as the reciprocal

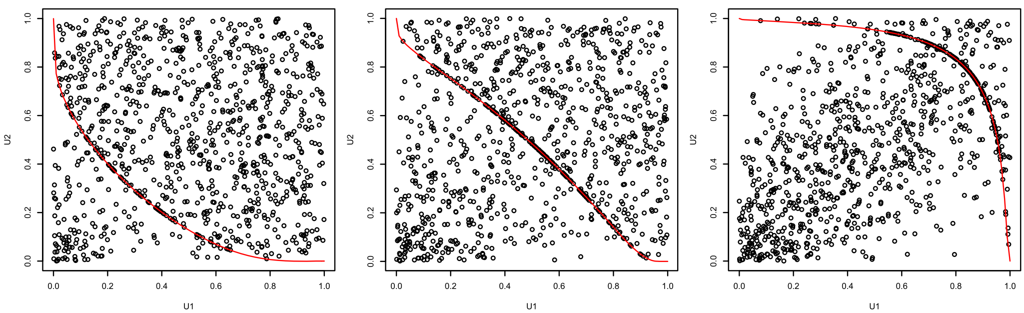

Figure 3. Random samples of size 1000 from the bivariate truncated Galambos copula from Example 9 with $\theta = 1.28$ and $\kappa = 1$ (left), $\kappa = 2$ (middle) and $\kappa = 10$ (right). The curve $F^{-1}(u_1) + F^{-1}(u_2) = \kappa$ is indicated in red.

Archimedean copula C_F. If ν_F has an atom at $\kappa \in (0, \infty)$, then

$$\Pr\{F^{-1}(U_1) + \cdots + F^{-1}(U_d) = \kappa\} > 0,$$

implying that C_F has a singular component. To see why this holds, let $\boldsymbol{Y}$ be the random vector distributed as H_F and ζ be the PRM with mean measure $\nu_F \times \sigma_d$, as in Proposition 4. Now fix some arbitrary $0 < \varepsilon < 1/\{d(d-1)\}$ and set

$$B_\varepsilon = \big\{\boldsymbol{s} \in \mathbb{S}_d : \forall_{i \in \{1,\dots,d-1\}} s_i \in [1/d - \varepsilon, 1/d + \varepsilon]\big\}$$

Clearly, if $\boldsymbol{s} \in B_\varepsilon$, then $s_d \in [1/d - (d-1)\varepsilon, 1/d + (d-1)\varepsilon] \subset [0, 1]$ by the choice of ε. As a consequence, $\sigma_d(B_\varepsilon) > 0$. Now set $\delta = \kappa\{1/d - (d-1)\varepsilon\}$, so that $0 < \delta < \kappa$, and introduce the following partition of $(0, \infty] \times \mathbb{S}_d$:

$$A_1 = (\kappa, \infty] \times \mathbb{S}_d, \quad A_2 = \{\kappa\} \times B_\varepsilon, \quad A_3 = \{\kappa\} \times B_\varepsilon^{\complement}, \quad A_4 = (\delta, \kappa] \times \mathbb{S}_d, \quad A_5 = (0, \delta] \times \mathbb{S}_d.$$

The event $\{Y_1 + \cdots + Y_d = \kappa\}$ occurs in particular if $\zeta(A_j) = 0$ for $j \in \{1, 3, 4\}$ and $\zeta(A_2) = 1$. Given that $\nu_F \times \sigma_d(A_j) \in (0, \infty)$ for $j \in \{1, 2, 3, 4\}$, one has

$$\Pr(Y_1 + \cdots + Y_d = \kappa) \ge \nu_F\big(\{\kappa\}\big) \times \sigma_d(B_\varepsilon) \times \prod_{j=1}^{4} e^{-\nu_F \times \sigma_d(A_j)} > 0,$$

as claimed. The exact computation of $\Pr(Y_1 + \cdots + Y_d = \kappa)$ seems cumbersome, however.

Next, note the following fact, whose proof is straightforward and therefore omitted.

Lemma 7. *Let $S \in \mathcal{S}_1$ be a generalized survival function of a radial measure ν with Williamson d-transform Λ. Then for any $\theta \in (0, \infty)$, $\theta S \in \mathcal{S}_1$ is the generalized survival function of $\nu_\theta = \theta\nu$. Furthermore, the Williamson d-transform of ν_θ is $\theta\Lambda$.*

In view of Lemma 7, parametric families of reciprocal Archimedean copulas can be created by first choosing a *baseline* reciprocal Archimedean generator F through a generalized survival function S as outlined above, and then considering the class of reciprocal Archimedean copulas generated by the Lehmann alternatives of F, viz. $\{F^\theta : \theta \in (0, \infty)\}$.

Example 10. For any $t \in (0, \infty)$ and $\theta \in (0, \infty)$, set $S_\theta(t) = \theta(e^{1/t} - 1)$. Straightforward calculations lead to the following expression for the generator of the corresponding ℓ_1-norm symmetric exponent measure, valid for all $t \in (0, \infty)$:

$$\Lambda_\theta(t) = \theta \int_t^\infty (1 - t/r)^{d-1} e^{1/r} (1/r)^2 \, dr = \theta e^{1/t} t^d \gamma(d, 1/t),$$

where for any $x \in (0, \infty)$, $\gamma(d, x) = \int_0^x z^{d-1} e^{-z} \, dz$ denotes the lower incomplete Gamma function. Consequently, the reciprocal Archimedean generator of this “reciprocal Gamma” family is given, for all $t \in (0, \infty)$, by

$$F_\theta(t) = \exp\{-\theta e^{1/t} t^d \gamma(d, 1/t)\}.$$

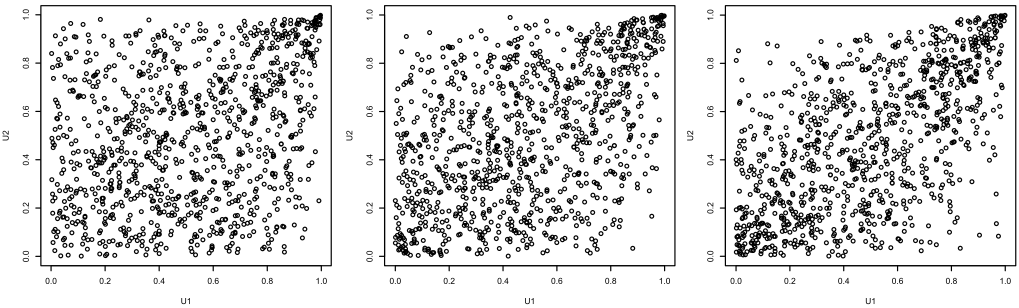

Figure 4. Random samples of size 1000 from the bivariate reciprocal Gamma family from Example 10 with $\theta = 0.1$ (left), $\theta = 1$ (middle) and $\theta = 100$ (right).

Figure 4 shows samples of size $n = 1000$ from this copula for various values of θ when $d = 2$. As the plots suggest, this family can only capture low degrees of dependence.

6.2. Specifying the exponent measure generator Λ

An alternative strategy for the construction of reciprocal Archimedean copulas consists of specifying the generator directly through a choice of $\Lambda \in \mathcal{L}_d^0$, that is, a d-monotone function on $(0, \infty)$ such that $\Lambda(t) \to 0$ as $t \to \infty$ and $\Lambda(t) \to \infty$ as $t \to 0$. By Theorem 2, the generalized survival function S_ν of the corresponding radial measure ν is then given, for all $t \in (0, \infty)$, by the inverse Williamson d-transform of Λ, viz.

$$S_\nu(t) = \sum_{k=0}^{d-2} \frac{(-1)^k \Lambda^{(k)}(t)}{k!} t^k + \frac{(-1)^{d-1} \Lambda_+^{(d-1)}(t)}{(d-1)!} t^{d-1}.$$

The main advantage of this strategy is the explicit form of the reciprocal Archimedean generator $F = e^{-\Lambda}$ and consequently of the reciprocal Archimedean copula itself. This makes Step 4 of Algorithm 1 easy to carry out. However, there are two major disadvantages. First, choosing a valid Λ is not straightforward, especially in higher dimensions, because the d-monotonicity condition is not easily met or verified. More importantly, the formula for S_ν is likely to be cumbersome, because it involves a number of higher order derivatives of Λ that may be tedious to compute explicitly. This in turn complicates matters in Step 2c of Algorithm 1, as the random generation therein requires the inverse of S_ν.

A prototype of this construction strategy is the reciprocal Clayton copula, already detailed in Examples 2, 4 and 6. The following example shows how it could be generalized.

Example 11. Let $\varphi : [0, 1] \to [0, \infty)$ be a completely monotone function such that $\varphi(1) = 0$ and $\varphi(t) \to \infty$ as $t \to 0$. Consider the function defined, for all $t \in (0, \infty)$, by

$$\Lambda(t) = \varphi(1 - e^{-t}).$$

Then Λ is clearly decreasing and such that $\Lambda(t) \to 0$ as $t \to \infty$ while $\Lambda(t) \to \infty$ as $t \to 0$. Through an application of Faà di Bruno's formula, one can also see that the kth derivative of Λ is given, for all $t \in (0, \infty)$, by

$$\Lambda^{(k)}(t) = \sum^{*} \frac{k!}{m_1! \cdots m_k!} \varphi^{(m)}\left(1 - e^{-t}\right) \prod_{j=1}^{k} \left\{ \frac{(-1)^{j+1}}{j!} e^{-t} \right\}^{m_j},$$

where $m = m_1 + \cdots + m_k$ and Σ^* denotes a sum over all k-tuples of non-negative integers $(m_1, \ldots, m_k)$ such that $m_1 + 2m_2 + \cdots + km_k = k$. In this expression, $(-1)^{j+1} e^{-t}$ is the jth derivative of $1 - e^{-t}$. Elementary manipulations lead to the simplified form

$$\Lambda^{(k)}(t) = (-1)^k \sum^{*} \frac{k!}{\prod_{j=1}^{k} m_j!(j!)^{m_j}} (-1)^m \varphi^{(m)}\left(1 - e^{-t}\right) e^{-mt},$$

which shows that $(-1)^k \Lambda^{(k)}$ is non-negative, owing to the fact that φ is completely monotone and hence $(-1)^m \varphi^{(m)} \geq 0$ for any integer $m \geq 0$. Consequently, $\Lambda \in \mathcal{L}_d^0$ for any $d \geq 2$ and $F = e^{-\Lambda}$ can generate a reciprocal Archimedean copula in any dimension.

A trivial example of completely monotone function φ is given, for fixed $\theta \in (0, \infty)$ and all $t \in \mathbb{R}$, by $\varphi(t) = -\theta \ln(t)$. The corresponding generator Λ is that of the reciprocal Clayton copula already considered in Examples 2, 4 and 6. Another example is provided by defining, for fixed $\theta \in (0, \infty)$ and all $t \in (0, \infty)$, $\varphi(t) = t^{-\theta} - 1$. One then has, for all $t \in (0, \infty)$, $\Lambda(t) = (1 - e^{-x})^{-\theta} - 1$ and $F(t) = \exp\{1 - (1 - e^{-x})^{-\theta}\}$.

Next, observe that the class $\mathcal{L}_d^0$ of valid generators Λ is closed under sums and products.

Lemma 8. *Let Λ_1, Λ_2 be arbitrary elements in $\mathcal{L}_d^0$. Then $\Lambda_1 \Lambda_2 \in \mathcal{L}_d^0$ and $\Lambda_1 + \Lambda_2 \in \mathcal{L}_d^0$.*

Proof. To verify that $\Lambda_1 \Lambda_2$ and $\Lambda_1 + \Lambda_2$ satisfy the limit conditions at 0 and ∞ and that $\Lambda_1 + \Lambda_2$ is d-monotone is straightforward. The fact that $\Lambda_1 \Lambda_2$ is d-monotone follows directly from Theorem 5 in [29]. □

The final example of this section shows how Lemma 8 can be used to create new families of reciprocal Archimedean copulas.

Example 12. For $t \in (0, \infty)$ and $\theta_1, \theta_2 \in (0, \infty)$, let $\Lambda_{\theta_1}(t) = t^{-1/\theta_1}$ and $\Lambda_{\theta_2}(t) = -\ln(1 - e^{-t})/\theta_2$ be the exponent measure generators corresponding to the Galambos and reciprocal Clayton copulas, respectively (see Examples 1–2). Consider the exponent measure generator $\Lambda_{\theta_1,\theta_2} = \Lambda_{\theta_1} + \Lambda_{\theta_2}$ whose corresponding reciprocal Archimedean generator is $F_{\theta_1,\theta_2} = F_{\theta_1} F_{\theta_2}$, where $F_{\theta_i} = e^{-\Lambda_j}$ for $j \in \{1, 2\}$.

To draw a random sample from the family generated by F_{θ_1,θ_2} one can use the fact that $H_{F_{\theta_1,\theta_2}} = H_{F_{\theta_1}} H_{F_{\theta_2}}$ is the distribution of $\boldsymbol{Y}_{\theta_1} \vee \boldsymbol{Y}_{\theta_2}$, where for $j \in \{1, 2\}$, $\boldsymbol{Y}_{\theta_j}$ is a random vector distributed as $H_{F_{\theta_j}}$. Alternatively, Algorithm 1 can be used directly. To this end, it is particularly

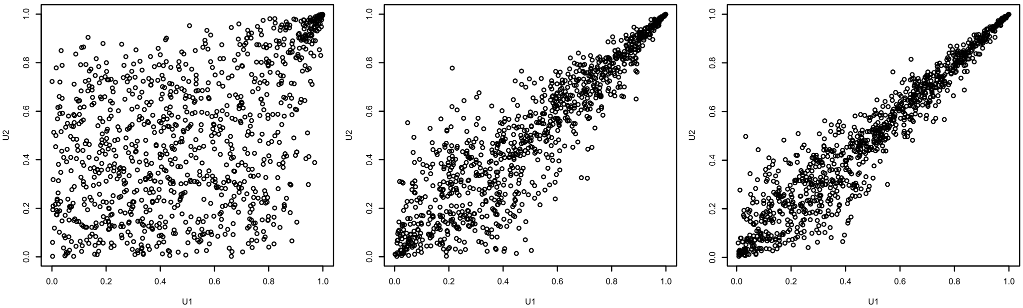

Figure 5. Random samples of size 1000 from the bivariate reciprocal mixture copula from Example 12 with $(\theta_1, \theta_2) = (1.28, 0.1)$ (left), $(5, 1)$ (middle) and $(8, 2)$ (right).

convenient that by Theorem 2, the generalized survival function of the radial measure corresponding to $\Lambda_{\theta_1,\theta_2}$ is of the form $S_{\theta_1,\theta_2} = S_{\theta_1} + S_{\theta_2}$, where for $j \in \{1, 2\}$ S_{θ_j} is the generalized survival function of the radial measure corresponding to Λ_{θ_j}. Eq. (12) then implies that the survival function of the radial variable $R_{k,i}$ that needs to be sampled from in Step 2c is a mixture, that is, for all $r \in (0, \infty)$,

$$\Pr(R_{k,i} > r) = \alpha_k \Pr(R^1_{k,i} > r) + (1 - \alpha_k) \Pr(R^2_{k,i} > r),$$

where the mixing probability is given by $\alpha_k = \{S_{\theta_1}(\Delta_k) - S_{\theta_1}(\Delta_{k-1})\}/\{S_{\theta_1}(\Delta_k) + S_{\theta_2}(\Delta_k) - S_{\theta_1}(\Delta_{k-1}) - S_{\theta_2}(\Delta_{k-1})\}$ and for $j \in \{1, 2\}$, $\Pr(R^j_{k,i} > r)$ is the survival function of the radial variable corresponding to S_{θ_j}, that is, as in Eq. (12) with S_{ν_F} replaced by S_{θ_j}.

Figure 5 shows samples of size 1000 for various values of θ_1 and θ_2 when $d = 2$. Although the right-hand picture suggests that there is both upper- and lower-tail dependence, it can be checked using Lemma 3 that $\lambda_U > 0$ but $\lambda_L = 0$.

7. The completely monotone case

As mentioned, for example, in [20] in a financial context, extendible dependence structures are often useful in high-dimensional problems. Reciprocal Archimedean copulas can be used in this context provided that the corresponding generator F belongs to $\mathcal{F}_d$ for all $d \geq 2$. The Galambos and reciprocal Clayton copulas satisfy this requirement, as do the families considered in Examples 11 and 12.

Theorem 1 implies that if $F \in \mathcal{F}_d$ for every $d \geq 2$, F must be logarithmically completely monotone on $(0, \infty)$, that is, $\Lambda = -\ln(F)$ must be completely monotone on $(0, \infty)$. By the well-known Hausdorff–Bernstein–Widder theorem [28], Theorem 12b, page 161, Λ is then the Laplace transform of a non-negative measure ω_F on $[0, \infty)$, that is, for all $t \in (0, \infty)$,

$$\Lambda(t) = \int_0^\infty e^{-st} \, \mathrm{d}\omega_F(s). \tag{15}$$

Because $\Lambda(t) \to 0$ when $t \to \infty$ and $\Lambda(t) \to \infty$ when $t \to 0$, the measure ω_F satisfies $\omega_F\{0\} = 0$ and $\omega_F[0, \infty) = \infty$.

Example 13. For the Galambos copula with parameter $\theta \in (0, \infty)$, $\Lambda_G(t) = t^{-1/\theta}$ for all $t \in (0, \infty)$; see Example 1. Setting, for all $s \in \mathbb{R}_+$, $\omega_{F_G}[0, s] = s^{1/\theta}/\Gamma(1 + 1/\theta)$, one finds

$$\int_0^\infty e^{-st} \omega_{F_G}(s) = \frac{1/\theta}{\Gamma(1 + 1/\theta)} \int_0^\infty e^{-st} s^{1/\theta - 1} \, ds = t^{-1/\theta} = \Lambda_G(t).$$

Example 14. For the reciprocal Clayton copula with parameter $\theta \in (0, \infty)$, $F_C(t) = (1 - e^{-t})^{1/\theta}$ for all $t \in (0, \infty)$; see Example 2. Expanding $\Lambda_C = -\ln(F_C)$ in Taylor series, one can see that the non-negative measure ω_{F_C} whose Laplace transform is Λ_{F_C} satisfies, for all $t \in (0, \infty)$,

$$\Lambda_C(t) = -\frac{1}{\theta} \ln(1 - e^{-t}) = \frac{1}{\theta} \sum_{k=1}^\infty \frac{e^{-kt}}{k} = \int_0^\infty e^{-ts} \, d\omega_{F_C}(s).$$

Therefore, ω_{F_C} is a discrete measure assigning mass $1/(\theta k)$ to every $k \in \mathbb{N}$.

Because of Eq. (15), reciprocal Archimedean copulas having a logarithmically completely monotone generator have an alternative form, as stated next and proved in Appendix E.

Proposition 5. *Suppose that $F \in \mathcal{F}_d$ is such that $\Lambda = -\ln(F)$ is completely monotone on $(0, \infty)$. Then, for all $u_1, \ldots, u_d \in (0, 1)$,*

$$C_F(u_1, \ldots, u_d) = \exp\left[\int_0^\infty \left[\prod_{k=1}^d \{1 - e^{-sF^{-1}(u_k)}\} - 1\right] \mathrm{d}\omega_F(s)\right].$$

The following result shows how the measure ω_F is related to the radial measure ν_F. It also gives an alternative stochastic representation of reciprocal Archimedean copulas with logarithmically completely monotone generators.

Proposition 6. *Suppose that $F \in \mathcal{F}_d$ is such that $\Lambda = -\ln(F)$ is completely monotone on $(0, \infty)$. Let ν_F be the radial measure corresponding to Λ through Eq. (10) and ω_F be such that Eq. (15) holds for all $t \in (0, \infty)$. Then the following statements hold.*

(i) *Let $g : (0, \infty) \times [0, \infty) \to (0, \infty]$ be the mapping defined, for all $x \in (0, \infty)$ and $w \in \mathbb{R}_+$, by $g(x, w) = x/w$. Let also ϵ_d denote the Erlang distribution with parameter $d \in \mathbb{N}$. Then the image measure of $\epsilon_d \times \omega_F$ by g is ν_F.*
(ii) *Let $h : (0, \infty)^d \times [0, \infty) \to E_d$ be the mapping defined, for all $\boldsymbol{s} \in (0, \infty)^d$ and $t \in \mathbb{R}_+$, by $h(\boldsymbol{x}, t) = \boldsymbol{x}/t$. Let also ϵ_1 denote the unit exponential distribution and $\xi = \sum_k \varepsilon_{(\boldsymbol{x}_k, w_k)}$ be a PRM on $(0, \infty)^d \times [0, \infty)$ with mean measure $\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F$. Then H_F is the distribution function of the random vector*

$$\boldsymbol{Z} = \max_k (\boldsymbol{x}_k / w_k) \vee \boldsymbol{0}.$$

Again, there is a clear parallel between this representation and that of Archimedean copulas with completely monotone generators in terms of multiplicative hazard models. In the present context, the measure ω_F plays the same role as the frailty distribution.

Finally, it is worth pointing out the relation between the Laplace transform (15) and the Williamson d-transform appearing in Theorem 2. To this end, suppose that ν is an arbitrary measure on $(0,\infty]$ such that $\nu(0,\infty]=\infty$, $\nu(r,\infty]<\infty$ for any $r\in(0,\infty)$, and $\nu\{\infty\}=0$. Furthermore, for any $d\geq 2$, let ν_d denote the image measure of ν by the mapping $x\mapsto dx$. Then the following holds, for any $t\in(0,\infty)$, by Lebesgue's Dominated Convergence theorem:

$$\begin{aligned}\lim_{d\to\infty}\mathfrak{W}_d(\nu_d)(t) &= \lim_{d\to\infty}\int_t^\infty\left(1-\frac{t}{r}\right)^{d-1}d\nu_d(r)\\ &= \lim_{d\to\infty}\int_{t/d}^\infty\left(1-\frac{t}{dr}\right)^{d-1}d\nu(r)\\ &= \int_0^\infty e^{-t/r}\,d\nu(r).\end{aligned}$$

In other words, the Williamson d-transform of ν_d, as $d\to\infty$, becomes the Laplace transform of the image measure of ν by the mapping $x\mapsto 1/x$.

8. The case $F^{-1}(0)>0$

In this section, we briefly comment on the condition $F^{-1}(0)=\sup\{x\in\mathbb{R}:F(x)=0\}=0$ that was imposed throughout this article. First recall from Remark 2 that while $F^{-1}(0)<0$ cannot occur, $F^{-1}(0)>0$ is indeed possible. For arbitrary $a\in(0,\infty)$, let $\mathcal{F}_d^a$ denote the set of all continuous distribution functions F such that $F^{-1}(0)=a$ and such that C_F given by (3) is a *bona fide* d-dimensional copula. Let also $\mathcal{C}_d^a=\{C_F:F\in\mathcal{F}_d^a\}$.

As a first observation, note that for any integer $d\geq 2$ and $a\in(0,\infty)$, the independence copula Π is included in $\mathcal{C}_d^a$. Indeed, let F be an arbitrary continuous distribution function with support $[a,b]$, where $b\leq 2a$. Then for arbitrary $\boldsymbol{u}\in[0,1]^d$ and $j\in\{1,\ldots,d\}$, $F^{-1}(u_j)\geq a$. Consequently, for any set $A\subseteq\{1,\ldots,d\}$ with $|A|\geq 2$, $F\{\sum_{j\in A}F^{-1}(u_j)\}=1$ and hence C_F given by (3) reduces to the independence copula.

Because $C_F=\Pi$ for any continuous F whose support is $[a,b]$ with $a\in(0,\infty)$ and $b\leq 2a$, Lemma 1 concerning the uniqueness of a reciprocal Archimedean generator no longer holds. It is also clear from this example that $\Lambda=-\ln(F)$ does not need to be d-monotone when $F\in\mathcal{F}_d^a$, and hence Proposition 1 does not extend to this case. Nevertheless, as stated below, d-monotonicity of Λ is sufficient and implies the max-id property. From the proof given in Appendix F, however, the exponent measure of H_F is no longer ℓ_1-norm symmetric.

Proposition 7. *Let Λ be a d-monotone map on (a,∞) for some $a\in(0,\infty)$ such that $\Lambda(t)\to 0$ as $t\to\infty$ and $\Lambda(t)\to\infty$ as $t\to a$. Then $F=e^{-\Lambda}\in\mathcal{F}_d^a$ and C_F given by (3) is max-id.*

For any $F \in \mathcal{F}_d$ and any $a \in (0, \infty)$, Proposition 7 implies that $F_a \in \mathcal{F}_d^a$, where F_a is given, for all $t \in \mathbb{R}$, by $F_a(t) = F(t - a)$. When $d = 2$, C_{F_a} is given, for all $u_1, \ldots, u_d \in [0, 1]$, by

$$C_{F_a}(u_1, \ldots, u_d) = \prod_{A \in \mathcal{P}_{d,o}} F\Big\{\sum_{k \in A} F^{-1}(u_k) + a\big(|A| - 1\big)\Big\} \Big/ \prod_{A \in \mathcal{P}_{d,e}} F\Big\{\sum_{k \in A} F^{-1}(u_k) + a\big(|A| - 1\big)\Big\}.$$

This shows that reciprocal Archimedean copulas in $\mathcal{C}_d$ can be extended using a "shift" parameter a. This is of interest in particular because if $x_F < \infty$ and $a \ge x_F$, then $C_{F_a} = \Pi$.

9. Discussion

The main purpose of this article was to introduce a new class of multivariate copulas that are tractable, interpretable, and relatively easy to simulate. In dimension $d \ge 2$, these copulas are of the form (3) for some distribution function F having support $[0, x_F]$ with $0 < x_F \le \infty$ whose logarithm $\Lambda = -\ln(F)$ is d-monotone. We propose to call them reciprocal Archimedean copulas because of their affinity with classical Archimedean copulas in the bivariate case; cf. (1) and (2). As was seen, however, the parallel between these two classes of copulas extends much further.

Given the popularity of the Archimedean class, one can easily imagine that reciprocal Archimedean copulas will gradually find applications in situations where it makes sense to use a positive, symmetric (exchangeable) dependence structure. Algorithm 1 ensures that random samples from members of this broad class of copulas can be generated with relative ease. In subsequent work, we hope to explore inferential issues associated with the use of reciprocal Archimedean copulas in multivariate data analysis. Reciprocal Archimedean copulas whose generators are such that $F^{-1}(0) > 0$ may also merit further research. However, given the unidentifiability issue mentioned in Section 8, inference within that class will require particular care.

Appendix A: Proofs of the results in Section 2

Proof of Lemma 1. Sufficiency is obvious. To show necessity, suppose that $F, G \in \mathcal{F}_d$ generate the same copula $C = C_F = C_G \in \mathcal{C}_d$. As any bivariate margin of C is of the form (1), one then has, for all $u_1, u_2 \in (0, 1)$,

$$F\{F^{-1}(u_1) + F^{-1}(u_2)\} = G\{G^{-1}(u_1) + G^{-1}(u_2)\}.$$

Equivalently, if s, t are arbitrary elements in the support of F, then $G^{-1} \circ F(s) + G^{-1} \circ F(t) = G^{-1} \circ F(s + t)$, which shows that $G^{-1} \circ F$ satisfies Cauchy's functional equation on the support of F. Given that $G^{-1} \circ F$ is non-decreasing, the only possible solution is of the form $G^{-1} \circ F(t) = \kappa t$ for some $\kappa \in (0, \infty)$. This concludes the argument. $\square$

The proof of Proposition 1 uses several basic properties of d-monotone functions, which are summarized below for convenience, along with a definition of the concept. For additional details, see, for example, [24,28,29].

Definition. Let $a, b \in \bar{\mathbb{R}}$ and $f = f^{(0)}$ be a real-valued function defined on (a, b). For each $k \in \mathbb{N}$, let $f^{(k)}$ denote the kth derivative of f when it exists.

(i) f is called 1-monotone if it is non-negative and non-increasing on (a, b).
(ii) f is called d-monotone if it is differentiable up to the order $d-2$ on (a, b) and if, for all $k \in \{0, \dots, d-2\}$ and $x \in (a, b)$, one has $(-1)^k f^{(k)}(x) \geq 0$, and further if $(-1)^{d-2} f^{(d-2)}$ is non-increasing and convex in (a, b).
(iii) If f has derivatives of all orders in (a, b) and if $(-1)^k f^{(k)}(x) \geq 0$ for all $k \in \mathbb{N}$ and x in (a, b), then f is said to be *completely monotone*.

Lemma A.1. *Let $a, b \in \bar{\mathbb{R}}$ and f be a real-valued function defined on (a, b). Let $\tilde{f}$ be a function defined at each $x \in (-b, -a)$ by $\tilde{f}(x) = f(-x)$. Further let $d \in \mathbb{N}$ be an integer. Then the following statements are equivalent*:

(i) *f is d-monotone on (a, b)*;
(ii) *f is non-negative and for all $k \in \{1, \dots, d\}$, $x \in (-b, -a)$ and $h_1, \dots, h_k \in (0, \infty)$ such that $-b < x + h_1 + \dots + h_k < -a$, one has*

$$\Delta_{h_k} \cdots \Delta_{h_1} \tilde{f}(x) = \sum_{B \subseteq \{1, \dots, k\}} (-1)^{k-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i\right) \geq 0,$$

where $\Delta_{h_k} \cdots \Delta_{h_1}$ denotes a sequential application of the first-order difference operator Δ_h given by $\Delta_h \tilde{f}(x) \equiv \tilde{f}(x+h) - \tilde{f}(x)$ whenever x and $x+h$ are in $(-b, -a)$.

Proof. This result is excerpted from Proposition A.1 in [24]. What remains to be shown is that for any $k \in \{1, \dots, d\}$, any $x \in (-b, -a)$ and any $h_1, \dots, h_k \in (0, \infty)$ such that $-b < x + h_1 + \dots + h_k < -a$, one has

$$\Delta_{h_k} \cdots \Delta_{h_1} \tilde{f}(x) = \sum_{B \subseteq \{1, \dots, k\}} (-1)^{k-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i\right).$$

This follows easily by induction on k. Indeed, for $k = 1$, the right-hand side reduces to $\tilde{f}(x + h_1) - \tilde{f}(x)$, which is precisely $\Delta_{h_1} \tilde{f}(x)$. Now suppose that the statement is true for all integers up to $k \in \{1, \dots, d-1\}$. Then for arbitrary $x \in (-b, -a)$ and $h_1, \dots, h_{k+1} \in (0, \infty)$ such that $-b < x + h_1 + \dots + h_{k+1} < -a$, one has

$$\begin{aligned}
&\Delta_{h_{k+1}} \cdots \Delta_{h_1} \tilde{f}(x) \\
&\quad = \Delta_{h_{k+1}} \sum_{B \subseteq \{1, \dots, k\}} (-1)^{k-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i\right) \\
&\quad = \sum_{B \subseteq \{1, \dots, k\}} (-1)^{k-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i + h_{k+1}\right) - \sum_{B \subseteq \{1, \dots, k\}} (-1)^{k-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i\right) \\
&\quad = \sum_{B \subseteq \{1, \dots, k+1\}} (-1)^{k+1-|B|} \tilde{f}\left(x + \sum_{i \in B} h_i\right),
\end{aligned}$$

as claimed. □

Proof of Proposition 1. First, note that the limits of Λ at 0 and infinity follow directly from the fact that $\Lambda = -\ln(F)$ and that $F(0) = 0$. Furthermore, if C_F given by (3) is a copula, Sklar's representation theorem guarantees that H_F given by (5) is a d-variate cumulative distribution function. Consequently, all lower-dimensional margins of H_F are distribution functions as well. Note that for every $k \in \{1, \ldots, d\}$, the k-dimensional margin of H_F is of the same form as H_F itself, that is, for all $x_1, \ldots, x_k \in [0, \infty)$, one has

$$\begin{aligned} H_{Fk}(x_1, \ldots, x_k) &= \lim_{\min(y_{k+1}, \ldots, y_d) \to \infty} H_F(x_1, \ldots, x_k, y_{k+1}, \ldots, y_d) \\ &= \prod_{A \in \mathcal{P}_{k,o}} F\Bigl(\sum_{i \in A} x_i\Bigr) \Big/ \prod_{A \in \mathcal{P}_{k,e}} F\Bigl(\sum_{i \in A} x_i\Bigr). \end{aligned}$$

Next, fix $k \in \{1, \ldots, d\}$ and $h_1, \ldots, h_k, t \in (0, \infty)$ such that $t - (h_1 + \cdots + h_k) > 0$. Because H_{Fk} is non-decreasing in its last argument, one has

$$H_F\Biggl(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k} h_i\Biggr) \le H_{Fk}\Biggl(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k-1} h_i\Biggr). \tag{A.1}$$

From the form of H_{Fk}, it follows directly that

$$H_{Fk}\Biggl(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k} h_i\Biggr) = \frac{\prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \notin A}} F(\sum_{i \in A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \in A}} F(t - \sum_{i \notin A} h_i - h_k)}{\prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \notin A}} F(\sum_{i \in A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \in A}} F(t - \sum_{i \notin A} h_i - h_k)}$$

and similarly that

$$H_{Fk}\Biggl(h_1, \ldots, h_{k-1}, t - \sum_{i=1}^{k-1} h_i\Biggr) = \frac{\prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \notin A}} F(\sum_{i \in A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \in A}} F(t - \sum_{i \notin A} h_i)}{\prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \notin A}} F(\sum_{i \in A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \in A}} F(t - \sum_{i \notin A} h_i)}.$$

Upon substitution into (A.1) and cancellation of common terms, the inequality reduces to

$$\frac{\prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \in A}} F(t - \sum_{i \notin A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \in A}} F(t - \sum_{i \notin A} h_i - h_k)}{\prod_{\substack{A \in \mathcal{P}_{k,e} \\ k \in A}} F(t - \sum_{i \notin A} h_i) \prod_{\substack{A \in \mathcal{P}_{k,o} \\ k \in A}} F(t - \sum_{i \notin A} h_i - h_k)} \ge 1.$$

Now observe that $\{B \subseteq \{1, \ldots, k\} : k - |B| \text{ is odd}\}$ is the same as

$$\begin{aligned} &\bigl\{B \subseteq \{1, \ldots, k\} : k \notin B \text{ and } k - |B| \text{ is odd}\bigr\} \\ &\quad \cup \bigl\{B \cup \{k\}, B \subseteq \{1, \ldots, k\} : k \notin B \text{ and } k - |B| \text{ is even}\bigr\} \end{aligned}$$

and that a similar identity holds for $\{B \subseteq \{1, \ldots, k\} : k - |B| \text{ is even}\}$. Then (A.1) becomes

$$\prod_{B \subseteq \{1,\ldots,k\}, k-|B| \text{ is odd}} F\Big(t - \sum_{i \in B} h_i\Big) \Big/ \prod_{B \subseteq \{1,\ldots,k\}, k-|B| \text{ is even}} F\Big(t - \sum_{i \in B} h_i\Big) \geq 1,$$

which is equivalent to

$$\sum_{B \subseteq \{1,\ldots,k\}} (-1)^{k-|B|} \Lambda\Big(t - \sum_{i \in B} h_i\Big) = \sum_{B \subseteq \{1,\ldots,k\}} (-1)^{k-|B|} \tilde{\Lambda}\Big(-t + \sum_{i \in B} h_i\Big) \geq 0,$$

where the map $\tilde{\Lambda} : (-\infty, 0) \to [0, \infty)$ is defined, for all $x \in (-\infty, 0)$, by $\tilde{\Lambda}(x) = \Lambda(-x)$. Given that this inequality holds for all $k \in \{1, \ldots, d\}$ and arbitrary $t, h_1, \ldots, h_k \in (0, \infty)$ with $t - (h_1 + \cdots + h_k) > 0$, the d-monotonicity of Λ follows directly from Lemma A.1. □

Proof of Corollary 1. Given arbitrary $x_1 < y_1$ and $x_2 < y_2$, Lemma A.1 implies that

$$\Lambda(x_1 + y_2) + \Lambda(x_2 + y_1) \leq \Lambda(x_1 + x_2) + \Lambda(y_1 + y_2).$$

This is equivalent to $F(x_1 + y_2)F(x_2 + y_1) \geq F(x_1 + x_2)F(y_1 + y_2)$ and to

$$\frac{F(x_1)F(x_2)F(y_1)F(y_2)}{F(x_1 + x_2)F(y_1 + y_2)} \geq \frac{F(x_1)F(x_2)F(y_1)F(y_2)}{F(x_1 + y_2)F(y_1 + x_2)}.$$

In view of the definition of H_F, this is equivalent to (6). □

Proof of Lemma 2. The formula for Blomqvist's beta is immediate from the definition, that is, $\beta(C_F) = -1 + 4C_F(1/2, 1/2)$. To derive the expression for Spearman's rho, first use integration by parts to check that

$$\int_0^\infty \frac{F(x_2)}{F(x_1 + x_2)}\, dF(x_2) = \frac{1}{2} + \frac{1}{2} \int_0^\infty \frac{F^2(x_2)}{F^2(x_1 + x_2)} f(x_1 + x_2)\, dx_2.$$

Upon substitution into the definition of $\rho(C_F)$, one then gets

$$\begin{aligned}
\rho(C_F) &= -3 + 12 \int_0^\infty \int_0^\infty H(x_1, x_2)\, dF(x_2)\, dF(x_1) \\
&= -3 + 6 \int_0^\infty F(x_1)\, dF(x_1) + 6 \int_0^\infty \int_0^\infty F(x_1) \frac{F^2(x_2)}{F^2(x_1 + x_2)} f(x_1) f(x_1 + x_2)\, dx_2\, dx_1 \\
&= 6 \int_0^\infty H^2(x_1, x_2) r(x_1) f(x_1 + x_2)\, dx_2\, dx_1.
\end{aligned}$$

Similarly,

$$\rho(C_F) = 6 \int_0^\infty H^2(x_1, x_2) r(x_2) f(x_1 + x_2)\, dx_2\, dx_1.$$

The formula reported in the statement of the lemma is simply the average of these two expressions. Finally, to obtain the formula for Kendall's tau, first write

$$\begin{aligned}\tau(C_F) &= -1 + 4\int_0^\infty \int_0^\infty H(x_1, x_2)\, dH(x_1, x_2)\\ &= -1 + 4\int_0^\infty \int_0^\infty H(x_1, x_2)h(x_1, x_2)\, dx_2\, dx_1,\end{aligned}$$

where, for all $x_1, x_2 \in (0, \infty)$,

$$h(x_1, x_2) = H(x_1, x_2)r'(x_1 + x_2) + H(x_1, x_2)\{r(x_1 + x_2) - r(x_1)\}\{r(x_1 + x_2) - r(x_2)\}.$$

Upon substitution into the definition of $\tau(C_F)$, one can check that the result holds if

$$2H^2(x_1, x_2)r'(x_1 + x_2) + 4H^2(x_1, x_2)\{r(x_1 + x_2) - r(x_1)\}\{r(x_1 + x_2) - r(x_2)\}$$

integrates to 1 over $(0, \infty)^2$, which is obvious as it is the density of H^2. □

Proof of Lemma 3. (i) If $\bar{F} \in RV_{-\alpha}$, then $x_F = \infty$. Consequently,

$$\lambda_U(C_F) = 2 - \lim_{t\to\infty} \frac{1 - F^2(t)/F(2t)}{1 - F(t)} = 2 - \lim_{t\to\infty} \frac{F(2t) - F^2(t)}{F(2t)\{1 - F(t)\}}.$$

Writing the numerator of the latter fraction as $F(2t) - 1 + \{1 - F(t)\}\{1 + F(t)\}$ and noting that $\{1 + F(t)\}/F(2t) \to 2$ as $t \to \infty$, one finds

$$\lambda_U(C_F) = \lim_{t\to\infty} \frac{1 - F(2t)}{F(2t)\{1 - F(t)\}} = 2^{-\alpha}.$$

By the same argument, it also follows that $\lambda_U(C_F) = 0$ if $x_F = \infty$ and $\bar{F}$ is rapidly varying. Finally, if $x_F < \infty$, one has

$$\lambda_U(C_F) = 2 - \lim_{t\uparrow x_F} \frac{1 + F(t)}{F(2t)} + \lim_{t\uparrow x_F} \frac{1 - F(2t)}{F(2t)\{1 - F(t)\}} = 0.$$

(ii) First note that F^* is a survival function whose upper endpoint is ∞, from the assumption that the support of F is of the form $[0, x_F]$. By definition of $\lambda_L(C_F)$, one has

$$\lambda_L(C_F) = \lim_{t\downarrow 0} \frac{F(t)}{F(2t)} = \lim_{t\to\infty} \frac{F^*(t)}{F^*(t/2)}.$$

This limit is either $2^{-\beta}$ if $F^* \in RV_{-\beta}$, or 0 if F^* is rapidly varying, as claimed. □

Appendix B: Proofs of the results in Section 3

Proof of Lemma 5. Let S_μ be the generalized survival function of an exponent measure $\mu \in \mathcal{M}_d$. The proof that S_μ satisfies (i)–(iii) is analogous to establishing the well-known properties of a survival function of a multivariate probability distribution. To show (i) it suffices, without loss of generality, to consider the case $x_1 \to \infty$. Then for all $x_2, \dots, x_d \in \mathbb{R}_+$,

$$\lim_{x_1 \to \infty} S_\mu(x_1, \dots, x_d) = \mu\big(\{\infty\} \times (x_2, \infty] \times \cdots \times (x_d, \infty]\big) = 0$$

from the fact that μ is an exponent measure; see Definition 2. Further, the property (ii) follows directly from the fact that σ-finite measures are continuous from above. Finally, to see that (iii) holds, note that for any $\boldsymbol{x} \leq \boldsymbol{y} \in E_d$,

$$\sum_{\boldsymbol{c} \in S_{\boldsymbol{x},\boldsymbol{y}}} \operatorname{sign}(\boldsymbol{c}) S_\mu(c_1, \dots, c_d) = \mu\big((x_1, y_1] \times \cdots \times (x_d, y_d]\big) \geq 0. \tag{B.1}$$

Conversely, suppose that $S : E_d \to [0, \infty)$ satisfies (i)–(iii). Let $\mathcal{E}$ be the collection of all finite unions of sets of the type $(\boldsymbol{x}, \boldsymbol{y}]$ or $\{\boldsymbol{z} \in E_d : z_k = 0, \forall_{i \neq k} x_i < z_i \leq y_i\}$ for all $\boldsymbol{x} \leq \boldsymbol{y} \in E_d$ and $k \in \{1, \dots, d\}$. Observe that $\mathcal{E}$ is a ring that generates the Borel σ-field on E_d and that Eqs. (B.1) and (8) define μ on $\mathcal{E}$. Given that μ is a pre-measure on $\mathcal{E}$ by property (ii), it can be uniquely extended by the Carathéodory process to a measure on the entire σ-field of E_d. To show that μ is an exponent measure, take any $\boldsymbol{x} > \boldsymbol{0}$ and call on Poincaré's identity to write

$$\begin{aligned} \mu[-\infty, \boldsymbol{x}]^{\complement} &= \mu\left(\bigcup_{k=1}^{d} \{\boldsymbol{z} \in E_d : z_k > x_k\}\right) \\ &= \sum_{k=1}^{d} (-1)^{k+1} \sum_{A \subseteq \{1,\dots,d\}, |A|=k} \mu\{\boldsymbol{z} \in E_d : \forall_{j \in A} z_j > x_j\}. \end{aligned} \tag{B.2}$$

In view of Condition (8), this may be rewritten as

$$\begin{aligned} \mu[-\infty, \boldsymbol{x}]^{\complement} &= \sum_{k=1}^{d} (-1)^{k+1} \sum_{A \subseteq \{1,\dots,d\}, |A|=k} \mu\{\boldsymbol{z} \in E_d : \forall_{j \in A} z_j > x_j, \forall_{j \notin A} z_j > 0\} \\ &= \sum_{k=1}^{d} (-1)^{k+1} \sum_{A \subseteq \{1,\dots,d\}, |A|=k} S(\boldsymbol{x}_A) < \infty, \end{aligned} \tag{B.3}$$

where $\boldsymbol{x}_A \in \mathbb{R}_+^d$ is a point whose jth coordinate equals x_j if $j \in A$ and zero otherwise. As S satisfies (i), the right-hand side tends to 0 as $\boldsymbol{x} \to \infty$. Thus $\mu \in \mathcal{M}_d$ by Lemma 4. □

Proof of Proposition 2. First, suppose that Λ is d-monotone and that $\Lambda(t) \to 0$ as $t \to \infty$. Set $\Lambda(\infty) = 0$ and define $S(\boldsymbol{x}) = \Lambda(x_1 + \cdots + x_d)$ for all $\boldsymbol{x} \in E_d$. To show that S is the generalized survival function of a measure $\mu \in \mathcal{M}_d$, first note that S satisfies properties (i)–(ii) because $\Lambda(t) \to 0$ as $t \to \infty$ and because Λ is continuous on $(0, \infty)$; see Lemma 2 in [24]. To establish property (iii), note that for any $\boldsymbol{x} > \boldsymbol{0}$ and $h_1, \ldots, h_d \in (0, \infty)$ such that $x_k - h_k > 0$ for all $k \in \{1, \ldots, d\}$, Eq. (B.1) implies that

$$\mu\big((x_1 - h_1, x_1] \times \cdots \times (x_d - h_d, x_d]\big) = \sum_{B \subseteq \{1,\ldots,d\}} (-1)^{d-|B|} \Lambda\left(\sum_{k=1}^d x_k - \sum_{k \in B} h_k\right).$$

Next, introduce the map $\tilde{\Lambda} : (-\infty, 0) \to [0, \infty)$ defined, for all $t \in (-\infty, 0)$, by $\tilde{\Lambda}(t) = \Lambda(-t)$. In view of Lemma A.1, one may conclude from the above calculation that

$$\mu\big((x_1 - h_1, x_1] \times \cdots \times (x_d - h_d, x_d]\big) = \Delta_{h_1} \cdots \Delta_{h_d} \tilde{\Lambda}\left(-\sum_{k=1}^d x_k\right) \geq 0.$$

Thus S defines a unique exponent measure $\mu \in \mathcal{M}_d$ by Lemma 5.

Conversely, suppose that $\mu \in \mathcal{M}_d$ is ℓ_1-norm symmetric with generator Λ_μ. Then

$$\lim_{t \to \infty} \Lambda_\mu(t) = \lim_{t \to \infty} S_\mu(t/d, \ldots, t/d) = \lim_{t \to \infty} \mu\big((t/d, \infty] \times \cdots \times (t/d, \infty]\big) = 0$$

because μ is an exponent measure. Next, define the map $\tilde{\Lambda}_\mu : (-\infty, 0) \to [0, \infty)$ at every $t \in (-\infty, 0)$ by $\tilde{\Lambda}_\mu(t) = \Lambda_\mu(-t)$. For any $k \in \{1, \ldots, d\}$, $x \in (-\infty, 0)$ and $h_1, \ldots, h_k \in (0, \infty)$ such that $x + h_1 + \cdots + h_k < 0$, one has

$$\begin{aligned}
\Delta_{h_1} \cdots \Delta_{h_k} \tilde{\Lambda}_\mu(x) &= \sum_{B \subseteq \{1,\ldots,k\}} (-1)^{k-|B|} \tilde{\Lambda}_\mu\left(x + \sum_{i \in B} h_i\right) \\
&= \sum_{j=0}^k (-1)^{k-j} \sum_{B \subseteq \{1,\ldots,k\}, |B|=j} \Lambda_\mu\left\{-(k-j)x/k - \sum_{i \in B}(x/k + h_i)\right\}.
\end{aligned}$$

Observing that the right-hand side is equal to

$$\sum_{j=0}^k (-1)^{k-j} \sum_{B \subseteq \{1,\ldots,k\}, |B|=j} \mu\left\{\boldsymbol{x} \in E_d : \forall_{i \notin B} x_i > -\frac{x}{k} \text{ and } \forall_{i \in B} x_i > -\frac{x}{k} - h_i\right\},$$

one may then conclude that

$$\Delta_{h_1} \cdots \Delta_{h_k} \tilde{\Lambda}_\mu(x) = \mu\big\{\boldsymbol{x} \in E_d : \forall_{i \in \{1,\ldots,k\}} x_i \in (-h_i - x/k, -x/k]\big\}.$$

As the last expression is clearly non-negative, Λ_μ is d-monotone by Lemma A.1. $\square$

Appendix C: Proofs of the results in Section 4

The proof of Theorem 2 relies on Lemma 9 below, which characterizes d-monotone functions using the results of [29]. Its proof is given for completeness; it proceeds along the same lines as the proof of Proposition 3.1 in [24].

Lemma 9. *A function f is d-monotone on $(0,\infty)$ if and only if there exists a unique Radon measure ν on $(0,\infty]$ such that, for all $t \in (0,\infty)$,*

$$f(t) = \int_t^\infty (1-t/r)^{d-1}\,\mathrm{d}\nu(r).$$

Letting $f_+^{(d-1)}$ denote the right-hand derivative of $f^{(d-2)}$, one also has, for any $s \in (0,\infty)$,

$$\nu(r,\infty] = \sum_{k=0}^{d-2} \frac{(-1)^k f^{(k)}(r)}{k!} r^k + \frac{(-1)^{d-1} f_+^{(d-1)}(r)}{(d-1)!} r^{d-1},$$

with $\nu(0,\infty] = \lim_{t\downarrow 0} f(t)$ and $\nu\{\infty\} = \lim_{t\to\infty} f(t)$.

Proof of Lemma 9. Theorems 1 and 2 of [29] yield that f is d-monotone on $(0,\infty)$ if and only if there exists a unique measure ν^* on $[0,\infty)$ such that

$$f(t) = \int_0^{1/t} (1-rt)^{d-1}\,\mathrm{d}\nu^*(r).$$

Its generalized distribution function γ is defined, for all $r \in \mathbb{R}_+$, by $\gamma(r) = \nu^*[0,r]$. Theorem 3 in [29] implies $\gamma(0-) = 0$, $\gamma(0) = f(\infty)$ and for all $r \in (0,\infty)$,

$$\gamma(r) = \sum_{k=0}^{d-2} \frac{(-1)^k f^{(k)}(1/r)}{k!} r^{-k} + \frac{(-1)^{d-1} f_-^{(d-1)}(1/r)}{(d-1)!} r^{-(d-1)},$$

where $f_-^{(d-1)}$ denotes the left-hand derivative of $f^{(d-2)}$. The term generalized distribution function is used because although γ is non-decreasing and right-continuous, it may not be that $\gamma(r) \to 1$ as $r \to \infty$; in fact, it may happen that γ is not bounded from above.

The statement of Lemma 9 then follows upon letting ν be the image measure of ν^* by the map $\phi : [0,\infty) \to (0,\infty]$ defined, for all $r \ge 0$, by $\phi(r) = 1/r$. Observe that ν is a Radon measure on $(0,\infty]$ if and only if for all $r \in (0,\infty)$, $\nu(r,\infty] < \infty$. In particular, the function f given in (9) is well defined and, for all $r \in (0,\infty)$, $\nu(r,\infty] = \nu^*[0,1/r) = \gamma(1/r-)$.

Finally, the fact that $f(t) \to \nu(0,\infty]$ as $t \to 0$ follows from the integral representation of f by Levi's Monotone Convergence theorem and $\lim_{t\to\infty} f(t) = \nu\{\infty\}$ follows from the proof of Theorem 1 in [29]. $\square$

Proof of Theorem 2. First, suppose that μ is a measure on E_d whose image measure by $\mathcal{T}$ is $\nu_\mu \times \sigma_d$, where ν_μ is a Radon measure on $(0,\infty]$ such that $\nu_\mu\{\infty\} = 0$. Because $\mathcal{T}$ is one-to-one,

μ is the image measure of $\nu_\mu \times \sigma_d$ by $\mathcal{T}^{-1}$. To see that μ satisfies Condition (8), note that for any $k \in \{1, \ldots, d\}$,

$$\begin{aligned}\mu\{\boldsymbol{x} \in E_d : x_k = 0\} &= \mu\left(\bigcup_{j=1}^{\infty}\{\boldsymbol{x} \in E_d : \forall_{i \neq k} x_j > z_{jn} \text{ and } x_k = 0\}\right) \\ &= \lim_{n\to\infty} \mu\{\boldsymbol{x} \in E_d : \forall_{j \neq k} x_j > z_{jn} \text{ and } x_k = 0\},\end{aligned}$$

where (z_n) is an arbitrary sequence such that $z_n > \mathbf{0}$ and $z_n \to \mathbf{0}$ as $n \to \infty$. The fact that $\mu\{\boldsymbol{x} \in E_d : x_k = 0\} = 0$ now follows because

$$\mu\{\boldsymbol{x} \in E_d : \forall_{j \neq k} x_j > z_{jn}, x_k = 0\} = \nu_\mu\left(\sum_{j \neq k} z_{jn}, \infty\right] \times \sigma_d\{\boldsymbol{s} \in \mathbb{S}_d : s_k = 0\} = 0.$$

Furthermore, for any $\boldsymbol{x} \in E_d$,

$$\begin{aligned}\mu(\boldsymbol{x}, \boldsymbol{\infty}] &= \nu_\mu \times \sigma_d\left\{(r, \boldsymbol{s}) : r > \sum_{k=1}^{d} x_k, \boldsymbol{s} > \boldsymbol{x}/r\right\} \\ &= \int_{x_1+\cdots+x_d}^{\infty}\left(1 - \sum_{k=1}^{d} x_k/r\right)^{d-1} \mathrm{d}\nu_\mu(r) \\ &\equiv \Lambda_\mu(x_1 + \cdots + x_d).\end{aligned} \tag{C.1}$$

Given that $\nu_\mu\{\infty\} = 0$, it follows from Lemma 9 that $\Lambda_\mu(t) \to 0$ as $t \to \infty$. Put together, μ is an ℓ_1-norm symmetric exponent measure with generator $\Lambda_\mu = \mathfrak{W}_d(\nu_\mu)$.

Conversely, suppose that μ is an ℓ_1-norm symmetric exponent measure with generator Λ_μ. Proposition 2 implies that Λ_μ is d-monotone on $(0, \infty)$ and such that $\Lambda_\mu(t) \to 0$ as $t \to \infty$. By Lemma 9, Λ_μ is the Williamson d-transform of a unique Radon measure ν_μ on $(0, \infty]$ such that $\nu_\mu\{\infty\} = 0$ and, for all $r \in (0, \infty)$, $\nu_\mu(r, \infty] = \mathfrak{W}_d^{-1}(\Lambda_\mu)(r)$. From (C.1), it follows that the generalized survival function of μ is the same as the generalized survival function of the image measure of $\nu_F \times \sigma_d$ by $\mathcal{T}^{-1}$. Lemma 5 and the fact that $\mathcal{T}$ is one-to-one thus imply that the image measure of μ by $\mathcal{T}$ is $\nu_\mu \times \sigma_d$, as claimed. □

Proof of Proposition 3. The result follows from the fact that if $C \in \mathcal{C}_d$ is extreme-value, F must be a univariate extreme-value distribution. Because all bivariate margins of C are identical and of the form (1) one can assume, without loss of generality, that $d = 2$. For arbitrary $t \in (0, \infty)$ and $u, v \in (0, 1)$, one then has $C^t(u^{1/t}, v^{1/t}) = C(u, v)$. Thus, for arbitrary $x, y \in (0, \infty)$, H_F given by (5) satisfies

$$H_F^t(x, y) = [C\{F(x), F(y)\}]^t = C^t[\{F^t(x)\}^{1/t}, \{F^t(y)\}^{1/t}] = C\{F^t(x), F^t(y)\}.$$

Given the specific form of C, however, this means that, for arbitrary $t, x, y \in (0, \infty)$, one has

$$\frac{F^t(x)F^t(y)}{F^t(x+y)} = \frac{F^t(x)F^t(y)}{F\{F^{-1} \circ F^t(x) + F^{-1} \circ F^t(y)\}}$$

or equivalently, $F^{-1} \circ F^t(x+y) = F^{-1} \circ F^t(x) + F^{-1} \circ F^t(y)$. Accordingly, the map $G_t = F^{-1} \circ F^t$ satisfies Cauchy's functional equation on $(0, \infty)$, and hence it is linear. Given that G_t is non-decreasing and that $G_t(0) = 0$, one can conclude that there exists a constant $\alpha_t \in (0, \infty)$ such that, for all $x \in (0, \infty)$, $G_t(x) = \alpha_t x$ or, equivalently, $F^t(x) = F(\alpha_t x)$. As this relationship holds for all $t \in (0, \infty)$, it follows that F is max-stable [26], and hence an extreme-value distribution whose support is $(0, \infty)$. One can further conclude from the Fisher–Tippett theorem that F is a Fréchet distribution Φ_θ for some $\theta \in (0, \infty)$, that is, $F(t) = \Phi_\theta(t) = \exp(-t^{-1/\theta})$ for all $t \in (0, \infty)$. From Example 1, the copula generated by F is the Galambos copula with parameter θ. □

Appendix D: Proof of Lemma 6

Denote by ν_F^* the restriction of the radial measure ν_F to $(\Delta, \infty]$ and let μ_F^* be the image measure of $\nu_F^* \times \sigma_d$ through the transformation $\mathcal{T}^{-1}$. If then follows from Theorem 2 that μ_F^* is an ℓ_1-norm symmetric exponent measure with generator given, for all $t \in (0, \infty)$, by $\Lambda^*(t) = \Lambda(t \vee \Delta)$. Proceeding as in the proof of Proposition 4, one can see that $\boldsymbol{Y}^*$ has a max-id distribution with joint distribution function given, for all $\boldsymbol{x} > \boldsymbol{0}$, by $H^*(\boldsymbol{x}) = e^{-\mu_F^*[-\infty, \boldsymbol{x}]^\complement}$. From the discussion just preceding Theorem 1, one can conclude that H^* is of the form (5) with F replaced by $F^* = e^{-\Lambda^*}$. As a result, one has, for all $\boldsymbol{x} \geq (\Delta, \dots, \Delta)$, $H^*(\boldsymbol{x}) = H_F(\boldsymbol{x})$. In particular, $\Pr(Y_1^* > \Delta, \dots, Y_d^* > \Delta) = \bar{H}_F(\Delta, \dots, \Delta)$, as claimed. Finally, the inequality follows from the fact that H_F is positively upper orthant dependent (PUOD); see, for example, Theorem 8.6 in [13].

Appendix E: Proofs of the results in Section 7

Proof of Proposition 5. From the Hausdorff–Bernstein–Widder theorem, one can write $\Lambda = -\ln(F)$ in the form (15) for some non-negative measure ω_F on $[0, \infty)$. Expression (3) then simplifies, for all $u_1, \dots, u_d \in (0, 1)$, to

$$C_F(u_1, \dots, u_d) = \exp\left\{ \sum_{A \in \mathcal{P}_{d,o} \cup \mathcal{P}_{d,e}} (-1)^{|A|} \int_0^\infty e^{-s \sum_{k \in A} F^{-1}(u_k)} \, \mathrm{d}\omega_F(s) \right\},$$

which yields the stated result because $\mathcal{P}_{d,o} \cup \mathcal{P}_{d,e} = \{A \subseteq \{1, \dots, d\} : A \neq \varnothing\}$ and

$$\prod_{k=1}^{d} \{1 - e^{-sF^{-1}(u_k)}\} = \sum_{A \subseteq \{1,\dots,d\}} \prod_{k \in A} \{-e^{-sF^{-1}(u_k)}\}$$

by the multinomial formula. □

Proof of Proposition 6. We prove part (i) by showing that the Williamson d-transform of the image measure of $\epsilon_d \times \omega_F$ by g, ν^*, say, is Λ. Indeed, for any $t \in (0, \infty)$,

$$\begin{aligned}\int_t^\infty \left(1 - \frac{t}{r}\right)^{d-1} d\nu^*(r) &= \int_0^\infty \int_{tw}^\infty \left(1 - \frac{tw}{x}\right)^{d-1} \frac{e^{-x} x^{d-1}}{\Gamma(d)}\, dx\, d\omega_F(w) \\ &= \int_0^\infty \frac{e^{-tw}}{\Gamma(d)} \int_{tw}^\infty (x - tw)^{d-1} e^{-x+tw}\, dx\, d\omega_F(w) \\ &= \int_0^\infty e^{-tw}\, d\omega_F(w) \\ &= \Lambda(t).\end{aligned}$$

The conclusion then follows from the fact, established in Lemma 9, that a Radon measure on $(0, \infty]$ is uniquely specified by its Williamson d-transform.

Part (ii) follows from the fact that by Proposition 3.7 in [26], $\tilde{\xi} = \sum_k \varepsilon_{\boldsymbol{x}_k/w_k}$ is a PRM on E_d with mean measure μ^*, which is the image measure of $\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F$ by h. Now μ^* clearly satisfies Eq. (8). Moreover, the generalized survival function of μ^* is given, for any $\boldsymbol{y} \in E_d$, by

$$\begin{aligned}S_{\mu^*}(\boldsymbol{y}) &= (\epsilon_1 \times \cdots \times \epsilon_1 \times \omega_F)\big\{\boldsymbol{x} \in (0, \infty)^d, w \geq 0 : \forall_{j \in \{1,\ldots,d\}} x_j > w y_j\big\} \\ &= \int_0^\infty \prod_{j=1}^d e^{-w y_j}\, d\omega_F(w) \\ &= \Lambda(y_1 + \cdots + y_d).\end{aligned}$$

In particular, therefore, μ^* coincides with the ℓ_1-norm symmetric exponent measure μ_F generated by Λ. The rest of the argument is analogous to the proof of Proposition 4. $\square$

Appendix F: Proof of Proposition 7

Fix $a \in (0, \infty)$, let $\boldsymbol{a} = (a, \ldots, a) \in \mathbb{R}^d$ and define $E_d^a = [\boldsymbol{a}, \boldsymbol{\infty}] \setminus \{\boldsymbol{a}\}$. For any $I \subsetneq \{1, \ldots, d\}$, define the function S_I at any $\boldsymbol{x} \in (\boldsymbol{a}, \boldsymbol{\infty}]$ by

$$S_I(\boldsymbol{x}) = \sum_{B \subseteq I} (-1)^{|I|-|B|} \Lambda\left\{\sum_{j \notin I} x_j + a\big(|I| - |B|\big)\right\}.$$

Now let $\tilde{\Lambda} : (-\infty, -a) \to [0, \infty)$ be given, for all $t < -a$, by $\tilde{\Lambda}(t) = \Lambda(-t)$. It then follows from Lemma A.1 that

$$S_I(\boldsymbol{x}) = \underbrace{\Delta_a \cdots \Delta_a}_{I \text{ times}} \tilde{\Lambda}\left(-\sum_{j \notin I} x_j - a|I|\right) \geq 0.$$

In order to construct a measure μ on E_d^a, consider the ring $\mathcal{E}^a$ of all finite unions of sets of the type $\{\boldsymbol{z} \in E_d^a : \forall_{j \in I}\ z_j = a, \forall_{j \notin I} x_j < z_j \leq y_j\}$ for all $I \subsetneq \{1, \ldots, d\}$ and all $\boldsymbol{x}, \boldsymbol{y} \in (\boldsymbol{a}, \boldsymbol{\infty}]$ with

$\boldsymbol{x} \le \boldsymbol{y}$. Then $\mathcal{E}^a$ generates the Borel σ-field on E_d^a. For any $I \subsetneq \{1, \dots, d\}$ and any $\boldsymbol{x}, \boldsymbol{y} \in (\boldsymbol{a}, \infty]$ with $\boldsymbol{x} \le \boldsymbol{y}$, set

$$\mu\{\boldsymbol{z} \in E_d^a : \forall_{j\in I}\ z_j = a, \forall_{j\notin I} x_j < z_j \le y_j\} = \sum_{\boldsymbol{c}\in S_{\boldsymbol{x},\boldsymbol{y}}^I} \operatorname{sign}(\boldsymbol{c}) S_I(\boldsymbol{c}),$$

where $S_{\boldsymbol{x},\boldsymbol{y}}^I = \{\boldsymbol{z} \in E_d^a : \forall_{j\in I}\ z_j = a, \forall_{j\notin I}\ z_j \in \{x_j, y_j\}\}$ and $\operatorname{sign}(\boldsymbol{c})$ is as in Lemma 5. The right-hand side can be easily rewritten as

$$\begin{aligned}
&\sum_{\boldsymbol{c}\in S_{\boldsymbol{x},\boldsymbol{y}}^I} \operatorname{sign}(\boldsymbol{c}) \sum_{B\subseteq I} (-1)^{|I|-|B|} \Lambda\Big\{\sum_{j\notin I} x_j + a\big(|I| - |B|\big)\Big\} \\
&\quad = \sum_{A\subseteq\{1,\dots,d\}} (-1)^{d-|A|} \Lambda\Big\{\sum_{j\in A\cap I^{\complement}} x_j + \sum_{j\in A^{\complement}\cap I^{\complement}} y_j + a\big(|I| - |A\cap I|\big)\Big\}.
\end{aligned} \tag{F.1}$$

Thus if one sets $t = \sum_{j\notin I} y_j + a|I|$, $h_j = a$ if $j \in I$, and $h_j = y_j - x_j$ if $j \in I^{\complement}$, then

$$\sum_{j\in A\cap I^{\complement}} x_j + \sum_{j\in A^{\complement}\cap I^{\complement}} y_j + a\big(|I| - |A\cap I|\big) = t - \sum_{j\in A} h_j > a.$$

By Lemma A.1(ii) and the d-monotonicity of Λ, the right-hand side of (F.1) equals

$$\sum_{A\subseteq\{1,\dots,d\}} (-1)^{d-|A|} \Lambda\Big(t - \sum_{j\in A} h_j\Big) = \Delta_{h_1} \cdots \Delta_{h_d} \tilde{\Lambda}(-t) \ge 0.$$

Therefore, μ is a pre-measure on $\mathcal{E}^a$ which can be extended to a measure on the entire σ-field of E_d^a by the Carathéodory process. Furthermore, for any $\boldsymbol{x} > \boldsymbol{a}$ and $A \subseteq \{1, \dots, d\}$ with $|A| = k \ge 1$, one has

$$\begin{aligned}
&\mu\big\{\boldsymbol{z} \in E_d^a : \forall_{j\in A} z_j > x_j\big\} \\
&\quad = \sum_{I\subseteq A^{\complement}} \mu\big\{\boldsymbol{z} \in E_d^a : \forall_{j\in A} z_j > x_j, \forall_{j\in I} z_j = a, \forall_{j\in A^{\complement}\cap I^{\complement}} z_j > a\big\} \\
&\quad = \sum_{I\subseteq A^{\complement}} \sum_{B\subseteq I} (-1)^{|I|-|B|} \Lambda\Big\{\sum_{j\in A} x_j + a\big(\big|A^{\complement}\big| - |B|\big)\Big\}.
\end{aligned}$$

Because the summand only depends on the cardinality of the sets I and B, the last expression simplifies to

$$\sum_{\ell=0}^{d-k} \sum_{m=0}^{\ell} \binom{d-k}{\ell} \binom{\ell}{m} (-1)^{\ell-m} \Lambda\Big\{\sum_{j\in A} x_j + a(d-k-m)\Big\}.$$

By interchanging the order of summation and setting $\ell^* = \ell - m$, one gets

$$\sum_{m=0}^{d-k} \binom{d-k}{m} \Lambda\left\{\sum_{j\in A} x_j + a(d-k-m)\right\} \sum_{\ell^*=0}^{d-k-m} \binom{d-k-m}{\ell^*} (-1)^{\ell^*}.$$

Clearly, the second summand vanishes unless $d-k-m=0$. Hence,

$$\mu\{z \in E_d^a : \forall_{j\in A} z_j > x_j\} = \Lambda\left(\sum_{j\in A} x_j\right).$$

From (B.2), it then follows that for any $\boldsymbol{x} > \boldsymbol{a}$,

$$\mu[-\infty, \boldsymbol{x}]^{\complement} = \sum_{k=1}^{d} (-1)^{k+1} \sum_{A\subset\{1,\dots,d\},|A|=k} \Lambda\left(\sum_{j\in A} x_j\right), \tag{F.2}$$

where $^{\complement}$ refers to the complement within E_d^a. Because $\Lambda(t) \to 0$ as $t \to \infty$, μ is an exponent measure by a straightforward extension of Lemma 4 to E_d^a stated in Section 5.3 of [26]. Note in passing that in contrast to the exponent measure constructed in the proof of Proposition 2 for a similar purpose,

$$\mu(E_d^a \setminus (\boldsymbol{a}, \infty]) = \infty, \qquad \mu(\boldsymbol{a}, \infty] = \Lambda(da) < \infty.$$

As μ is an exponent measure on E_d^a, the distribution function defined, for all $\boldsymbol{x} \geq \boldsymbol{a}$, by

$$H_\mu(\boldsymbol{x}) = e^{-\mu[-\infty,\boldsymbol{x}]^{\complement}}$$

is max-id. Because $\Lambda(t) \to \infty$ as $t \to a$, the margins of H_μ are continuous. From (F.2), H_μ is of the form (5) and its copula is reciprocal Archimedean with generator F, which concludes the argument.

Acknowledgments

Funding in partial support of this work was provided by the Canada Research Chairs Program, the Natural Sciences and Engineering Research Council (RGPIN/39476–2011, RGPIN/06801–2015), the Canadian Statistical Sciences Institute, and the Fonds de recherche du Québec – Nature et technologies (2015–PR–183236).

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Received October 2016 and revised May 2017