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Probabilistic proof of product formulas for Bessel functions

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We write, for geometric index values, a probabilistic proof of the product formula for spherical Bessel functions. Though our proof looks elementary in the real variable setting, it has the merit to carry over without any further effort to Bessel-type hypergeometric functions of one matrix argument, thereby avoid complicated arguments from differential geometry. Moreover, the representative probability distribution involved in the last setting is shown to be closely related to the symmetrization of upper-left corners of Haar-distributed orthogonal matrices. Analysis of this probability distribution is then performed and in case it is absolutely continuous with respect to Lebesgue measure on the space of real symmetric matrices, we derive an invariance-property of its density. As a by-product, Weyl integration formula leads to the product formula for Bessel-type hypergeometric functions of two matrix arguments.

Keywords: conditional independence; hypergeometric functions; matrix-variate normal distribution; product formula

1. Reminder and motivation

The spherical Bessel function j_ν of index ν is defined for all complex z and all $\nu > -1$ by Watson [15]

$$j_\nu(z) = \sum_{l=0}^{+\infty} \frac{(-1)^l}{(\nu+1)_l l!}\left(\frac{z}{2}\right)^{2l},$$

where $(\nu+1)_l := \Gamma(\nu+l+1)/\Gamma(\nu+1)$ denotes the usual Pochhammer symbol. It provides a basic example of one-variable special function satisfying a product formula that opened the way to a rich harmonic analysis. More precisely, for $\nu \geq -1/2$ and nonnegative real numbers x, y, z, it is well known that

$$j_\nu(xy) j_\nu(zy) = \int_{\mathbb{R}_+} j_\nu(\xi y) \tau_{x,z}^\nu(\mathrm{d}\xi), \tag{1.1}$$

where $\tau_{x,z}^\nu$ is a compactly-supported probability distribution. Recall that for $\nu > -1/2$, (1.1) is a trivial consequence of the addition theorem for Bessel functions (see, for instance, Chapter XI in Watson [15]) while it obviously holds for $\nu = -1/2$ since $j_{-1/2}(z) = \cos(z)$. Nevertheless, for an integer $p \geq 1$ and for the so-called geometrical index values $\nu = (p/2) - 1$, (1.1) may be

1350-7265

derived from the following Poisson-type integral representation

$$j_{(p/2)-1}\big(|v|\big) = \int_{S^{p-1}} \mathrm{e}^{\mathrm{i}\langle v,s\rangle}\sigma_1(\mathrm{d}s), \qquad v \in \mathbb{R}^p, \tag{1.2}$$

where σ_1 is the uniform distribution on the unit sphere S^{p-1} and $\langle\cdot,\cdot\rangle$, $|\cdot|$ are respectively the Euclidean inner product and the associated Euclidean norm in $\mathbb{R}^p$. Indeed, if we set $|v| = y$, then

$$j_{(p/2)-1}\big(x|v|\big) j_{(p/2)-1}\big(z|v|\big) = \int_{\mathbb{R}^p} \mathrm{e}^{\mathrm{i}\langle v,s\rangle}(\sigma_x \star \sigma_z)(\mathrm{d}s),$$

where σ_x, σ_z are the uniform distributions on spheres of radii x, z, respectively. But according to Ragozin [11], Corollary 5.2, page 1149, the probability distribution $\sigma_x \star \sigma_z$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^p$ and due to its rotational invariance it has a radial density. The use of spherical coordinates yields then (1.1). Avoiding techniques from differential geometry like the ones used to prove the absolute continuity of $\sigma_x \star \sigma_z$, we write a probabilistic proof of (1.1) for geometric index values and supply a probabilistic interpretation of $\tau_{x,z}^{(p/2)-1}$. Our starting point is the elementary fact that the conditional distribution of a standard normal vector N in $\mathbb{R}^p$ given its radius $|N|$ is the uniform distribution on the sphere of radius $|N|$. The product of two spherical Bessel functions turns towards the conditional independence of two independent standard normal vectors N_1, N_2 relative to the σ-field generated by their radii $|N_1|, |N_2|$ (Revuz [13]). The representative probability distribution $\tau_{x,z}^{(p/2)-1}$ is then seen to be the conditional distribution of the radial part $|N_1 + N_2|$ given $(|N_1| = x, |N_2| = z)$. In fact, $N_1 + N_2$ is again distributed as a standard Gaussian vector (up to a constant) and its angular part is independent from both radii $|N_1|$ and $|N_2|$. The reader will easily realize from the ingredients needed in the proof that choosing any multivariate stable distribution in $\mathbb{R}^p$ whose density is a radial function does not alter our proof. But the Fourier transform of a radial function is again radial therefore the choice restricts uniquely to isotropic or rotationally invariant stable distributions (whose Lévy exponents are given up to a constant by $v \mapsto |v|^\alpha, \alpha \in (0, 2]$ (Sato [14], page 86)).

Our proof has also the merit to carry over after mild modifications to some matrix analogues of spherical Bessel functions, requiring no knowledge of the theory of Gelfand pairs and their spherical functions. Those we consider here are known as Bessel-type hypergeometric functions of one and two $m \times m$ real symmetric matrix arguments. This is by no means a loss of generality since product formulas over the complex division algebra may be easily derived along the same lines. For functions of one matrix argument, the proof is identical to that written for $j_{(p/2)-1}$. Besides, the representative probability distribution is seen to be the conditional distribution of the radial part of the sum of two independent $p \times m$ $(p \geq m)$ standard matrix-variate normal distributions given the radial part of each. We shall prove that this conditional distribution is closely related to the distribution of the $m \times m$ upper-left corner of an orthogonal matrix of size p, whence its absolute continuity (with respect to Lebesgue measure) is deduced for $p \geq m + 1$. For these values of p, one easily derives the product formula for functions of two arguments using Weyl integration formula for the space of real symmetric matrices. As a matter of fact, the corresponding representative probability distribution has an analogous description in terms of singular values rather than matrices. Besides, when $p \geq 2m$, a result due to Collins provides a detailed description of the distribution of the upper-left corner of an orthogonal matrix (see remark at the end of the paper), agreeing with the variable change formula given in Lemma 3.7, page 495 in Herz [9]. Note finally that since Bessel-type hypergeometric functions of two matrix arguments we consider here are instances of generalized Bessel functions associated with B-type root systems (see the last chapter in Chybiryakov *et al.* [3]), then our approach resembles the one carried for proving Theorem 5.16 (ii) in Biane, Bougerol and O'Connell [1].

The paper is organized as follows. In the next section, we consider spherical Bessel functions $j_{(p/2)-1}$ and prove (1.1) for geometric index values. In Section 3, we extend our proof to Bessel-type hypergeometric functions of one real symmetric matrix argument. In the last section, we perform a detailed analysis of the representative probability distribution: it is absolutely continuous for $p \geq m+1$ and its density enjoys a certain averaged bi-invariance property with respect to the orthogonal group. The product formula for functions of two real symmetric matrix arguments follows then from Weyl integration formula on the space of real symmetric matrices.

2. Product formula for spherical Bessel functions

All random variables occurring below are defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we denote $\mathbb{E}$ the corresponding expectation. Furthermore, for the σ-field $\sigma(X)$ generated by a random variable X, we write

$$\mathbb{E}[\cdot|X] \qquad \text{for } \mathbb{E}\big[\cdot|\sigma(X)\big],$$

and we recall that all equalities involving conditional expectations hold $\mathbb{P}$-almost surely. Let N be a standard normal vector[1] in $\mathbb{R}^p$ and let $N = R\Theta$ be its polar decomposition ($R > 0$ and $\Theta \in S^{p-1}$). Then, R and Θ are independent and Θ is uniformly distributed on S^{p-1}. It follows that for any $v \in \mathbb{R}^p$

$$\mathbb{E}\big[e^{i\langle v,N\rangle}|R\big] = \int_{S^{p-1}} e^{i\langle v,Rs\rangle}\sigma_1(ds) = j_{(p/2)-1}\big(|v|R\big).$$

In fact, if X, Y are independent random variables valued in some measurable spaces and if $\mathcal{D}_Y$ stands for the distribution of Y, then

$$\mathbb{E}\big[f(X,Y)|X\big] = \int f(X,y)\mathcal{D}_Y(dy)$$

for any bounded Borel function f (see Revuz [13], page 108, Exercise 4.27).

Now, let N_1, N_2 be two independent standard normal vectors in $\mathbb{R}^p$ with polar decompositions $N_1 = R_1\Theta_1$, $N_2 = R_2\Theta_2$ respectively, and consider the product σ-field $\sigma(R_1, R_2)$ generated by R_1, R_2. Then, the independence of N_1 and N_2 implies that (Revuz [13])

$$\mathbb{E}\big[e^{i\langle v,N_1\rangle}|R_1\big] = \mathbb{E}\big[e^{i\langle v,N_1\rangle}|R_1, R_2\big],$$
$$\mathbb{E}\big[e^{i\langle v,N_2\rangle}|R_2\big] = \mathbb{E}\big[e^{i\langle v,N_2\rangle}|R_1, R_2\big].$$

[1] Its coordinates are independent centered normal distributions with unit variance.

Besides, N_1, N_2 are conditionally independent relative to $\sigma(R_1, R_2)$ (see Revuz [13], page 109, Exercise 4.32). In fact, one has for any bounded Borel function $f:\mathbb{R}^p \to \mathbb{R}$

$$\mathbb{E}\big[f(N_2)|N_1, R_1, R_2\big] = \mathbb{E}\big[f(N_2)|R_2\big] = \mathbb{E}\big[f(N_2)|R_1, R_2\big].$$

Thus,

$$\mathbb{E}\big[e^{i\langle v, N_1\rangle}|R_1\big]\mathbb{E}\big[e^{i\langle v, N_2\rangle}|R_2\big] = \mathbb{E}\big[e^{i\langle v, N_1+N_2\rangle}|R_1, R_2\big].$$

Write $N_1 + N_2 := R_3\Theta_3$, then $N_1 + N_2$ is (up to a constant factor) a standard normal vector so that Θ_3 is uniformly distributed on S^{p-1} and is independent from R_3. We claim that:

Proposition 2.1. Θ_3 *is independent from* $\sigma(R_1, R_2)$.

Proof. Let $f : S^{p-1} \to \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be bounded Borel functions, then the independence of N_1, N_2 yields

$$\begin{aligned}
&\mathbb{E}\big[f(\Theta_3)g(R_1, R_2)\big] \\
&\quad = \mathbb{E}\left[f\left(\frac{N_1+N_2}{|N_1+N_2|}\right)g\big(|N_1|, |N_2|\big)\right] \\
&\quad = \int_0^\infty \int_0^\infty F(r_1, r_2)\,dr_1\,dr_2 \int_{S^{p-1}\times S^{p-1}} f\left(\frac{r_1\theta_1 + r_2\theta_2}{|r_1\theta_1 + r_2\theta_2|}\right)\sigma_1(d\theta_1)\sigma_1(d\theta_2),
\end{aligned}$$

where

$$F(r_1, r_2) := \frac{1}{2^{p-2}\Gamma^2(p/2)}(r_1r_2)^{p-1}e^{-(r_1^2+r_2^2)/2}g(r_1, r_2).$$

Let $\nu_{r_1,r_2}(d\theta)$ be the pushforward of $\sigma_1 \otimes \sigma_1$ under the map

$$(\theta_1, \theta_2) \mapsto \frac{r_1\theta_1 + r_2\theta_2}{|r_1\theta_1 + r_2\theta_2|},$$

then

$$\int_{S^{p-1}\times S^{p-1}} f\left(\frac{r_1\theta_1 + r_2\theta_2}{|r_1\theta_1 + r_2\theta_2|}\right)\sigma_1(d\theta_1)\sigma_1(d\theta_2) = \int_{S^{p-1}} f(\theta)\nu_{r_1,r_2}(d\theta).$$

But ν_{r_1,r_2} is obviously invariant under the action of $O(p)$, therefore $\nu_{r_1,r_2} = \sigma_1$ since σ_1 is the unique distribution on S^{p-1} enjoying the rotational invariance property. □

We also need the following lemma.

Lemma 2.2. *Let* V, X, Y *be random variables such that* Y *and* (X, V) *are independent. Then, for any bounded Borel function* f

$$\mathbb{E}\big[f(X, Y)|V\big] = \int \mathbb{E}\big[f(X, y)|V\big]\mathcal{D}_Y(dy).$$

Proof. This fact is easily proved for bounded functions $f(x, y) = g(x)h(y)$ and then extended to bounded Borel functions using the monotone class theorem (Revuz [12], page 5). $\square$

Combining the proposition and the lemma, one gets

$$\mathbb{E}\bigl[\mathrm{e}^{\mathrm{i}\langle v, N_1+N_2\rangle} | R_1, R_2\bigr] = \int_{S^{p-1}} \mathbb{E}\bigl[\mathrm{e}^{\mathrm{i}\langle v, R_3 s\rangle} | R_1, R_2\bigr]\sigma_1(\mathrm{d}s).$$

Finally, let $\mu_{R_3|(R_1,R_2)}$ be a regular version of the conditional distribution of R_3 given (R_1, R_2), then Fubini theorem entails

$$j_{(p/2)-1}\bigl(|v|R_1\bigr) j_{(p/2)-1}\bigl(|v|R_2\bigr) = \int_{\mathbb{R}_+} j_{(p/2)-1}\bigl(|v|\xi\bigr)\mu_{R_3|(R_1,R_2)}(\mathrm{d}\xi).$$

Thus, (1.1) is proved and $\tau_{x,z}^{(p/2)-1}$ fits $\mu_{R_3|(R_1,R_2)}$ on the event $\{R_1 = x, R_2 = z\}$ as explained in the following remark.

Remark 2.1. Let Φ be the angle between Θ_1, Θ_2: $\cos\Phi = \langle\Theta_1, \Theta_2\rangle$. Then

$$R_3 = \sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\Phi}.$$

But the independence of Θ_1, Θ_2 entails for any real w

$$\begin{aligned}
\mathbb{E}\bigl[\mathrm{e}^{\mathrm{i}w\cos\Phi}\bigr] &= \int_{S^{p-1}}\int_{S^{p-1}} \mathrm{e}^{\mathrm{i}w\langle s,t\rangle}\sigma_1(\mathrm{d}s)\sigma_1(\mathrm{d}t) \\
&= \int_{S^{p-1}} j_{(p/2)-1}\bigl(w|t|\bigr)\sigma_1(\mathrm{d}t) \\
&= j_{(p/2)-1}(w) = \frac{\Gamma(p/2)}{\Gamma(1/2)\Gamma((p-1)/2)} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}w\xi}\bigl(1-\xi^2\bigr)^{(p-3)/2}\,\mathrm{d}\xi,
\end{aligned}$$

where we used Lemma 5.4.4, page 195 in Dunkl and Xu [5]. Performing the variable change

$$u = \sqrt{x^2 + z^2 + 2xz\xi}, \qquad \xi \in [-1, 1],$$

one recovers the density of $\tau_{x,z}^{(p/2)-1}$ derived in Proposition A.5, page 1153 in Ragozin [11].

3. Product formula for Bessel-type hypergeometric functions of one real symmetric matrix argument

In this section, we consider matrix-variate normal distributions rather than vectors. Doing so leads to a product formula for Bessel-type hypergeometric functions of one real symmetric matrix argument (see below). To this end, we recall from Chikuse [2], Chapter I, the following needed facts. Let $p \geq m \geq 1$ and let N be a real matrix-variate $p \times m$ standard normal distribution, that is a $p \times m$ matrix whose entries are independent centered normal distributions with unit variance. Then N admits almost surely a unique polar decomposition $N = Z(N^T N)^{1/2} := ZH$. Moreover, Z and H are independent, H is almost surely invertible and Z is uniformly distributed on the real Stiefel manifold

$$\Sigma_{p,m} := \{A \in M_{p,m}(\mathbb{R}),\, A^T A = \mathbf{I}_m\},$$

where $M_{p,m}(\mathbb{R})$ is the space of $p \times m$ real matrices. Let $O(p)$ be the orthogonal group, then $\Sigma_{p,m}$ is a homogeneous space $\Sigma_{p,m} \approx O(p)/O(p-m)$. It thereby admits a unique $O(p)$-invariant distribution we shall denote $\sigma_{p,m}$. More precisely, $\sigma_{p,m}$ is the pushforward of the Haar distribution on $O(p)$ under the map

$$O \mapsto Oe_{p,m}, \qquad e_{p,m} := I_m \oplus 0_{p-m,m}.$$

Hence, for any $C \in M_{p,m}(\mathbb{R})$

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}\,\mathrm{tr}(C^T N)} | H\big] = \int_{\Sigma_{p,m}} \mathrm{e}^{\mathrm{i}\,\mathrm{tr}(C^T sH)} \sigma_{p,m}(\mathrm{d}s) = \int_{\Sigma_{p,m}} \mathrm{e}^{\mathrm{i}\,\mathrm{tr}(HC^T s)} \sigma_{p,m}(\mathrm{d}s).$$

Now, let N_1, N_2 be two independent $p \times m$ matrix-variate standard normal distributions with respective polar decomposition $N_1 = Z_1 H_1$, $N_2 = Z_2 H_2$. Then, by considering the product σ-field $\sigma(H_1, H_2)$ generated by H_1, H_2 we easily derive

$$\mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T N_1)} | H_1\big]\mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T N_2)} | H_2\big] = \mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T (N_1+N_2))} | H_1, H_2\big]. \tag{3.1}$$

Since $N_1 + N_2$ is up to a constant factor a $p \times m$ matrix-variate standard normal distribution, then it admits almost surely a polar decomposition $N_1 + N_2 = Z_3 H_3$, where Z_3 is uniformly distributed on $\Sigma_{p,m}$ and is independent from H_3. Similarly to the case $m = 1$, one proves that Z_3 is independent from $\sigma(H_1, H_2)$ (analogue of proposition 2.1) using the following variable change formula (Faraut and Korányi [7], Proposition XVI.2.1, page 351): let $\mathrm{d}A$ be the Lebesgue measure on $M_{p,m}(\mathbb{R})$, let $S_m^+(\mathbb{R})$ be the set of real positive definite matrices with Lebesgue measure $\mathrm{d}r$ and $\gamma = (p/2) - 1 - [m(m-1)]/2$. Then

$$\int_{M_{p,m}(\mathbb{R})} f(A)\,\mathrm{d}A = \int_{\Sigma_{p,m}} \int_{S_m^+(\mathbb{R})} f(s\sqrt{r})\big[\det(r)\big]^\gamma \sigma_{p,m}(\mathrm{d}s)\,\mathrm{d}r.$$

Accordingly and with the help of Lemma 2.2, one gets

$$\mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T Z_3 H_3)} | H_1, H_2\big] = \int_{\Sigma_{p,m}} \mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T sH_3)} | H_1, H_2\big] \sigma_{p,m}(\mathrm{d}s),$$

and if $\mu_{H_3|(H_1,H_2)}$ is the conditional distribution of H_3 given (H_1, H_2), then Fubini theorem entails

$$\mathbb{E}\big[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T Z_3 H_3)} | H_1, H_2\big] = \int_{S_m^+(\mathbb{R})} \bigg[\int_{\Sigma_{p,m}} \mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T s\xi)} \sigma_{p,m}(\mathrm{d}s)\bigg] \mu_{H_3|(H_1,H_2)}(\mathrm{d}\xi).$$

Using Herz [9], (3.5), page 493, one sees that

$$\mathbb{E}\left[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T N)}|H\right] = \int_{\Sigma_{p,m}} \mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(HC^T s)}\sigma_{d,m}(\mathrm{d}s) = {}_0F_1\left(\frac{p}{2}; -\left(HC^T CH\right)\right),$$

where ${}_0F_1$ is the Bessel-type hypergeometric function of one real symmetric argument and of geometrical index value $(p/2)$ (it reduces when $m = 1$ to $j_{(p/2)-1}$ (Muirhead [10])). Finally, (3.1) yields the product formula

$$\begin{aligned} &{}_0F_1\left(\frac{p}{2}; -H_1C^T CH_1\right){}_0F_1\left(\frac{p}{2}; -H_2C^T CH_2\right) \\ &\quad = \int_{S_m^+(\mathbb{R})} {}_0F_1\left(\frac{p}{2}; -\xi C^T C\xi\right)\mu_{H_3|(H_1,H_2)}(\mathrm{d}\xi). \end{aligned}$$

4. Absolute continuity of $\mu_{H_3|(H_1,H_2)}$ and product formula for Bessel-type hypergeometric functions of two matrix arguments

4.1. Absolute continuity of $\mu_{H_3|(H_1,H_2)}$

In contrast to the case $m = 1$, the absolute-continuity of $\mu_{H_3|(H_1,H_2)}$ is not obvious and needs a careful analysis we perform below.

Proposition 4.1. *For any $p \geq m + 1$, $\mu_{H_3|(H_2,H_1)}$ is absolutely continuous with respect to the Lebsegue measure on $S_m(\mathbb{R})$ and its density, say $f_{(H_1,H_2)}(A)$, satisfies*:

$$\begin{aligned} &\int_{O(m)\times O(m)} f_{(O_1H_1O_1^T, O_2H_2O_2^T)}\left(O_3^T AO_3\right)\mathrm{d}O_1 \otimes \mathrm{d}O_2 \\ &\quad = \int_{O(m)\times O(m)} f_{(O_1H_1O_1^T, O_2H_2O_2^T)}(A)\,\mathrm{d}O_1 \otimes \mathrm{d}O_2 \end{aligned} \tag{4.1}$$

almost surely for any $O_3 \in O(m)$, where $\mathrm{d}O_i$, $i \in \{1, 2\}$, are two copies of the Haar distribution on $O(m)$. For $p = m$, $\mu_{H_3|(H_2,H_1)}$ is singular.

Proof. Since

$$(H_3)^2 = (H_1)^2 + (H_2)^2 + H_1Z_1^T Z_2H_2 + H_2Z_2^T Z_1H_1$$

then $\mu_{H_3|(H_2,H_1)}$ is the pushforward of $\sigma_{p,m} \otimes \sigma_{p,m}$ under the map

$$(Z_1, Z_2) \mapsto \sqrt{(H_1)^2 + (H_2)^2 + H_1Z_1^T Z_2H_2 + H_2Z_2^T Z_1H_1}$$

for fixed H_1, H_2, where for a positive semi-definite matrix A, $\sqrt{A}$ is its square root. But from the very definition of $\sigma_{p,m}$, $\mu_{H_3|(H_1,H_2)}$ is the pushforward of the Haar distribution $\mathrm{d}O \otimes \mathrm{d}O$ on $O(p) \times O(p)$ under the map

$$(O_1, O_2) \mapsto \sqrt{(H_1)^2 + (H_2)^2 + H_1 e_{p,m}^T O_1^T O_2 e_{p,m} H_2 + H_2 e_{p,m}^T O_2^T O_1 e_{p,m} H_1}$$

or equivalently

$$(O_1, O_2) \mapsto \sqrt{(H_1)^2 + (H_2)^2 + H_1 e_{p,m}^T O_1 O_2 e_{p,m} H_2 + H_2 e_{p,m}^T O_2^T O_1^T e_{p,m} H_1}$$

since $\mathrm{d}O$ is invariant under $O \mapsto O^T$. Besides, the random variable $O_1 O_2 \in O(p)$ is Haar distributed since it is $O(p)$-invariant. As a matter of fact, $\mu_{H_3|(H_1,H_2)}$ is the pushforward of $\mathrm{d}O$ under the map

$$O \mapsto \sqrt{(H_1)^2 + (H_2)^2 + H_1 e_{p,m}^T O e_{p,m} H_2 + H_2 e_{p,m}^T O^T e_{p,m} H_1}.$$

Now observe that for fixed H_1, H_2,

$$O \mapsto (H_1)^2 + (H_2)^2 + H_1 e_{p,m}^T O e_{p,m} H_2 + H_2 e_{p,m}^T O^T e_{p,m} H_1$$

is a Lipschitz map from $O(p)$ into $S_m(\mathbb{R})$ whose differential is affine. Moreover, $O(p)$ and $S_m(\mathbb{R})$ are real analytic manifolds such that $\dim O(p) = p(p-1)/2$, $\dim S_m(\mathbb{R}) = m(m+1)/2$. As a matter of fact:

- If $p = m+1$, then $\dim O(m+1) = \dim S_m(\mathbb{R})$ and Theorem 3.2.5, page 244 in Federer [8] implies that the pushforward of the Haar distribution on $O(p)$ under this map is absolutely continuous with respect to the Lebesgue measure on $S_m(\mathbb{R})$.
- If $p \geq m+2$, then $\dim O(p) > \dim S_m(\mathbb{R})$ and Theorem 3.2.12, page 249 in Federer [8] yields the same conclusion.

Now, since the Jacobian of the transformation $A \mapsto \sqrt{A}$ on the space of positive definite matrices is proportional to $\det(A)^{-1/2}$, then it suffices to prove (4.1) for the conditional distribution of $H_3^2|(H_2, H_1)$. But if g denotes its density then for any $O_1, O_2, O_3 \in O(m)$, $g_{(O_1 H_1 O_1^T, O_2 H_2 O_2^T)}(O_3^T A O_3)$ is the density of the random variable (for fixed H_1, H_2)

$$\begin{aligned} & O_3 O_1 (H_1)^2 O_1^T O_3^T + O_3 O_2 (H_2)^2 O_2^T O_3^T \\ & \quad + O_3 O_1 H_1 O_1^T Z_1^T Z_2 O_2 H_2 O_2^T O_3^T + O_3 O_2 H_2 O_2^T Z_2^T Z_1 O_1 H_1 O_1^T O_3^T \end{aligned}$$

which can be written as

$$\begin{aligned} & (O_3 O_1)(H_1)^2 \big(O_1^T O_3^T\big) + (O_3 O_2)(H_2)^2 \big(O_2^T O_3^T\big) \\ & \quad + (O_3 O_1) H_1 \big(O_1^T O_3^T\big) \big(Z_1 O_3^T\big)^T \big(Z_2 O_3^T\big)(O_3 O_2) H_2 \big(O_2^T O_3^T\big) \\ & \quad + (O_3 O_2) H_2 \big(O_2^T O_3^T\big) \big(Z_2 O_3^T\big)^T \big(Z_1 O_3^T\big)(O_3 O_1) H_1 \big(O_1^T O_3^T\big). \end{aligned}$$

But since $\sigma_{p,m}$ is invariant under the right action of $O(m)$ (Chikuse [2], page 28) and since the Haar distribution $\mathrm{d}O$ is $O(m)$-bi-invariant, then the $f_{(H_1,H_2)}$ satisfies (4.1). Finally, since $\dim O(m) < \dim S_m(\mathbb{R})$ then Theorem 3.2.5 in Federer [8] shows that for $p = m$, $\mu_{H_3|(H_1,H_2)}$ is singular with respect to the Lebesgue measure on $S_m(\mathbb{R})$. □

Remark 4.1. Note that

$$e_{p,m}^T O e_{p,m} = \Lambda_m \oplus 0_{p-m,p-m}$$

where Λ_m is the upper-left $m \times m$ corner of the orthogonal matrix O. According to Collins [4], Remark 2.1, page 118, if $p \geq 2m$ then the distribution of Λ_m is absolutely continuous with respect to the Lebesgue measure on $M_{m,m}(\mathbb{R})$: its density is given by

$$\det\bigl(\mathbf{I}_m - AA^T\bigr)^{(p-2m-1)/2} \mathbf{1}_{\{\|A\|<1\}},$$

where $\|\cdot\|$ is the matrix norm induced by the Euclidian norm $|\cdot|$. This fact should be compared with Lemma 3.7, page 495 in Herz [9].

4.2. Product formula for functions of two matrix arguments

Let $p \geq m + 1$ so that $\mu_{H_3|(H_2,H_1)}$ is absolutely continuous with respect to Lebesgue measure on $S_m(\mathbb{R})$. Then one derives a product formula for the Bessel-type hypergeometric functions of two real symmetric matrix arguments and of geometrical index values $p/2$, $p \geq 1$: if A is a real positive semi-definite matrix and $C \in M_{p,m}(\mathbb{R})$, then these functions are related to those of one real symmetric matrix argument by

$$\begin{aligned} &{}_0F_1\left(\frac{p}{2}; A; -C^T C\right) \\ &\quad = \int_{O(m)} {}_0F_1\left(\frac{p}{2}; -O\sqrt{A}O^T\bigl(C^T C\bigr)O\sqrt{A}O^T\right) \mathrm{d}O \end{aligned} \tag{4.2}$$

where $\mathrm{d}O$ is now the Haar distribution on $O(m)$ (Theorem 7.3.3, page 260 in Muirhead [10]). Keeping the same notations used in the previous section, one has

$${}_0F_1\left(\frac{p}{2}; A; -C^T C\right) = \int_{O(m)} \mathbb{E}\bigl[\mathrm{e}^{2\mathrm{i}\,\mathrm{tr}(C^T N)} | H = O\sqrt{A}O^T\bigr] \mathrm{d}O$$

which in turn implies that for any positive semi-definite matrices A, B and any $C \in M_{p,m}(\mathbb{R})$

$$\begin{aligned} &{}_0F_1\left(\frac{p}{2}; A; -C^T C\right) {}_0F_1\left(\frac{p}{2}; B; -C^T C\right) \\ &\quad = \int_{O(m)\times O(m)} \int_{S_m^+(\mathbb{R})} {}_0F_1\left(\frac{p}{2}; -\xi C^T C \xi\right) \mu_{H_3|(O_1\sqrt{A}O_1^T, O_2\sqrt{B}O_2^T)}(\mathrm{d}\xi)\, \mathrm{d}O \otimes \mathrm{d}O. \end{aligned}$$

Recall now that $f_{(H_1,H_2)}$ denotes the density of $\mu_{H_3|(H_1,H_2)}$. Then Weyl integration formula for $S_m(\mathbb{R})$ (Faraut [6], Theorem 10.1.1, page 232), (4.1) and Fubini theorem entail

$$\begin{aligned}
&\int_{O(m)\times O(m)}\int_{S_m^+(\mathbb{R})} {}_0F_1\left(\frac{p}{2}; -\xi C^T C\xi\right) f_{(O_1\sqrt{A}O_1^T, O_2\sqrt{B}O_2^T)}(\xi)\,\mathrm{d}\xi\otimes \mathrm{d}O\otimes \mathrm{d}O\\
&\quad = c_m\int_{O(m)\times O(m)}\int_{O(m)\times\mathbb{R}_+^m} {}_0F_1\left(\frac{p}{2}; -ODO^T\left(C^TC\right)ODO^T\right)\\
&\qquad \times f_{(O_1\sqrt{A}O_1^T, O_2\sqrt{B}O_2^T)}\left(ODO^T\right)V(D)\,\mathrm{d}D\otimes \mathrm{d}O\otimes \mathrm{d}O\otimes \mathrm{d}O\\
&\quad = c_m\int_{O(m)\times\mathbb{R}_+^m} {}_0F_1\left(\frac{p}{2}; -ODO^T\left(C^TC\right)ODO^T\right)\\
&\qquad \times \left\{\int_{O(m)\times O(m)} f_{(O_1\sqrt{A}O_1^T, O_2\sqrt{B}O_2^T)}(D)\,\mathrm{d}O\otimes \mathrm{d}O\right\}V(D)\,\mathrm{d}D\otimes \mathrm{d}O,
\end{aligned}$$

where $D = \mathrm{diag}(\lambda_1 > \lambda_2 > \cdots > \lambda_m)$ is a positive definite diagonal matrix,

$$V(D) := \prod_{1\le n<j\le m}(\lambda_n - \lambda_j), \qquad \mathrm{d}D = \prod_{j=1}^m \mathrm{d}\lambda_j,$$

and c_m is a normalizing constant. By the virtue of (4.2), one gets

$${}_0F_1\left(\frac{p}{2}; A; -C^TC\right){}_0F_1\left(\frac{p}{2}; B; -C^TC\right) = c_m\int_{\mathbb{R}_+^m} {}_0F_1\left(\frac{p}{2}; D^2; -C^TC\right)\kappa_{A,B}(D)\,\mathrm{d}D,$$

where

$$\kappa_{A,B}(D) := V(D)\mathbf{1}_{\{\lambda_1>\cdots>\lambda_m>0\}}\int_{O(m)\times O(m)} f_{(O_1\sqrt{A}O_1^T, O_2\sqrt{B}O_2^T)}(D)\,\mathrm{d}O\otimes \mathrm{d}O.$$

Finally, one performs a change of variable $\lambda_i \mapsto \sqrt{\lambda_i}$, $1\le i\le m$ in order to get the product formula:

$$\begin{aligned}
&{}_0F_1\left(\frac{p}{2}; A; -C^TC\right){}_0F_1\left(\frac{p}{2}; B; -C^TC\right)\\
&\quad = \frac{c_m}{2^m}\int_{\lambda_1>\cdots>\lambda_m>0} {}_0F_1\left(\frac{p}{2}; D; -C^TC\right)\frac{\kappa_{A,B}(\sqrt{D})}{\sqrt{\lambda_1\cdots\lambda_m}}\prod_{i=1}^m \mathrm{d}\lambda_i.
\end{aligned}$$

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