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Weak noise and non-hyperbolic unstable fixed points: Sharp estimates on transit and exit times

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We consider certain one dimensional ordinary stochastic differential equations driven by additive Brownian motion of variance ε^2. When $\varepsilon = 0$ such equations have an unstable non-hyperbolic fixed point and the drift near such a point has a power law behavior. For $\varepsilon > 0$ small, the fixed point property disappears, but it is replaced by a random escape or transit time which diverges as $\varepsilon \searrow 0$. We show that this random time, under suitable (easily guessed) rescaling, converges to a limit random variable that essentially depends only on the power exponent associated to the fixed point. Such random variables, or laws, have therefore a universal character and they arise of course in a variety of contexts. We then obtain quantitative sharp estimates, notably tail properties, on these universal laws.

Keywords: martingale theory; Schrödinger equation; stochastic differential equations; unstable non-hyperbolic fixed points; WKB analysis

1. Introduction

1.1. Effect of noise on non-hyperbolic unstable points

Noise perturbations on dynamical systems lead to a variety of phenomena and many are of crucial interest in understanding the dynamics of real systems [15,18]. Here we focus on a basic issue that has been repeatedly addressed in various domains [1,5,19,24]: the effect of noise on stationary non-hyperbolic points of one dimensional Ordinary Differential Equations (ODE). The basic question we have in mind is easily stated at an informal level: consider the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = -U'(X_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t, \tag{1.1}$$

where $\varepsilon \geq 0$, $\{W_t\}_{t\geq 0}$ is a standard Brownian motion, and $U(\cdot)$ is a smooth function such that $U'(0) = U''(0) = 0$, that is 0 is a non-hyperbolic fixed point for the case $\varepsilon = 0$. We also require such fixed point to be unstable: the cases we have in mind are for example

$$U(x) = -\frac{x^3}{6} \quad \text{and} \quad U(x) = \sin(x) - x, \tag{1.2}$$

that is the case in which 0 is a saddle-node fixed point, and

$$U(x) = -\frac{x^4}{4} \quad \text{and} \quad U(x) = -\big(1 - \cos(x)\big)^2, \tag{1.3}$$

that is the case is which 0 is a symmetric non-hyperbolic unstable point: we may focus on these example for the sake of informal discussion and we refer to them as the cases $d = 3$ and $d = 4$, in conformity with the rest of the paper in which we will address the case in which $U(x)$ is *roughly* proportional to x^d in a neighborhood of zero.

Switching from $\varepsilon = 0$ to $\varepsilon > 0$ will have the obvious drastic effect on the solution. Nonetheless, if ε is small, it will require a long time to leave the neighborhood of the origin. Two comments are in order:

(1) in general (1.1) does not admit a global (strong) solution: in fact for the first case in both (1.2) and (1.3) for $\varepsilon > 0$ and any choice of X_0 the solution to (1.1) has a finite explosion time (see, e.g., [16], Section 5.5.C, or [14]);
(2) the cases of $d = 3$ and 4 or, more generally, d odd or even, are different and in the former case we will be interested in $X_0 \in [-\infty, 0)$ so that $\lim_{t\to\infty} X_t = 0$ for $\varepsilon = 0$, while in the even d case the most interesting choice is $X_0 = 0$.

We are after understanding the distribution of the time for going through the saddle point (for d odd) and the distribution of the time of escape from 0 (for d even), in the small ε limit. More precisely:

- In the cases in (1.2), consider the first hitting time $\tau_{a,\varepsilon}$ of $a \in (0, \infty]$ or $a \in (0, \pi)$ (according to whether we consider the first or second case) for $X_0 \in [-\infty, 0)$ or $X_0 \in (-\pi, 0)$. It is easy to see that $\lim_{\varepsilon\to 0} \tau_{a,\varepsilon} = +\infty$ and by scaling argument is not difficult to guess that $\tau_{a,\varepsilon} \approx \varepsilon^{-2(d-2)/d}$ (e.g., [5,19,24], the argument is also given explicitly in Section 2.1) and it is natural to expect that $\varepsilon^{2(d-2)/d}\tau_{a,\varepsilon}$ converges in law as $\varepsilon \searrow 0$ to a random variable T_d that does not depend on a, nor on X_0, nor on whether we have chosen the first or second example. This has been argued for example in [5,19,24] where this claim is substantiated by explicit computations of mean and variance of T_d and by numerical computations.
- In the cases in (1.3), consider for $X_0 = 0$ the first hitting time $\tau_{a,\varepsilon}$, of $\pm a$ with a either in $(0, \infty]$ or in $(0, \pi)$ according to which of the cases we consider. Once again scaling arguments [1,8] suggest that $\varepsilon^{2(d-2)/d}\tau_{a,\varepsilon}$ converges in law as $\varepsilon \searrow 0$ to a random variable T_d that does not depend on a nor on which of the two cases we have chosen.

The applied literature based on (1.1) with the type of potentials we are looking at is extremely vast, and the focus on understanding T_d is often at the heart of the analyses. We mention here for example the relevance of the odd d case (saddle-node) in the context of modeling excitable systems [18,19,24] and in this context, it is very natural to consider the extension to the *weakly tilted case* of the left inset in Figure 1: the saddle node can be in fact viewed as the critical or marginal case of a saddle-node bifurcation (this is going to be taken up in Section 1.2). We signal also the recent [6] for a related time dependent problem with applications to hysteresis. The even d case is instead motivated by a variety of real world phenomena (e.g., laser instabilities [8], and we suggest to consult the introduction of [1] for an overview of applications) and, from a more theoretical viewpoint, by the analysis of anomalous fluctuations at criticality and here

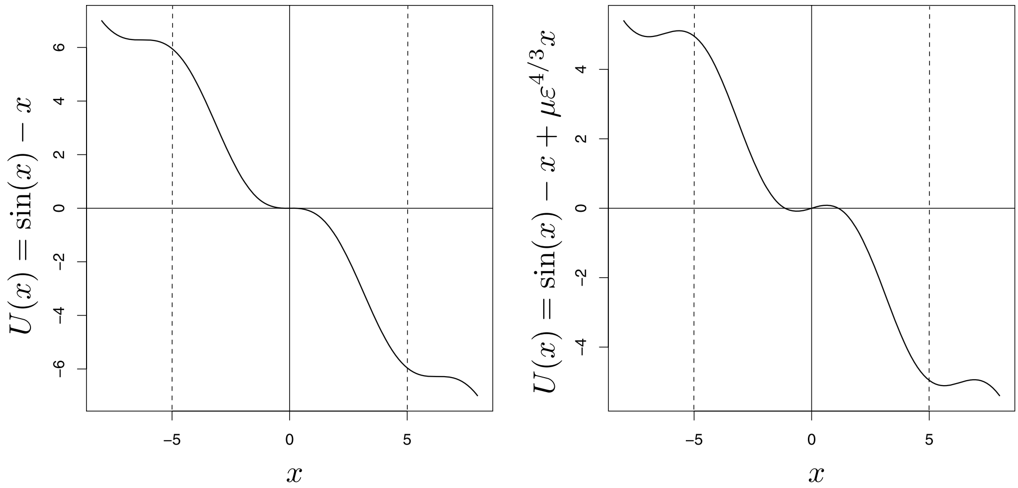

Figure 1. A case with $d = 3$. The potential on the left has saddle points at integer multiples of 2π. We just focus on the saddle point at 0 and we aim at capturing the leading asymptotic behavior of the hitting time of 5, or any other point in $(0, 2\pi)$, starting from -5, or any other point in $(-2\pi, 0)$. For ε small, but positive, the effect of the noise if essentially negligible except in a small neighborhood of 0, which turns out to be $O(\varepsilon^{2/3})$: in this neighborhood the noise eventually drives X_t through the saddle point, but this transit event takes a long time (of order $\varepsilon^{-2/3}$). The phenomenon does not change qualitatively for suitable ε-small perturbations (figure on the right), but note that the scaling in ε of the term containing μ has been carefully chosen so that also this term contributes to determine the limit transit time that in fact will be a new random variable $T_{3,\mu}$, see Section 2.

again the analysis of *nearly-critical* systems naturally leads to consider instances like the one in the right inset of Figure 2. Regardless of d being even or odd, and referring to the right insets of Figure 1 and Figure 2, for $\mu > 0$, respectively $\mu < 0$, the fixed point 0 becomes linearly stable, respectively unstable, but observe that the term containing μ vanishes as $\varepsilon \searrow 0$.

As we are going to argue further in Section 1.2, $\mu \neq 0$ is a very relevant generalization, but for the sake of clarity let us stick to the case $\mu = 0$ still for a while. Actually, the random variable T_d is a limiting universal random variable behind a very basic mechanism due to the interaction of noise and non-linearity: it is the first and foremost quantifier of how a weak stochastic perturbation makes a diffusion go through a saddle point or how it leads to the escape from a *degenerate* unstable point. And in fact there have been several attempts to determine fine properties of the distribution of T_d in the literature beyond computing the first two moments (see, e.g., [5,8]), but the results appear to be confined to uncontrolled approximations and numerical observations that fail, in particular, to capture for example the probability of observing large and small values of T_d. The purpose of this work is to present a rigorous treatment of the convergence statement $\mathcal{L} - \lim_{\varepsilon \searrow 0} \varepsilon^{2(d-2)/d} \tau_{a,\varepsilon} = T_d$: not surprisingly we will see that T_d can be directly characterized (recall we are still only talking about the case $\mu = 0$) as the explosion time for

$$dY_t = c_d Y_t^{d-1} \, dt + dW_t, \tag{1.4}$$

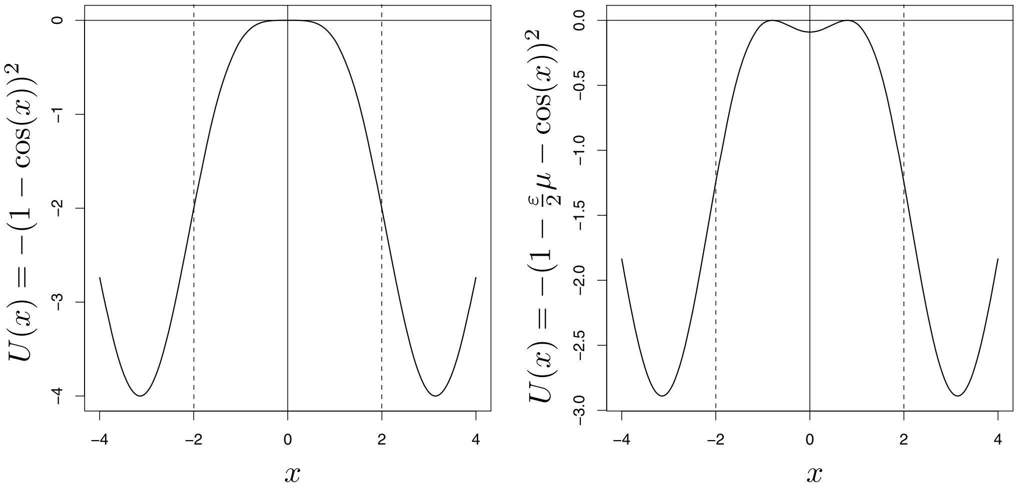

Figure 2. A case with $d = 4$. The potential on the left has unstable non-hyperbolic points at integer multiples of 2π. We just focus on the unstable point at 0 and we aim at capturing the leading asymptotic behavior of the hitting time of $\{-2, 2\}$, starting from 0. For ε small, but positive, the effect of the noise if essentially negligible except in a small neighborhood of 0, which turns out to be $O(\varepsilon^{1/2})$: in this neighborhood the noise eventually drives X_t away from 0: for this specific potential X_t will reach a neighborhood of π or $-\pi$ and will stay there for a time which to leading order is an exponential random variable with mean $\exp(c\varepsilon^{-1})$, for some $c > 0$ that can be computed by Large Deviations technics [12]. The escape from 0 happens instead on a much shorter time scale (ε^{-1}). The phenomenon does not change qualitatively for suitable ε-small perturbations (figure on the right), but the very same considerations at the end of the caption of Figure 1 apply in this case too.

with $Y_0 = 0$ for d even and $Y_0 = -\infty$ (see Section 2 for a precise definition) for d odd ($c_3 = 1/2$ and $c_4 = 1$ in the examples in this introduction). Moreover, our purpose is to obtain sharp quantitative estimates on the law of T_d, notably sharp estimates on the probability of observing small and large values of T_d and regularity estimates on the density of T_d.

1.2. A generalized set-up: Near critical cases

As already mentioned above and visually presented in Figures 1 and 2, it is very natural to consider the more general set-up of considering ε-dependent potentials, covering thus nearly-critical situations. This actually leads to families of limit transit or escape times associated to equations of the form

$$dY_t = \sum_{i=1}^{d} c_i Y^{i-1}\,dt + dW_t. \tag{1.5}$$

While our approach can cover all these cases, the analysis would be heavy and not particularly transparent. We therefore decide to deal with the cases in which only the coefficients $c_d = 1$ and

$c_2 = -\mu$ are non-zero for d even, and $c_d = 1/2$ and $c_1 = -\mu/2$ are non-zero for d odd. In fact, in some cases full details of the analysis will be given only in the cases $d = 3$ and $d = 4$. The arising limit SDEs for which we will be interested in the explosion times, starting respectively from $-\infty$ and 0, are

$$\mathrm{d}Y_t = \frac{1}{2}\big(Y_t^2 - \mu\big)\,\mathrm{d}t + \mathrm{d}W_t \quad \text{and} \quad \mathrm{d}Y_t = \big(Y_t^3 - \mu Y_t\big)\,\mathrm{d}t + \mathrm{d}W_t. \tag{1.6}$$

Note that the drift part of these two SDEs are the normal form, respectively, of a saddle-node bifurcation and of a subcritical pitchfork bifurcation, and of course μ is the bifurcation parameter.

It is worth pointing out that generalizing the odd d case to go toward (1.5) is really just a matter of going in detail through the analysis, whereas generalizing the analysis in the even d case to non-even limit potentials – note that the potential in the right-hand side of (1.6), $-x^4/4 + \mu x^2/2$, is indeed even – requires a slightly different analysis, because then the explosion to $\pm\infty$ does not happen with the same probability (nevertheless, we are able to adapt our approach also to these cases [13]).

Let us conclude the Introduction by remarking that in the case $d = 2$, for example $U(x) = -x^2/2$, the point 0 is hyperbolic unstable. The noise induced escape from an hyperbolic unstable point happens on times that to leading order behave like $\log(1/\varepsilon)/|U''(0)|$ as can be easily guessed by solving explicitly the linearized equation. Therefore T_2 is, to leading order, just a positive constant: a detailed treatment can be found for example in [2], along with the treatment of the subleading correction, which is random.

2. Set-up and main results

2.1. General set-up and rescaling

For $\mu \in \mathbb{R}$ we consider a family of C^1 potentials $\{U_{\mu,\varepsilon}\}_{\varepsilon>0}$, with $U'_{\mu,\varepsilon}(\cdot)$ locally Lipschitz, for which further assumptions will be given just below (of course the four potentials in Figures 1 and 2 fall into the realm of our analysis). We consider the strong solution X to the stochastic differential equation

$$X_0 = x_0^{(\varepsilon)}, \qquad \mathrm{d}X_t = -(U_{\mu,\varepsilon})'(X_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t. \tag{2.1}$$

Equation (2.1) can be rewritten by performing a rescaling. For this let us introduce $Y_t := \varepsilon^{-2/d} X_{\varepsilon^{-2(d-2)/d}t}$, $B_t = \varepsilon^{(d-2)/d} W_{\varepsilon^{-2(d-2)/d}t}$, $t \geq 0$ (so $B_\cdot$ is also a standard BM), $y_0^{(\varepsilon)} := \varepsilon^{-2/d} x_0^{(\varepsilon)}$, and $V_{\mu,\varepsilon}(y) = \varepsilon^{-2} U_{\mu,\varepsilon}(\varepsilon^{2/d} y)$ so that $(V_{\mu,\varepsilon})'(y) = \varepsilon^{2(1-d)/d}(U_{\mu,\varepsilon})'(\varepsilon^{2/d} y)$. Then Y is the strong solution to the stochastic differential equation

$$Y_0 = y_0^{(\varepsilon)}, \qquad \mathrm{d}Y_t = -(V_{\mu,\varepsilon})'(Y_t)\,\mathrm{d}t + \mathrm{d}B_t. \tag{2.2}$$

The scaling exponent has been chosen in particular so that $V_{\mu,\varepsilon}$ has a non-trivial limit as $\varepsilon \searrow 0$ and for this we give the following assumption.

Assumption 2.1. *Let*

$$V_\mu(y) := \begin{cases} -\dfrac{1}{2d} y^d + \dfrac{\mu}{2} y & \text{if } d \text{ is odd}, \\ -\dfrac{1}{d} y^d + \dfrac{\mu}{2} y^2 & \text{if } d \text{ is even}. \end{cases} \tag{2.3}$$

We assume that the family of (differentiable, with locally Lipschitz derivative) functions $\{U_{\mu,\varepsilon}\}_{\varepsilon>0}$ – recall that $V_{\mu,\varepsilon}(\cdot) = \varepsilon^{-2} U_{\mu,\varepsilon}(\varepsilon^{2/d}\cdot)$ – is such that, for any $A > 0$

$$\lim_{\varepsilon \searrow 0} \| V_{\mu,\varepsilon} - V_\mu \|_{\infty,[-A,A]} = 0, \tag{2.4}$$

with, for $B \subset \mathbb{R}$, $\|f\|_{\infty,B} := \sup_{x\in B} |f(x)|$. We assume in addition that there exists a $b \in (0,\infty]$, $A > 0$ and an increasing continuous function $\psi : (0,\infty) \to (0,\infty)$ with the property $\int_0^\infty \mathrm{d}x/\psi(x) < \infty$, such that when ε is small enough,

$$\psi\big(|y|\big) \le \begin{cases} -(V_{\mu,\varepsilon})'(y) & \text{if } d \text{ is odd}, \\ -\operatorname{sign}(y)(V_{\mu,\varepsilon})'(y) & \text{if } d \text{ is even}, \end{cases} \tag{2.5}$$

for every y such that $|y| \in [A, b\varepsilon^{-2/d})$.

Recall that we focus on how fast our initial diffusion X (with potential $U_{\mu,\varepsilon}$) travels from $x_0^{(\varepsilon)} < 0$, notably in the case in which $\lim_{\varepsilon \searrow 0} x_0^{(\varepsilon)} < 0$, to $a > 0$ (for d odd), or how long it takes to go from 0, or very nearby, to $\pm a$ (for d even). Let us be precise about the initial condition:

Assumption 2.2. *For d odd and $b \in (0,\infty)$ (recall that $b \in (0,\infty]$ is chosen as in Assumption 2.1) we require*

$$\liminf_{\varepsilon \searrow 0} x_0^{(\varepsilon)} > -b \quad \text{and} \quad \varepsilon^{-2/d} x_0^{(\varepsilon)} = y_0^{(\varepsilon)} \overset{\varepsilon \searrow 0}{\longrightarrow} -\infty \tag{2.6}$$

and if $b = \infty$ it is sufficient to require only the second of the two conditions in (2.6). For d even instead we require

$$x_0^{(\varepsilon)} = \mathrm{o}\big(\varepsilon^{2/d}\big). \tag{2.7}$$

We are after

$$\tau_{a,\varepsilon}(X) := \begin{cases} \inf\{t:\ X_t = a\} & \text{if } d \text{ is odd}, \\ \inf\{t:\ |X_t| = a\} & \text{if } d \text{ is even}. \end{cases} \tag{2.8}$$

It is necessary to assume

$$b \ge a, \tag{2.9}$$

simply because we make no hypothesis on $\{U_{\mu,\varepsilon}\}(x)$ for $|x| > b$.

Equation (2.4) expresses that, as $\varepsilon \searrow 0$, $U_{\mu,\varepsilon}$ has a very precise limiting behaviour in any small neighborhood of 0 of the form $[-A\varepsilon^{2/d}, A\varepsilon^{2/d}]$. Equation (2.5) instead says that $U'_{\mu,\varepsilon}$ is *sufficiently superlinear* with the correct sign in $(-b, b) \setminus [-A\varepsilon^{2/d}, A\varepsilon^{2/d}]$: this guarantees that, in the odd d case, the rescaled Y diffusion reaches $-A$ in a finite time – recall that the initial condition is *infinitely far* from the origin, cf. (2.6) – and that it will escape *infinitely far* to the right once A is reached, again in a finite time. Analogous observations hold for the even d case.

2.2. Main results: Convergence and sharp estimates on generating functions

Given a random variable $Z \geq 0$, we write $\Phi_Z(\lambda) := \mathbb{E}[\exp(\lambda Z)]$: $\Phi_Z(\cdot)$ is the moment generating function of Z (with slight abuse of notation we use this terminology without further assumptions on Z, notably without assuming the existence of the moments). Note also that $\Phi_Z(-\cdot)$ is the Laplace transform of Z, but later on we will often use generating function and Laplace transform as synonymous. Of course $\Phi_Z(\lambda) \leq 1$ for $\lambda \leq 0$ and $\Phi_Z(\cdot)$ is convex and non-decreasing. We set

$$\lambda_0 = \lambda_0(Z) := \sup\{\lambda:\ \Phi_Z(\lambda) < \infty\}, \tag{2.10}$$

so $\Phi_Z(\cdot)$ is well defined and analytic in $\{\lambda \in \mathbb{C}:\ \Re(\lambda) < \lambda_0\}$.

We are now ready to state the basic convergence result.

Theorem 2.3. *Under Assumptions 2.1 and 2.2, we have*

$$\mathcal{L} - \lim_{\varepsilon \searrow 0} \varepsilon^{2(d-2)/d} \tau_{a,\varepsilon}(X) =: T_{d,\mu}. \tag{2.11}$$

Let us start with the estimates on the moment generating function: of course Theorem 2.3 is equivalent to

$$\lim_{\varepsilon \searrow 0} \mathbb{E}\big[\exp\big(\lambda \varepsilon^{2(d-2)/d} \tau_a(X)\big)\big] = \Phi_{T_{d,\mu}}(\lambda), \tag{2.12}$$

for every $\lambda < 0$. But we will see that $\Phi_{T_{d,\mu}}(\lambda) < \infty$ also for some $\lambda > 0$, that is, with the notation that we have introduced, $\lambda_0(T_{d,\mu}) > 0$. In order to state our result on $\lambda_0(T_{d,\mu})$ and on the behavior of $\Phi_{T_{d,\mu}}(\lambda)$ near $\lambda_0(T_{d,\mu})$, we introduce the Schrödinger operator L with domain $C^2(\mathbb{R};\mathbb{R})$:

$$Lu(x) = -\tfrac{1}{2}u''(x) + q_{d,\mu}(x)u(x), \tag{2.13}$$

where $q_{d,\mu}(\cdot)$ is a polynomial function, more precisely:

$$q_{d,\mu}(x) := \tfrac{1}{2}\big(V'_\mu(x)\big)^2 - \tfrac{1}{2}V''_\mu(x). \tag{2.14}$$

In particular, we have

$$q_{3,\mu}(x) = \tfrac{1}{2}x + \tfrac{1}{8}\big(x^2 - \mu\big)^2 \quad \text{and} \quad q_{4,\mu}(x) = \tfrac{1}{2}\big(x^3 - \mu x\big)^2 + \tfrac{3}{2}x^2. \tag{2.15}$$

Classical deep results, see, for example, [7,25] and Section 4, ensure that the equation $Lu = \lambda u$ has a (classical) solution that is in $\mathbb{L}^2(\mathbb{R};\mathbb{R})$ if and only if $\lambda = \widetilde{\lambda}_j$, with $\widetilde{\lambda}_0 < \widetilde{\lambda}_1 < \cdots$. Therefore, $\widetilde{\lambda}_0$ is the bottom of the spectrum of L. This fact will be used for d odd.

A slightly different result will be needed for even d and this is connected to the fact that the question we ask differs according to the parity of d. When d is even, we consider in fact the spectrum of L on the domain $[0, \infty)$, or equivalently on $(-\infty, 0]$, with the boundary condition $u(0) = 1$ and $u'(0) = 0$; classical results, see, for example, [7,25] and Section 5, warrant that the spectrum (in $\mathbb{L}^2[0, \infty)$) of L is discrete and the eigenvalues $\widetilde{\lambda}_j$ form an increasing sequence of real numbers like in the previous case. Actually, in both the even and odd case we will be mainly interested in $\widetilde{\lambda}_0$ (however note that $\widetilde{\lambda}_1$ does play a rôle in the precision of our approximation in Proposition 2.7 below). It is practical to introduce the richer notation $\widetilde{\lambda}_{0,d}, \widetilde{\lambda}_{1,d}$ and the reader should keep in mind that the even and odd cases correspond to different spectral problems, both because the two Schrödinger operator differ and because the domains and boundary conditions also differ.

Theorem 2.4. *$\lambda_0 := \lambda_0(T_{d,\mu})$ is (strictly) positive and it coincides with $\widetilde{\lambda}_{0,d}$. Moreover, there exists a positive constant $C_{d,\mu}$ such that*

$$\Phi_{T_{d,\mu}}(\lambda) \overset{\lambda \nearrow \lambda_0}{\sim} \frac{C_{d,\mu}}{\lambda_0 - \lambda}. \tag{2.16}$$

Actually $\Phi_{T_{d,\mu}}(\cdot)$ can be extended to the whole of $\mathbb{C}$ as a meromorphic function: in particular, (2.16) is therefore saying that $\Phi_{T_{d,\mu}}(\cdot)$ has a simple pole at λ_0 with residue $-C_{d,\mu}$.

See Corollary 4.5 and Corollary 5.3 for the precise values of $C_{d,\mu}$, respectively in the odd and even cases: full details are given in the cases $d = 3$ and 4, but the generalization is straightforward.

Instead the asymptotic behavior for $\lambda \to -\infty$ of $\Phi_{T_{d,\mu}}(\lambda)$ – the next result – is even more explicit, but in this case the expressions for general d are rather cumbersome (the expression for the leading term inside the exponential can nonetheless be generalized in a straightforward way, see below Corollary 4.6 and Remark 4.10 for odd d, and Corollary 5.4 for even d). So the precise statement is restricted to $d = 3$ and 4.

Theorem 2.5. *For $\lambda \to -\infty$, we have*

$$\Phi_{T_{3,\mu}}(\lambda) = \big(1 + O\big(|\lambda|^{-1/4}\big)\big) \exp\big(-C_{3/4}|\lambda|^{3/4} - \mu C_{1/4}|\lambda|^{1/4}\big), \tag{2.17}$$

where $C_{3/4} = 3\Gamma(-(3/4))^2/(2^{9/4}\sqrt{2\pi})$ and $C_{1/4} = 2^{1/4}\Gamma(3/4)^2\sqrt{2/\pi}$. Moreover, in the same limit

$$\Phi_{T_{4,\mu}}(\lambda) = 2^{-1/4}|\lambda|^{1/4}\big(1 + O\big(|\lambda|^{-1/3}\big)\big) \exp\big(-C_{2/3}|\lambda|^{2/3} - \mu C_{1/3}|\lambda|^{1/3} - \tfrac{1}{6}\mu^2\big), \tag{2.18}$$

with $C_{2/3}$ and $C_{1/3}$ positive constants explicitly given in (5.18) in terms of elliptic integrals of first and second kind.

2.3. Tail probabilities, existence and smoothness of density

By Tauberian arguments one can extract from Theorem 2.4 the behavior of $\mathbb{P}(T_{d,\mu} > t)$ for t large and Theorem 2.5 yields precise Laplace estimates on $\mathbb{P}(T_{d,\mu} < t)$ for $t \searrow 0$. Here is the result:

Corollary 2.6. *We have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(T_{d,\mu} > t) = -\lambda_0(T_{d,\mu}), \tag{2.19}$$

and

$$\lim_{t\searrow 0} t^{d/(d-2)} \log \mathbb{P}(T_{d,\mu} < t) = -a_d, \tag{2.20}$$

with

$$a_d := \frac{d-2}{d}\left(\frac{dC_{d/(2(d-1))}}{2(d-1)}\right)^{2(d-1)/(d-2)}. \tag{2.21}$$

Equation (2.19) is a direct consequence of Theorem 2.4 and the Tauberian result ([21], Th. 3). Instead (2.20) follows from Theorem 2.5 and de Bruijn's Tauberian Theorem ([4], Th. 4.12.9).

Of course Corollary 2.6 loses quite some information with respect to Theorem 2.4 and Theorem 2.5. Theorem 2.4 actually suggests that the right-tail of $T_{d,\mu}$ should be close to the tail of an exponential random variable of parameter λ_0 translated by $(1/\lambda_0)\log(C_{d,\mu}/\lambda_0)$. Theorem 2.5 yields sharp asymptotic, notably μ dependent, behaviors of which there is no trace in (2.20).

We can actually strongly improve (2.19) of Corollary 2.6, and the first step goes through establishing the existence of the density $\mathtt{f}_{T_{d,\mu}}(\cdot)$ of $T_{d,\mu}$ (see Figures 3, 4). We can do better than this, in the sense that we can establish not only the existence of the density, but also its analyticity properties and its asymptotic behaviour at ∞. In fact, we will first establish that, with the standard notation for the characteristic function $\varphi_X(s) := \mathbb{E}\exp(\mathrm{i}sX)$ of a random variable X, there exists $c > 0$ such that as $s \to \pm\infty$

$$\left|\varphi_{T_{d,\mu}}(s)\right| = \mathrm{O}\big(\exp\big(-c|s|^{d/(2(d-1))}\big)\big), \tag{2.22}$$

which we prove in Corollary 4.6 and the discussion following it for $d = 3$, and in Corollary 5.4 for $d = 4$.

It is a very standard (Fourier analysis) result that (2.22) entails that the density $\mathtt{f}_{T_{d,\mu}}$ exists and it can be chosen to be C^∞. But in Corollary 4.6 and in Corollary 5.4 we actually prove a result that is substantially stronger than the bound (2.22), in the sense that we know the asymptotic behavior of $\Phi_{T_{d,\mu}}(\cdot)$ along any ray in the complex plane and therefore we know in which sector the Laplace transform decays to zero, and how fast, at infinity. This implies both a stronger result on the regularity of $\mathtt{f}_{T_{d,\mu}}$ and, coupled to Theorem 2.4, a sharp result on $\mathtt{f}_{T_{d,\mu}}(t)$ for $t \to \infty$: this is the content of the next statement.

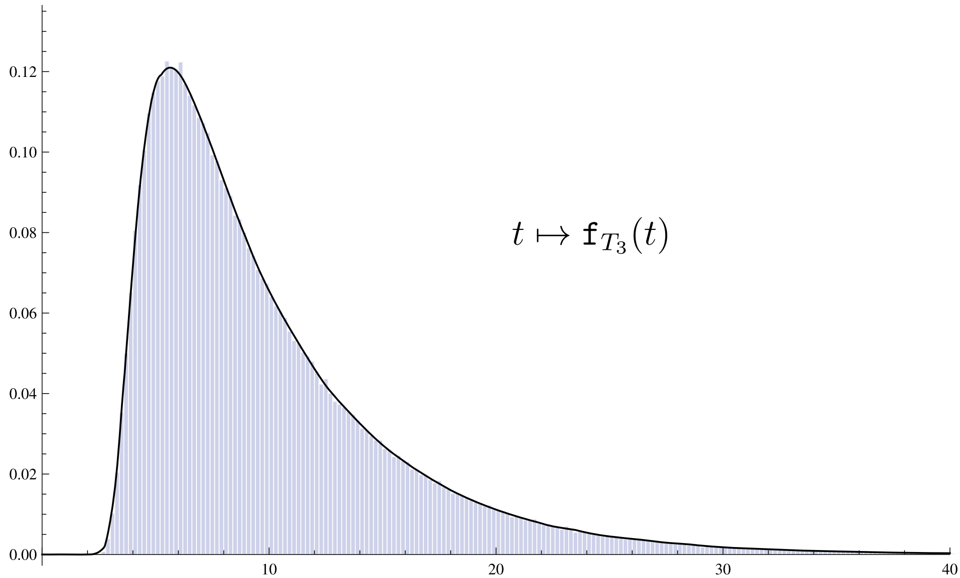

Figure 3. The density of T_3, from the next section it is rather called $T_{3,0}$, plotted via the histogram representation of a sample of 5×10^5 (independent) realizations of the random variable. The sample has been obtained by exploiting the representation formula in the first line of (3.27), but it is particularly challenging to simulate correctly extreme values (see however the log-plot in the inset of Figure 4). The empirical mean of the sample we have used is 9.939..., against the exact value 9.952..., and the empirical standard deviation is 5.70... against the exact value 5.74... (the first two moments of T_3 have explicit expressions involving Γ functions [24]).

Proposition 2.7. $\mathtt{f}_{T_{d,\mu}}(\cdot)$ *is real analytic except at* 0 *and it can be extended to an analytic function in the cone*

$$\left\{z \in \mathbb{C}\colon \Re(z) > 0, \left|\arg(z)\right| < \pi\left(\frac{1}{2} - \frac{1}{d}\right)\right\}. \tag{2.23}$$

Moreover for $t \to \infty$

$$\mathtt{f}_{T_{d,\mu}}(t) = C_{d,\mu} \exp\bigl(-\lambda_0(T_{d,\mu})t\bigr) + O\bigl(\exp(-bt)\bigr), \tag{2.24}$$

for any choice of $b \in (\widetilde{\lambda}_{0,d}, \widetilde{\lambda}_{1,d}) = (\lambda_0(T_{d,\mu}), \widetilde{\lambda}_{1,d})$.

The substantial difference between obtaining $t \to \infty$ and $t \searrow 0$ estimates on the density is that in the first case the leading behavior is directly linked to a pole of the Laplace transform of the density (see [11] for more details), while in the second case an essential singularity enters the game. As a matter of fact it is not difficult to show that $\mathtt{f}_{T_{d,\mu}}(t)$ goes quickly to zero as $t \searrow 0$, but even simply extending (2.20) to $\mathtt{f}_{T_{d,\mu}}(t)$ appears to be rather challenging: this can be approached by proving a *slow-decrease* property for $\mathtt{f}_{T_{d,\mu}}(\cdot)$ near zero, so that the Tauberian Theorem ([4], Th. 4.12.11), would apply, but we do not have such an estimate.

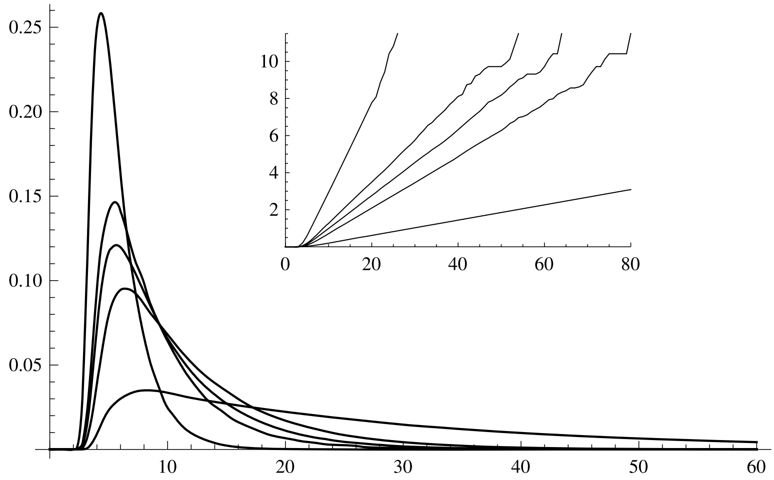

Figure 4. The main plot is the one of $t \mapsto \mathtt{f}_{T_{3,\mu}}$, for $\mu = -1, 0, -0.2, 0, 0.2, 1$: for $\mu = -1$, respectively for $\mu = +1$, $T_{3,\mu}$ is more concentrated (resp., more spread out). In the inset instead there is $t \mapsto -\log \mathbb{P}(T_{3,\mu} > t)$, for the same values of μ. The curves in the main figure have been obtained by a smoothing procedure on (five) samples each of size 10^5. The same samples have been used for the second figure and the plot is the linear interpolation of the discrete graph obtained for t that takes integer values.

2.4. Organization of the paper, with a sketch of the approach

In Section 3.1, we apply results from martingale theory to prove Theorem 2.3 along with a representation of the Laplace transform of the law of $T_{d,\mu}$, that is $\Phi_{T_{d,\mu}}(\lambda)$ for $\lambda \le 0$, as ratio of asymptotic values of the solution $f_\lambda(\cdot)$ of a suitable ODE, for $\lambda \le 0$. The representation formula has the form

$$\mathbb{E}\big[\exp(\lambda T_{d,\mu})\big] = \frac{f_\lambda(y_0)}{f_\lambda(+\infty)}, \tag{2.25}$$

with $y_0 = -\infty$ for odd d and $y_0 = 0$ for even d.

In Section 4, we make a thorough analysis of the ODE solved by $f_\lambda(\cdot)$, for odd d and for the sake of clarity we give full details only for $d = 3$. Via a standard transformation such ODE is mapped into a Schrödinger equation for which the analysis is carried out. It is at this level that the spectral properties of the arising Schrödinger operator play a rôle, but the questions we are asking are rather different from the standard ones that are typically addressed for such an operator. In fact, much of the analysis takes place out of the spectrum, so for functions that are not in $\mathbb{L}^2$. Moreover, we analyze the solutions of the arising Schrödinger equation to establish that the right-hand side of (2.25) is the ratio of entire functions (of λ), and it is therefore meromorphic, so that, in particular, (2.25) holds also for $\lambda \in \mathbb{C}$ with $\Re(\lambda)$ smaller than the first pole of the function in the right-hand side of (2.25) (and such a pole is real: actually, the set of the poles of the right-hand side of (2.25) coincides with the spectrum of the Schrödinger operator we are working with). By exploiting (2.25), extended as explained to complex values of λ, we will then obtain the proofs of Theorem 2.4, of Theorem 2.5 and of Proposition 2.7, for d odd.

In Section 5, we go again through the arguments for the case of even d.

3. Martingales, Laplace transforms, and convergence

3.1. Martingales and Laplace transforms

One should note that most objects of interest introduced in the paragraph below such as functions u, s_ε or processes Y, M in fact depend on $V_{\mu,\varepsilon}$ (in particular they all depend on the two parameters μ and ε). For bookkeeping purposes this will not appear in our notation but the reader should keep this dependency in mind.

For $t \geq 0$, write $\mathcal{F}_t = \sigma(B_s, s \leq t)$. By (2.2) and a direct application of Itô's formula, if the function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is C^2 and satisfies

$$\partial_t u(t, y) - \partial_y u(t, y) V'_{\mu,\varepsilon}(y) + \frac{1}{2} \partial_y^2 u(t, y) = 0, \tag{3.1}$$

then

$$u(t, Y_t) = u\big(0, y_0^{(\varepsilon)}\big) + \int_0^t u'(s, Y_s) \, \mathrm{d}B_s. \tag{3.2}$$

Moreover, if τ is any $(\mathcal{F}_t)$-stopping time then $(M_t := u(t \wedge \tau, Y_{t\wedge\tau}))_{t\geq 0}$ is a $(\mathcal{F}_t)$-local martingale, whose quadratic variation at $t \geq 0$ equals $\int_0^{t\wedge\tau} (u'(s, Y_s))^2 \, \mathrm{d}s$. Here are some standard facts (for a proof see, e.g., Theorem 4.7 in [17]):

Proposition 3.1. *If, for all $t \geq 0$*

$$\mathbb{E}\left[\int_0^{t\wedge\tau} \big(u'(s, Y_s)\big)^2 \, \mathrm{d}s\right] < \infty, \tag{3.3}$$

then M is a square integrable martingale. Moreover, if

$$\mathbb{E}\left[\int_0^{\tau} \big(u'(s, Y_s)\big)^2 \, \mathrm{d}s\right] < \infty, \tag{3.4}$$

then M is bounded in $\mathbb{L}^2$, hence uniformly integrable.

We recall also the Doob's Optional Stopping Theorem: if M is a uniform integrable martingale and τ a stopping time, then $M_\tau \in \mathbb{L}^1$ and $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$.

A first application of Proposition 3.1: A localization lemma

Proposition 3.1 directly yields exit probabilities from a strip. This is useful only when d is odd: in the even d case we know a priori that $X_{t\wedge\tau_{a,\varepsilon}}$ stays in $[-a, a]$, but in the odd d case $X_{t\wedge\tau_{a,\varepsilon}}$ can in principle go arbitrarily far off to the left, even where we cannot control it anymore (i.e.,

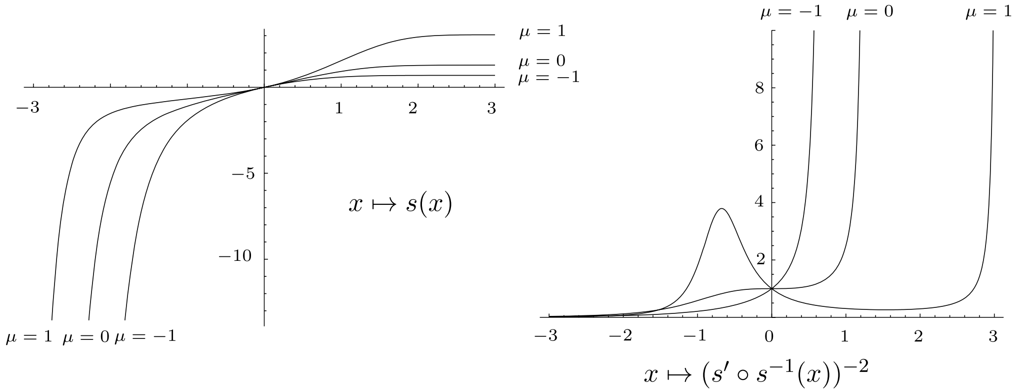

Figure 5. The plot of the scale function $s(\cdot)$ ($d = 3$) and the function $(s' \circ s^{-1}(\cdot))^{-2}$, both for $\mu = -1$, 0 and $+1$. The latter function is the one appearing in (3.27): while for $\mu < 0$ such a function is increasing and the large values are close to the divergence, that is close to $x = s(\infty)$, for $\mu > 0$ another bump appears and this accounts for the presence of a valley in the potential that (weakly) traps the diffusion and yields longer transit times.

to the left of $-b$). So, let us fix d odd and argue that this happens with negligible probability: we will first do this under the assumption $y_0^{(\varepsilon)} \stackrel{\varepsilon \searrow 0}{\longrightarrow} y_0 \in (-\infty, 0)$. Note that this is in contrast with Assumption 2.2, that requires $y_0 = -\infty$: we will in fact first treat the case y_0 finite and then show how to let $y_0 \to -\infty$.

In order to control the probability of an excursion far off to the left, we introduce a new stopping time for d odd: for $-b\varepsilon^{-2/d} \le l_\varepsilon < y_0^{(\varepsilon)} < r_\varepsilon \le b\varepsilon^{-2/d}$ let us set

$$\sigma = \sigma_{l_\varepsilon, r_\varepsilon}(Y) := \inf\bigl\{t \ge 0 \colon\ Y_t \in \{l_\varepsilon, r_\varepsilon\}\bigr\}. \tag{3.5}$$

It is easily checked that σ is a stopping time and that it is (exponentially) integrable. Let us introduce also (see Figure 5)

$$s_\varepsilon(y) := \int_0^y \exp\bigl(2V_{\mu,\varepsilon}(u)\bigr)\,\mathrm{d}u, \qquad y \in \mathbb{R}. \tag{3.6}$$

One easily checks that s_ε satisfies (3.1), thus $(s_\varepsilon(Y_{t\wedge\sigma}))_{t\ge 0}$ is a local martingale. In addition, for any $t \le \sigma$, $|Y_t| \le \max(|l_\varepsilon|, |r_\varepsilon|)$ so that $|s'_\varepsilon(Y_t)| \le C$. Thus

$$\mathbb{E}\biggl[\int_0^\sigma \bigl(s'_\varepsilon(Y_t)\bigr)^2\,\mathrm{d}t\biggr] \le C^2\mathbb{E}[\sigma], \tag{3.7}$$

hence $(s_\varepsilon(Y_{t\wedge\sigma}))_{t\ge 0}$ is a martingale, bounded in $\mathbb{L}^2$, so Doob's Optional Stopping Theorem implies $\mathbb{E}[s_\varepsilon(Y_\sigma)] = s_\varepsilon(y_0)$. But $Y_\sigma \in \{l_\varepsilon, r_\varepsilon\}$ so

$$\mathbb{P}(Y_\sigma = r_\varepsilon) = \frac{s_\varepsilon(y_0^{(\varepsilon)}) - s_\varepsilon(l_\varepsilon)}{s_\varepsilon(r_\varepsilon) - s_\varepsilon(l_\varepsilon)} = \frac{\int_{l_\varepsilon}^{y_0^{(\varepsilon)}} \exp(2V_{\mu,\varepsilon}(u))\,\mathrm{d}u}{\int_{l_\varepsilon}^{r_\varepsilon} \exp(2V_{\mu,\varepsilon}(u))\,\mathrm{d}u}. \tag{3.8}$$

From now on, we set

$$l_\varepsilon := -\varepsilon^{-2/d} \times \begin{cases} (b - x_0)/2 & \text{if } b < \infty, \\ 2x_0 & \text{if } b = \infty, \end{cases} \qquad r_\varepsilon := a\varepsilon^{-2/d}. \tag{3.9}$$

We can apply (3.8) to the above set-up, with $y_0^{(\varepsilon)} = \varepsilon^{-2/d} x_0$, $x_0 \in (-b, 0)$, and $\sigma_\varepsilon := \sigma_{l_\varepsilon, r_\varepsilon}(Y)$, and find

$$\mathbb{P}(Y_{\sigma_\varepsilon} = r_\varepsilon) = \frac{\int_{l_\varepsilon}^{y_0^{(\varepsilon)}} \exp(2V_{\mu,\varepsilon}(u))\,\mathrm{d}u}{\int_{l_\varepsilon}^{r_\varepsilon} \exp(2V_{\mu,\varepsilon}(u))\,\mathrm{d}u} \xrightarrow{\varepsilon \searrow 0} 1. \tag{3.10}$$

Therefore, we have the following lemma.

Lemma 3.2. *For d odd and l_ε as in* (3.9), *we have*

$$\lim_{\varepsilon \searrow 0} \mathbb{P}\big(\sigma_{l_\varepsilon, r_\varepsilon}(Y) \neq \tau_{a\varepsilon^{-2/d}}(Y)\big) = 0. \tag{3.11}$$

Thus, the convergence in distribution of $\varepsilon^{2(d-2)/d}\tau_a(X)$ is equivalent to that of $\varepsilon^{2(d-2)/d} \times \sigma_{-c,a}(X)$, for a wise choice of the positive constant c.

Informal martingale approach to the Laplace transform of τ

A classical approach to computing the Laplace transform (or equivalently the moment generating function) of the distribution of a hitting time is through martingales. When interested in the Laplace transform of $\tau_{a\varepsilon^{-2/d}}(Y) = \varepsilon^{2(d-2)/d}\tau_a(X)$ the game is to find a martingale of the form

$$M_t := \exp(\lambda t) f_\lambda(Y_t), \tag{3.12}$$

with the additional condition that $(M_{t \wedge \tau_{a\varepsilon^{-2/d}}(Y)})_{t \geq 0}$ is uniformly integrable.

By Itô's formula, if $f_{\lambda,\varepsilon}$ satisfies

$$\tfrac{1}{2} f''_{\lambda,\varepsilon}(x) - (V_{\mu,\varepsilon})'(x) f'_{\lambda,\varepsilon}(x) + \lambda f_{\lambda,\varepsilon}(x) = 0, \tag{3.13}$$

then

$$\mathrm{d}M_t = \exp(\lambda t) f'_{\lambda,\varepsilon}(Y_t)\,\mathrm{d}B_t. \tag{3.14}$$

At this stage the analysis of odd and even d differ. Let us consider the odd d case: even for $\lambda < 0$, which we assume, it is in general false that

$$\mathbb{E}\bigg[\int_0^{\tau_{a\varepsilon^{-2/d}}(Y)} \exp(2\lambda s)\big(f'_{\lambda,\varepsilon}(Y_s)\big)^2\,\mathrm{d}s\bigg] < \infty. \tag{3.15}$$

As a matter of fact (3.13) has a two dimensional space of solutions and it is conceivable that a choice has to be made at this stage so that (3.15) holds true, for example that

$\sup_{y \le a\varepsilon^{-2/d}} |f'_{\lambda,\varepsilon}(y)| < \infty$. Let us proceed assuming (3.15): we are then dealing with a martingale bounded in $\mathbb{L}^2$, so (cf. Proposition 3.1) $\mathbb{E}[M_{\tau_{a\varepsilon^{-2/d}}(Y)}] = M_0$. We therefore find

$$\mathbb{E}\big[\exp\big(\lambda\tau_{a\varepsilon^{-2/d}}(Y)\big)\big] = \frac{f_{\lambda,\varepsilon}(y_0^{(\varepsilon)})}{f_{\lambda,\varepsilon}(a\varepsilon^{-2/d})}. \tag{3.16}$$

Of course, the difficulty is that the ordinary differential equation (3.13) is not explicit (and its solution $f_{\lambda,\varepsilon}$ even less so). Indeed the equation depends on $V_{\mu,\varepsilon}(\cdot)$ which itself may depend on our parameter ε in a non-trivial manner. Thus, we are quite far from a satisfactory formula, or even from a formula *tout court*.

Heuristically nonetheless one can formally let $\varepsilon \searrow 0$ in (3.13) and in (3.16) (recall $V_{\mu,\varepsilon}(\cdot) \to V_\mu(\cdot)$, and $-2V'_\mu(x) = x^{d-1} - \mu$)

$$f''_\lambda(x) + \big(x^{d-1} - \mu\big) f'_\lambda(x) + 2\lambda f_\lambda(x) = 0, \tag{3.17}$$

which has to be supplied by appropriate boundary conditions, and, if we call T_{y_0} the limit variable, we should have

$$\mathbb{E}\big[\exp(\lambda T_{y_0})\big] = \frac{f_\lambda(y_0)}{f_\lambda(+\infty)}, \tag{3.18}$$

which is the formal limit of (3.16). Note that, in view of the right-hand side of (3.18), it is sufficient to determine $f_\lambda(\cdot)$ up to a multiplicative constant (we still have one degree of freedom though!).

Finally, one should not forget that in the odd d case we are really interested in sending y_0 to $-\infty$.

Since making rigorous all the steps we have just outlined does not seem to be easy, we take the following alternative path:

- Instead of working directly with Y, we go back to (3.6) and work with the martingale $s_\varepsilon(Y)$, which in turn can be transformed into a time-changed Brownian motion, thanks to Dubins–Schwarz Theorem. We do this for $Y_0 = y_0$, or in the slightly generalized case of $y_0^{(\varepsilon)}$ converging to y_0. This gives an amenable formula for $\tau_{a\varepsilon^{-2/d}}(Y)$ (rather, it gives an amenable formula for $\sigma_{l_\varepsilon,r_\varepsilon}(Y)$ in the odd d case, but recall Lemma 3.2). This step is performed in Section 3.2.
- We can pass to the limit in this formula, see Section 3.3, establishing thus that $\tau_{a\varepsilon^{-2/d}}(Y)$ converges in law as $\varepsilon \searrow 0$ to a limit variable that we call T_{y_0}, and this for every allowed choice of $V_{\mu,\varepsilon}$.
- Convergence in law is actually equivalent to the convergence of $\mathbb{E}[\exp(\lambda\tau_{a\varepsilon^{-2/d}}(Y))]$ to $\mathbb{E}[\exp(\lambda T_{y_0})]$ for every $\lambda < 0$. But we can compute $\mathbb{E}[\exp(\lambda\tau_{a\varepsilon^{-2/d}}(Y))]$ by making a judicious choice of $V_{\mu,\varepsilon}(\cdot)$, that is simply $V_{\mu,\varepsilon}(\cdot) = V_\mu(\cdot)$ (cf. (2.3)), so that (3.13) becomes (3.17) and we have gotten rid of the ε dependence in (3.13). There is still a priori an obstacle in making the steps (3.12)–(3.16) rigorous: selecting the right solution of (3.17), since there is one degree of freedom. But the crucial condition (3.15) does require some boundedness condition on $f'(y)$ and since $f(\cdot)$ is smooth this amounts to require this for $y \to -\infty$. As

a matter of fact solutions to (3.17) can have only certain asymptotic behaviors (this is one of the instances in which the WKB analysis plays a role), so actually requiring that $f(y)$ is bounded as $y \to -\infty$, implies that $f'(y) \to 0$, and in the end we will consider the solution of (3.17) such that $f'(-\infty) = 0$ and (say) $f(0) = 1$.

- Finally, in Section 3.4, we will send $y_0 \to -\infty$ in the left-hand side of (3.16) by performing a direct SDE estimate (the non-Lipschitz character of $V_\mu(\cdot)$ makes the time spent by the diffusion to go from $-\infty$ to y_0 arbitrarily small as $y_0 \to -\infty$).

3.2. Scale function, time-changed Brownian motion

Let $Z_t = s_\varepsilon(Y_t)$ with $s_\varepsilon(\cdot)$ given in (3.6) and $V(\cdot) = V_{\mu,\varepsilon}(\cdot)$. We also write $s(y) := s_0(y) = \int_0^y \exp(2V_\mu(u))\,\mathrm{d}u$. Of course s_ε is C^2 and increasing, and by (2.4), one directly verifies that, for any $\mathtt{K} \subset \mathbb{R}$ compact, $\|s_\varepsilon - s\|_{\infty,\mathtt{K}}$ vanishes as $\varepsilon \searrow 0$. Moreover, for any a such that $0 < a \le b$, and for any $y \in [0, a\varepsilon^{-2/d}]$,

$$\big|s_\varepsilon(y) - s(y)\big| \le A\|s_\varepsilon - s\|_{\infty,[0,A]} + \int_A^{a\varepsilon^{-2/d}} \big(\exp\big(2V_{\mu,\varepsilon}(u)\big) + \exp\big(2V_\mu(u)\big)\big)\,\mathrm{d}u. \tag{3.19}$$

Thanks to Assumption 2.1, as $A \to \infty$, the second term in the right-hand side tends to 0 uniformly in $\varepsilon \in [0, \varepsilon_0]$ for small enough ε_0. Thus, $\|s_\varepsilon - s\|_{\infty,[0,a\varepsilon^{-2/d}]}$ vanishes as $\varepsilon \searrow 0$. A very similar estimate shows that $\|s'_\varepsilon - s'\|_{\infty,[-A,a\varepsilon^{-2/d}]}$ vanishes in the same limit. This actually directly implies for d even

$$\lim_{\varepsilon \searrow 0} \|s_\varepsilon - s\|_{\infty,[-a\varepsilon^{-2/d},a\varepsilon^{-2/d}]} = 0, \qquad \lim_{\varepsilon \searrow 0} \big\|s'_\varepsilon - s'\big\|_{\infty,[-a\varepsilon^{-2/d},a\varepsilon^{-2/d}]} = 0, \tag{3.20}$$

while for d odd (3.20) holds with $[-a\varepsilon^{-2/d}, a\varepsilon^{-2/d}]$ replaced by $[-A, a\varepsilon^{-2/d}]$. The same argument also implies that

$$s_\varepsilon\big(a\varepsilon^{-2/d}\big) \overset{\varepsilon \searrow 0}{\longrightarrow} \int_0^\infty \exp\big(2V_\mu(u)\big)\,\mathrm{d}u =: s(\infty), \tag{3.21}$$

and for d even the same holds also for $s_\varepsilon(-a\varepsilon^{-2/d})$. On the other hand, by using once again (2.5), we find for d odd that for any $l \in (-b, 0)$,

$$s_\varepsilon\big(l\varepsilon^{-2/d}\big) \overset{\varepsilon \searrow 0}{\longrightarrow} \int_0^{-\infty} \exp\left(-\frac{u^d}{d} + \mu u\right)\mathrm{d}u = -\infty. \tag{3.22}$$

Recall now (cf. (3.6)) that $(Z_t)_{t\ge 0}$ is a continuous local martingale, started at $s_\varepsilon(y_0^{(\varepsilon)})$, and with quadratic variation

$$\langle Z\rangle_t = \int_0^t \exp\big(4V_\mu(Y_s)\big)\,\mathrm{d}s = \int_0^t \big((s_\varepsilon)' \circ (s_\varepsilon)^{-1}\big)^2(Z_s)\,\mathrm{d}s, \tag{3.23}$$

for every $t \ge 0$. Dubins–Schwarz Theorem (see, e.g., Theorem 5.5 in [17]) then directly leads to the following lemma.

Lemma 3.3. *On an enlarged probability space there exists a standard Brownian motion $\beta^{(\varepsilon)}$ such that*

$$Z_t = Z_0 + \beta^{(\varepsilon)}_{\gamma_t^{(\varepsilon)}} = s_\varepsilon\bigl(y_0^{(\varepsilon)}\bigr) + \beta^{(\varepsilon)}_{\gamma_t^{(\varepsilon)}}, \tag{3.24}$$

with

$$\gamma_t^{(\varepsilon)} = \int_0^t \bigl(g_\varepsilon(Z_s)\bigr)^2 \,\mathrm{d}s, \quad \textit{and} \quad g_\varepsilon(x) = (s_\varepsilon)' \circ (s_\varepsilon)^{-1}(x). \tag{3.25}$$

Moreover, the (continuous, increasing) inverse of $\gamma^{(\varepsilon)}$ is

$$A_t^{(\varepsilon)} = \inf\bigl\{u \geq 0\colon\ \gamma_u^{(\varepsilon)} > t\bigr\} = \int_0^t g_\varepsilon\bigl(\beta_s^{(\varepsilon)} + s_\varepsilon(y_0)\bigr)^{-2} \,\mathrm{d}s. \tag{3.26}$$

3.3. The limit for a restricted class of initial conditions

We start with the case when Y_0 is a point $y_0 \in \mathbb{R}$ not depending on ε.

Assumption 3.4. *Suppose $y_0^{(\varepsilon)} = y_0$ for every $\varepsilon > 0$.*

Proposition 3.5. *Under Assumption 3.4 and assuming without loss of generality that $y_0 \geq 0$ in the even case, as $\varepsilon \searrow 0$, $\varepsilon^{2(d-2)/d}\tau_a(X)$ converges in distribution towards*

$$\begin{aligned} T_{y_0} &:= \int_0^{\tau_{s(\infty)-s(y_0)}(\beta)} \bigl(s' \circ s^{-1}\bigl(\beta_s + s(y_0)\bigr)\bigr)^{-2} \,\mathrm{d}s \\ &= \int_{-l}^{s(\infty)} \bigl(s' \circ s^{-1}(y)\bigr)^{-2} \ell^{y-s(y_0)}_{\tau_{(s(\infty)-s(y_0))(\beta)}}(\beta) \,\mathrm{d}y, \end{aligned} \tag{3.27}$$

where $l = \infty$ (resp., $l = s(\infty)$) in the odd (resp., even) case, $\ell_t^x(\beta)$ is the local time of β at level x, up to time t.

Proof. Let us focus on the odd d case: the even case is almost identical. Thanks to (3.11), it is equivalent to look at the convergence in distribution, as $\varepsilon \searrow 0$, of $\sigma_{l_\varepsilon, r_\varepsilon}(Y)$. Note that this step is superfluous in the even d case. Using that s_ε is increasing, and Lemma 3.3, we find

$$\sigma_{l_\varepsilon, r_\varepsilon}(Y) \overset{\text{(law)}}{=} A^{(\varepsilon)}_{\sigma_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)}(\beta^{(\varepsilon)})}. \tag{3.28}$$

Since we are only interested in the distribution of the above quantity, we may as well use a generic Brownian motion β for any ε, thus

$$\begin{aligned} \sigma_{l_\varepsilon, r_\varepsilon}(Y) &\overset{\text{(law)}}{=} \int_0^{\sigma_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)}(\beta)} g_\varepsilon\bigl(\beta_s + s_\varepsilon(y_0)\bigr)^{-2} \,\mathrm{d}s \\ &= \int_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0)}^{s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)} g_\varepsilon\bigl(x + s_\varepsilon(y_0)\bigr)^{-2} \ell^x_{\sigma_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)}(\beta)}(\beta) \,\mathrm{d}x. \end{aligned} \tag{3.29}$$

As $\varepsilon \searrow 0$, by (3.21), $s_\varepsilon(r_\varepsilon) \to s(\infty)$, and by (3.22), $s_\varepsilon(l_\varepsilon) \to -\infty$. Proceeding as for (3.20), we find that for any $l \in (-b, 0)$ and for any $A < s(\infty)$,

$$\left\| (g_\varepsilon)^{-2} - g^{-2} \right\|_{\infty,[s_\varepsilon(-\varepsilon^{-2/d} l), A]} \xrightarrow{\varepsilon \searrow 0} 0, \qquad \text{with } g(x) := \left(s' \circ s^{-1}\right)(x), \tag{3.30}$$

see Figure 5. Now, the (deterministic) convergence of $s_\varepsilon(r_\varepsilon)$ and $s_\varepsilon(l_\varepsilon)$ easily implies that for almost every trajectory of β,

$$\sigma_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)}(\beta) \xrightarrow{\varepsilon \searrow 0} \sigma_{-\infty, s(\infty)-s(y_0)}(\beta) = \tau_{s(\infty)-s(y_0)}(\beta). \tag{3.31}$$

Moreover, $(\ell_t^x(\beta))_{t \geq 0, x \in \mathbb{R}}$ is a.s. jointly continuous in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ (see, e.g., Theorem VI.1.7 in [22]). Thus, for fixed positive K, T, the above implies that uniformly over $x \in [-K, K]$, almost surely

$$\ell^x_{\sigma_{s_\varepsilon(l_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(r_\varepsilon)-s_\varepsilon(y_0)}(\beta) \wedge T}(\beta) \xrightarrow{\varepsilon \searrow 0} \ell^x_{\tau_{s(\infty)-s(y_0)}(\beta) \wedge T}(\beta), \tag{3.32}$$

where we have used that a continuous function on an open domain (in this case of $\mathbb{R}^2$) is uniformly continuous on a compact subset of the domain. It only remains to note that

$$\lim_{T \to \infty} \mathbb{P}\left(\tau_{s(\infty)-s(y_0)}(\beta) \geq T/2\right) = 0 \quad \text{and} \quad \lim_{K \to \infty} \mathbb{P}\left(\sup\left\{|x| \colon \ell_T^x > 0\right\} > K\right) = 0, \tag{3.33}$$

to ensure that, uniformly over $x \in \mathbb{R}$, we have the convergence in probability

$$\ell^x_{\sigma_{s_\varepsilon(g_\varepsilon)-s_\varepsilon(y_0), s_\varepsilon(d_\varepsilon)-s_\varepsilon(y_0)}(\beta)}(\beta) \xrightarrow{\varepsilon \searrow 0} \ell^x_{\tau_{s(\infty)-s(y_0)}(\beta)}(\beta), \tag{3.34}$$

and therefore also the convergence in probability

$$\int_{s_\varepsilon(g_\varepsilon)}^{s_\varepsilon(d_\varepsilon)} g_\varepsilon(y)^{-2} \ell^{y-s_\varepsilon(y_0)}_{\tau_{s_\varepsilon(d_\varepsilon)-s_\varepsilon(y_0)}(\beta)}(\beta)\, \mathrm{d}y \xrightarrow{\varepsilon \searrow 0} \int_{-\infty}^{s(\infty)} g(y)^{-2} \ell^x_{\tau_{s(\infty)-s(y_0)}(\beta)}(\beta)\, \mathrm{d}y. \tag{3.35}$$

This concludes the proof of Proposition 3.5. □

Corollary 3.6. *Fix* $\lambda \leq 0$.

For d *odd, assume that there exists a solution to*

$$f_\lambda''(y) + 2V_\mu'(y) f_\lambda'(y) + 2\lambda f_\lambda(y) = 0, \tag{3.36}$$

such that $\lim_{y \to -\infty} f_\lambda'(y) = 0$.

For d *even instead we consider a solution of* (3.36) *such that* $f_\lambda'(0) = 0$.

Then in both cases, $\lim_{y \to \infty} f_\lambda(y) = f_\lambda(\infty) \in (0, \infty)$ *exists and we have*

$$\mathbb{E}\left[\exp(\lambda T_{y_0})\right] = \frac{f_\lambda(y_0)}{f_\lambda(\infty)}. \tag{3.37}$$

Note that, in the odd case, (3.37) implies the uniqueness of $f_\lambda(\cdot)$ up to a multiplicative constant (which will be actually chosen in the proofs, by prescribing the precise asymptotic behavior of $f_\lambda(\cdot)$ near $-\infty$): the existence will be proven in Section 4.

When d is even, note that $f_\lambda(\cdot)$ is an even function, so T_{y_0} coincides in law with T_{-y_0}. Moreover, we have $T_0 = T_{d,\mu}$ with the notation in Theorem 2.3.

Proof of Corollary 3.6. Recall our reasoning from paragraph 3.1. We are now in position to get around the difficulty of the fact that the ODEs (3.13) may depend on ε in a non-trivial way. Indeed from Proposition 3.5, we know that the limit T_{y_0} *does not depend* on our choice of the family of potentials $\{V_{\mu,\varepsilon}\}_{\varepsilon>0}$ (as long as they satisfy Assumption 2.1). To characterize the distribution of the limit, we may therefore as well choose $V_{\mu,\varepsilon} = V_\mu$ for any $\varepsilon > 0$. For this particular choice of the family of potentials, the ODE becomes (3.36). In such a case, for any $\varepsilon > 0$

$$\mathbb{E}\big[\exp\big(\lambda\tau_{a\varepsilon^{-2/d}}(Y)\big)\big] = \frac{f_\lambda(y_0)}{f_\lambda(a\varepsilon^{-2/d})}. \tag{3.38}$$

Since $\lambda \le 0$ we can let $\varepsilon \searrow 0$ in the left-hand side, and by Proposition 3.5, the limit is $\mathbb{E}[\exp(\lambda T_{y_0})]$. This implies that $f_\lambda(\cdot)$ has a non-zero limit at infinity, and that (3.37) holds. □

Remark 3.7. Results of Proposition 3.5, Corollary 3.6 are unchanged when Assumption 3.4 is relaxed into the weaker $y_0^{(\varepsilon)} \to y_0$ as $\varepsilon \searrow 0$. Indeed, in the above proofs, when before letting $\varepsilon \searrow 0$, one simply need to replace occurrences of y_0 with $y_0^{(\varepsilon)}$.

Remark 3.8. Even if this meant changing the scale of β, we could also have used a result of MacGill [20] (important steps in Macgill's paper are outlined in [22], Exercise XI.2.7) which expresses directly the Laplace transform of an additive functional of local times, such as the one in (3.27). The reader who would care to check this alternate argument would then rather find that

$$\mathbb{E}\big[\exp(\lambda T_{y_0})\big] = \frac{1}{h_\lambda(1)}, \tag{3.39}$$

where h_λ is the unique solution to

$$\begin{gathered} h_\lambda''(x) + 2\lambda\big(s(\infty) - s(-y_0)\big)^2 h_\lambda(x) g\big(x\big(s(\infty) - s(-y_0)\big) + s(-y_0)\big) = 0, \\ h_\lambda'(-\infty) = 0, \qquad h_\lambda(0) = 1, \end{gathered} \tag{3.40}$$

but in fact, setting, for any $t \in \mathbb{R}$,

$$f_\lambda(t) = h_\lambda\left(\frac{s(-t) - s(-y_0)}{s(\infty) - s(-y_0)}\right) \Big/ h_\lambda\left(\frac{-s(-y_0)}{s(\infty) - s(-y_0)}\right) \tag{3.41}$$

allows to exactly recover the result of the corollary.

3.4. Proof of Theorem 2.3

The framework is now the one of Theorem 2.3, that is Assumptions 2.1, 2.2 are in force, in particular for odd d we have $y_0^{(\varepsilon)} \to -\infty$, which represents a novelty with respect to what we have done up to now. Therefore we start with a result that addresses this issue.

Lemma 3.9. *The limits in distribution of T_y, as $y \to -\infty$, and of $\tau_{a\varepsilon^{-2/d}}(Y)$, as $\varepsilon \searrow 0$, exist and coincide.*

Proof. Write Y^y for the process Y satisfying (2.2) and $Y_0 = y$, and simply observe that for any y such that $y_0^{(\varepsilon)} \le y \le a\varepsilon^{-2/d}$

$$\tau_{a\varepsilon^{-2/d}}\big(Y^{y_0^{(\varepsilon)}}\big) = \tau_y\big(Y^{y_0^{(\varepsilon)}}\big) + \tau_{a\varepsilon^{-2/d}}\big(Y^y\big). \tag{3.42}$$

By Proposition 3.5, the second term in the above sum converges in distribution to T_y as $\varepsilon \searrow 0$. The proof will therefore be completed if we show that $\tau_y(Y^{y_0^{(\varepsilon)}})$, the time taken by our rescaled diffusion to go from $y_0^{(\varepsilon)}$ to some fixed y, becomes negligible when $y \to -\infty$, uniformly over small ε.

The strategy of proof is quite straightforward: for y large (and negative), the diffusion process Y_t, while it stays in $[y_0^{(\varepsilon)}, y]$, is dominated (with overwhelming probability) by the drift term. But it is straightforward to see that, under Assumption 2.1, such a non-Lipschitz drift term drives the solution from arbitrarily far to the left to y in a finite time, which can even be made arbitrarily small by choosing $|y|$ sufficiently large. To detail these steps let us introduce for $M > 0$ the event

$$\Omega_M := \big\{\big|B(t)\big| \le M \text{ for every } t \in [0, 1]\big\}, \tag{3.43}$$

where $B(\cdot)$ is the Brownian motion that drives Y. Of course the probability of Ω_M tends to one as M becomes large. Let us work with $M = |y|/4$ and let us assume that Ω_M is verified and that $|y| \ge A$, A given in Assumption 2.1. As a first step let us introduce U_t strong solution of $\mathrm{d}U_t = \psi(|U_t|)\,\mathrm{d}t + \mathrm{d}B_t$ for $U_0 = y_0^{(\varepsilon)}$: note that, in view of Assumption 2.1, there is no loss of generality in assuming that $\psi(\cdot)$ is not simply continuous, but smooth. Moreover in view of Lemma 3.2 the shape of $V_{\mu,\varepsilon}(y)$ for $y \le b\varepsilon^{-2/d}$ is inessential, so we may as well assume that $V'_{\mu,\varepsilon}(y) \le -\psi(|y|)$ also for $y \le b\varepsilon^{-2/d}$. Then one directly checks by analyzing $\mathrm{d}(Y_t - U_t)$ that $Y_t \ge U_t$ for every $t \le \zeta := \inf\{t\colon\, U_t \ge y\}$, because $y \le -A$. Therefore, ζ gives an upper bound on $\tau_y(Y^{y_0^{(\varepsilon)}})$. Let us then focus on $U_\cdot$ and let us set $Z_t := U_t - B_t$, so $Z_\cdot$ is differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t} Z_t = \psi\big(|Z_t + B_t|\big) \ge \psi(-Z_t - M), \tag{3.44}$$

where the inequality holds for $t \in [0, \zeta \wedge 1]$ and on Ω_M: note in fact that for $t \le \zeta$ we have $Z_t + B_t = U_t \le y < 0$ so, since we have also $t \le 1$, $|Z_t + B_t| = -Z_t - B_t \ge -Z_t - M$. By integrating the differential inequality (3.44), we see that on Ω_M

$$\begin{aligned} \zeta \wedge 1 &\le \int_{y_0^{(\varepsilon)}}^{Z_{\zeta\wedge 1}} \frac{\mathrm{d}u}{\psi(-u-M)} \le \int_{y_0^{(\varepsilon)}}^{y+M} \frac{\mathrm{d}u}{\psi(-u-M)} \\ &= \int_{|y|/2}^{-y_0^{(\varepsilon)}-|y|/4} \frac{\mathrm{d}u}{\psi(u)} \le \int_{|y|/2}^{\infty} \frac{\mathrm{d}u}{\psi(u)}, \end{aligned} \tag{3.45}$$

where we used an obvious change of variable and $M = |y|/4$ to get the equality of the second line. Since the rightmost term above can be made arbitrarily small by choosing $|y|$ large, ζ is at most of this size and the proof is complete. □

Proof of Theorem 2.3. For even d, Theorem 2.3 follows directly from Proposition 3.5 once we take into account Remark 3.7. Note that in this case T_0 of Proposition 3.5 is the random variable $T_{d,\mu}$ of Theorem 2.3. For odd d instead Proposition 3.5 has to be combined with Lemma 3.9: the random variable identified by the limit(s) in Lemma 3.9 is $T_{d,\mu}$. □

We are also ready to generalize Corollary 3.6 to cover the case $y_0 = -\infty$ when d is odd.

Corollary 3.10. *Under the same assumptions as in Corollary* 3.6 (*in particular recall that* $\lambda \le 0$), *when d is odd the limits* $\lim_{y\to\pm\infty} f_\lambda(y) =: f_\lambda(\pm\infty) \in (0,\infty)$ *exist and*

$$\mathbb{E}\big[\exp(\lambda T_{d,\mu})\big] = \frac{f_\lambda(-\infty)}{f_\lambda(\infty)}. \tag{3.46}$$

Proof. The fact that $\lim_{y\to\infty} f_\lambda(y) = f_\lambda(\infty) \in (0,\infty)$ has been already proven in Corollary 3.6 and it is exactly from (3.37) that we restart. Since $\lambda \le 0$, this is just a matter of taking $y_0 \to -\infty$ in (3.37), by using the fact that the limit of the left-hand side exists by Proposition 3.5. □

Let us conclude the section with two remarks: the first is particularly important.

Remark 3.11. For sake of conciseness, we have chosen to focus on convergence in law of $\{\varepsilon^{2/d}\tau_a(X)\}_{\varepsilon>0}$ and on properties of the limit $T_{d,\mu}$.

So far we have proven the convergence in law. In Section 4.1 below, we will check that assumptions of Corollary 3.6 are satisfied (see Remark 4.1 below), which ensures that

$$\lim_{\varepsilon\searrow 0} \mathbb{E}\big[\exp\big(\lambda\varepsilon^{2/d}\tau_a(X)\big)\big] = \frac{f_\lambda(l_d)}{f_\lambda(\infty)}, \tag{3.47}$$

with $l_d = -\infty$ for d odd and $l_d = 0$ for d even, but *a priori* only for $\lambda \le 0$. We then resort to analytic continuation arguments to show that

$$\Phi_{T_{d,\mu}}(\lambda) = \mathbb{E}\big[\exp(\lambda T_{d,\mu})\big] = \frac{f_\lambda(l_d)}{f_\lambda(\infty)}, \tag{3.48}$$

holds also for $\lambda > 0$: precisely, it holds for every $\lambda < \lambda_0 = \lambda_0(T_{d,\mu})$ (which was defined in the beginning of Section 2.2). Theorem 2.3 states in addition that it coincides with $\widetilde{\lambda}_{0,d}$, the detailed proof of this fact is given in Section 4 for d odd, and in Section 5 for d even.

More precisely we will show that the functions of λ appearing in the denominator and numerator of the rightmost side of (3.48) can both be extended to the whole complex plane as entire functions (e.g., in the odd case, see (4.2), (4.8), (4.9), and Theorem 4.4 below), therefore the only singularities of the rightmost side of (3.48) are poles and so it is analytic (in particular) in $\{\lambda \in \mathbb{C}\colon \Re(\lambda) < \widetilde{\lambda}\}$; in principle $\widetilde{\lambda} = \min\{\Re(\lambda)\colon f_\lambda(\infty) = 0\}$, but it turns out all the poles are real so $\widetilde{\lambda}$ is itself a pole.

Of course $\widetilde{\lambda} \geq 0$, but $\widetilde{\lambda} = 0$ is excluded too since $|\Phi_{T_{d,\mu}}(\lambda)| \leq 1$ for every $\lambda \in \mathrm{i}\mathbb{R}$, and this is incompatible with the fact that the singularity of the rightmost side of (3.48) is a pole.

On the other hand, by definition of λ_0 one directly sees that $\Phi_{T_{d,\mu}}(\cdot)$ is analytic in $\{\lambda \in \mathbb{C}\colon \Re(\lambda) < \lambda_0\}$. So (3.48) holds for $\lambda < \widetilde{\lambda} \wedge \lambda_0$ and we are left with proving that $\widetilde{\lambda} = \lambda_0$. For this, we observe that $\widetilde{\lambda} < \lambda_0$ means that $\Phi_{T_{d,\mu}}(\lambda)$ blows up as $\lambda \nearrow \widetilde{\lambda}$, which is impossible by the definition of λ_0. On the other hand, $\widetilde{\lambda} > \lambda_0$ cannot hold either: to show this assume $\widetilde{\lambda} > \lambda_0$. If $\Phi_{T_{d,\mu}}(\lambda_0) = \infty$, the contradiction is immediate. Let us therefore assume also that $\Phi_{T_{d,\mu}}(\lambda_0) < \infty$ and observe that, since the expression in the rightmost term in (3.48) has radius of convergence $\widetilde{\lambda} - \lambda_0 > 0$ at λ_0, the nth derivative in λ of this term is $O(\delta^{-n})$, for any $\delta < (\widetilde{\lambda} - \lambda_0)$. But, with the notation $T = T_{d,\mu}$, this implies $\mathbb{E}[T^n \exp(\lambda_0 T)] = O(\delta^{-n} n!)$, from which one directly extracts that $\mathbb{E}[\exp(\lambda T)] < \infty$ for $\lambda < \lambda_0 + \delta$, which contradicts the assumption. Therefore $\widetilde{\lambda} = \lambda_0$ and, for any $\lambda \in \mathbb{C}$ such that $\Re(\lambda)$ is smaller than this value, (3.48) holds.

Remark 3.12. It is natural to wonder about the validity of (3.47) for some $\lambda > 0$, possibly for all $\lambda < \lambda_0(T_{d,\mu})$, as Theorem 2.4 may suggest. This however, it is in general false: in fact, it may be that $\mathbb{E}[\exp(\lambda \varepsilon^{2/d} \tau_a(X))] = \infty$ for every $\lambda > 0$ and every $\varepsilon > 0$, see Figure 6.

Other, more subtle phenomena may happen and they are notably connected to the fact that we are making assumptions only on the limit behavior of $V_{\mu,\varepsilon}$ and not on $V'_{\mu,\varepsilon}$. We make the choice not to go further toward this direction and we just stress that stronger assumptions on $U_{\mu,\varepsilon}$, or on $V_{\mu,\varepsilon}$, are needed to establish (3.47), that is the convergence of the moment generating function, for $\lambda \in (0, \lambda_0)$.

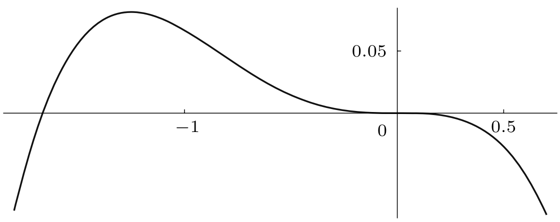

Figure 6. The choice $U_{0,\varepsilon}(x) = -\frac{1}{6}x^3 - \frac{1}{10}x^4$, independent of ε, fulfills Assumption 2.1 with rescaled potential $V_0(y) = -\frac{1}{6}y^3$. Our results apply to the diffusion $X_\cdot$, cf. (2.1), with $X_0 = -1$, and we know that $\varepsilon^{2/3}\tau_{a,\varepsilon}(X)$ (cf. (2.8)), any $a > 0$, converges as $\varepsilon \searrow 0$ to $T_{3,0}$ and this limit random variable has exponential moments (Theorem 2.5). However, since $X_\cdot$ reaches $-\infty$ with positive probability, hence $\mathbb{P}(\tau_{a,\varepsilon}(X) = \infty) > 0$ for every $\varepsilon > 0$ and therefore $\mathbb{E}[\exp(\lambda \varepsilon^{2/d} \tau_{a,\varepsilon}(X))] = \infty$ for every $\varepsilon > 0, \lambda > 0$.

4. ODE analysis: The saddle node case

For sake of conciseness, as we explained in Section 2, we restrict to $d = 3$, even if what we present can be generalized to arbitrary odd d (see Remark 4.10 below – only for Corollary 4.7 below does the generalization become cumbersome). Corollary 3.6 and Corollary 3.10 tell us that we are interested in a specific solution to (3.36): in this section we show the existence of such a solution and prove a number of quantitative results, that yield a proof of Theorem 2.4 and Theorem 2.5 for the saddle node case.

4.1. Schrödinger equation: Mapping, basic facts

For conformity with the ODE literature [7,23,25], we set $\eta = 2\lambda$ and focus, for $\mu \in \mathbb{R}$ and $\eta \in \mathbb{C}$, on solutions $g = g_\eta$ to

$$g_\eta''(x) + \left(x^2 - \mu\right)g_\eta'(x) + \eta g_\eta(x) = 0, \tag{4.1}$$

which of course is just a rewriting of (3.36). More precisely we look for a solution g_η to (4.1) such that $g_\eta'(-\infty) = 0$. We will actually look for a solution g_η that is bounded at $-\infty$, and we will then see that such a solution satisfies $g_\eta'(-\infty) = 0$. In any case, this boundedness requirement can (and, we will see, does) determine $g_\eta(\cdot)$, but only up to a multiplicative constant. This is a side issue for the moment since (for $\eta \le 0$, recall Corollary 3.10)

$$\Phi_{T_{3,\mu}}(\lambda) = \Phi_{T_{3,\mu}}(\eta/2) = \mathbb{E}\left[\exp\left(\frac{\eta}{2}T_{3,\mu}\right)\right] = \frac{g_\eta(-\infty)}{g_\eta(\infty)}, \tag{4.2}$$

whenever such a solution g_η exists.

Note that if we set

$$g_\eta(x) := \exp\left(V_\mu(x)\right)u_\eta(x) = \exp\left(\mu\frac{x}{2} - \frac{x^3}{6}\right)u_\eta(x), \tag{4.3}$$

then $u = u_\eta$ – when the context is clear we drop the subscript η from the notation – solves the Schrödinger equation

$$u''(x) - Q_\eta(x)u(x) = 0 \tag{4.4}$$

with

$$Q_\eta(x) := \left(V_\mu'\right)^2(x) - V_\mu''(x) - \eta = x + \tfrac{1}{4}\left(x^2 - \mu\right)^2 - \eta =: q(x) - \eta. \tag{4.5}$$

The ODE (4.4) has been intensively studied, see in particular [7,25] and the more recent [23]. Therefore we start by reminding some of the results which we need in our study, and then we state our results.

For $k \in \{-2, -1, 0, 1, 2, 3\}$ define the open sector $\mathcal{G}_k = \{x \in \mathbb{C}\colon |\arg(x) - \frac{k\pi}{3}| < \frac{\pi}{6}\}$.z Two of these sectors are of particular interest: $\mathcal{G}_0$ contains $(0, \infty)$, while $\mathcal{G}_3$ contains $(-\infty, 0)$, see

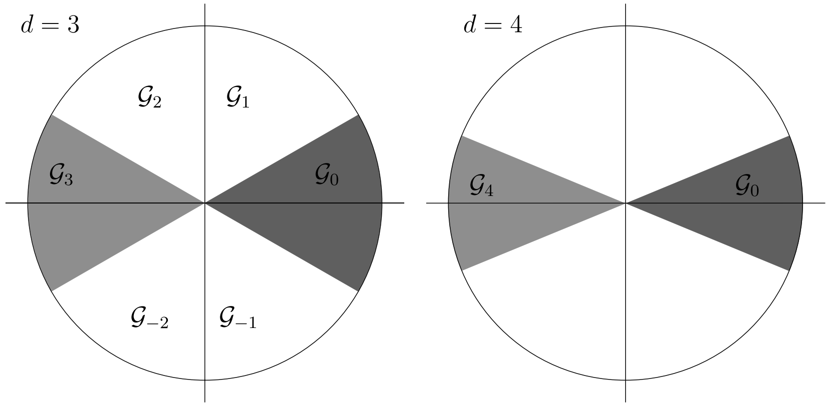

Figure 7. The sectors of $\mathbb{C}$ relevant for the study of (4.4), for the cases $d=3$ and $d=4$. It is not difficult to understand why these sectors are singled out: for $|x|$ large $Q_\eta(x) \sim c_d x^m$ ($c_d > 0$ and $m = 2d-2$). Via a quick WKB analysis (made for example, at an informal level like in [3], pages 486–487) one readily realizes that the leading contribution to the asymptotic behavior for $|x|$ large is $\exp(c'_d x^{(m+2)/2})$ (with $c'_d = 2c_d/(m+2)$), at least if $\Re(x^{(m+2)/2}) \neq 0$. If we consider $x = r\exp(\mathrm{i}\theta)$, $r>0$, $\Re(x^{(m+2)/2}) = 0$ if $\theta = \pi(1+2k)/(2d)$, for some integer k. So the asymptotic limits switch behavior when *crossing* the rays corresponding to the *critical* angles separating the sectors.

Figure 7 and its caption for further explanation. By [23], Th. 6.1 and Th. 7.1, for each $\mathcal{G}_k$, there exists a unique solution to (4.4), called the *subdominant solution in the sector* $\mathcal{G}_k$, and denoted $u_{\mathrm{sub},\mathcal{G}_k,\eta} = u_{\mathrm{sub},\mathcal{G}_k}$, which, up to a multiplicative constant, is characterized by

$$\lim_{\substack{|x|\to\infty, x\in\mathcal{G}_k \\ \arg(x)\ \text{fixed}}} u_{\mathrm{sub},\mathcal{G}_k}(x) = 0. \tag{4.6}$$

The solution $u_{\mathrm{sub},\mathcal{G}_3}$ which is subdominant along the negative real line, will be of particular interest for us and it is fully (i.e., not just up to a multiplicative constant) characterized by

$$\lim_{\substack{|x|\to\infty, x\in\mathcal{G}_3 \\ \arg(x)\ \text{fixed}}} \exp\left(-\frac{x^3}{6} + \frac{\mu}{2}x\right) u_{\mathrm{sub},\mathcal{G}_3}(x) = 1. \tag{4.7}$$

So we choose to work with the solution $g(\cdot) = g_\eta(\cdot)$

$$g_\eta(x) = \exp\left(-\frac{x^3}{6} + \mu\frac{x}{2}\right) u_{\mathrm{sub},\mathcal{G}_3}(x) \overset{x\to-\infty}{\sim} 1, \tag{4.8}$$

where the asymptotic equivalence is of course a particular case of (4.7). So, with this choice, namely $g_\eta(-\infty) = 1$, the expression (4.2) becomes somewhat simpler and, above all, we can directly apply results in [23] that grant analyticity. In fact Theorem 6.1 and Theorem 7.1 in [23] ensure that for any $k \in \{-2, -1, 0, 1, 2, 3\}$,

$$u_{\mathrm{sub},\mathcal{G}_k}(x) \text{ is entire with respect to } (\mu, \eta, x). \tag{4.9}$$

Remark 4.1. In [23], Th. 6.1, and explanations following it one finds an expansion to all orders of $u'_{\mathrm{sub},\mathcal{G}_3}$ in the same limit as in (4.7). More precisely, as $|x| \to \infty$, $x \in \mathcal{G}_3$, $\arg(x)$ fixed,

$$u'_{\mathrm{sub},\mathcal{G}_3}(x) = \left(\frac{x^2 - \mu}{2} + \mathrm{o}(1)\right) \exp\left(\frac{x^3}{6} - \frac{\mu}{2}x\right). \tag{4.10}$$

Observe that one formally obtains (4.10) by differentiating (4.7). This actually says that $f_\lambda(\cdot) = g_{2\lambda}(\cdot)$ fulfills the assumptions in Corollary 3.6. This settles the issue: g_η is the solution of (4.1) we are interested in.

Note that the uniqueness (this time, up to a multiplicative constant) of $u_{\mathrm{sub},\mathcal{G}_k}$ implies that any solution of (4.4) which is not a multiple of $u_{\mathrm{sub},\mathcal{G}_k}$ does not go to 0 when $|x| \to \infty$, $x \in \mathcal{G}_k$, $\arg(x)$ fixed. But more than that is true (see [23], page 19): any solution of (4.4) which is not a multiple of $u_{\mathrm{sub},\mathcal{G}_k}$ is *dominant* in $\mathcal{G}_k$, that is it tends to ∞ when $|x| \to \infty$, $x \in \mathcal{G}_k$, $\arg(x)$ fixed. We can even be more precise: still from [23], page 19, we learn that if a solution is subdominant in $\mathcal{G}_k$, it is dominant in the neighboring sectors $\mathcal{G}_{k\pm 1}$. Therefore, by their distinct behavior at infinity, $u_{\mathrm{sub},\mathcal{G}_k}$ and (say) $u_{\mathrm{sub},\mathcal{G}_{k+1}}$ are independent and form a basis (of course this implies that any solution that is not subdominant in a sector is necessarily dominant).

There is a priori no canonical choice of a dominant solution in a sector, but it is useful to make one and ours is $u_{\mathrm{dom},\mathcal{G}_k} := u_{\mathrm{sub},\mathcal{G}_{k+1}}$. This has the advantage that $u_{\mathrm{dom},\mathcal{G}_k,\eta}(x) = u_{\mathrm{dom},\mathcal{G}_k}(x)$ is entire in η and x. Of course we have

$$u_{\mathrm{sub},\mathcal{G}_3} = a u_{\mathrm{sub},\mathcal{G}_0} + b u_{\mathrm{dom},\mathcal{G}_0}, \tag{4.11}$$

for a unique choice of a and b, once η is fixed, and with such a choice we have also that a and b are entire functions of η. This follows simply from the fact that a and b are determined by

$$\begin{cases} u_{\mathrm{sub},\mathcal{G}_3}(0) = a u_{\mathrm{sub},\mathcal{G}_0}(0) + b u_{\mathrm{dom},\mathcal{G}_0}(0), \\ u'_{\mathrm{sub},\mathcal{G}_3}(0) = a u'_{\mathrm{sub},\mathcal{G}_0}(0) + b u'_{\mathrm{dom},\mathcal{G}_0}(0), \end{cases} \tag{4.12}$$

which of course is solvable since the Wronskian matrix of $\{u_{\mathrm{sub},\mathcal{G}_0}(0), u_{\mathrm{dom},\mathcal{G}_0}(0)\}$ is invertible. In Section 4.5, we will provide the precise asymptotic behaviour of $u_{\mathrm{dom},\mathcal{G}_0}$ at $+\infty$ and this will be a key step to establish the analyticity of $\eta \mapsto g_\eta(\infty)$.

4.2. The spectrum of the Schrödinger operator

A natural question to ask is wether one can indeed have $b = 0$, that is, if it can be that

$$u_{\mathrm{sub},\mathcal{G}_3} \propto u_{\mathrm{sub},\mathcal{G}_0}. \tag{4.13}$$

For this note, the sharp asymptotic behavior of all the subdominant solutions is known. In fact [23], Th. 6.1, Th. 7.1, in analogy with (4.7)

$$\lim_{\substack{|x|\to\infty, x\in\mathcal{G}_0 \\ \arg(x)\,\text{fixed}}} x^2 \exp\left(\frac{x^3}{6} - \frac{\mu}{2}x\right) u_{\text{sub},\mathcal{G}_0}(x) = 1, \tag{4.14}$$

in which of course we have made a precise choice of the multiplicative constant, but the key point is that (4.14), coupled of course with (4.7), implies that if (4.13) is satisfied, then we have a solution in $\mathbb{L}^2(\mathbb{R})$. So we have found an eigenfunction of the differential operator $u \mapsto -u'' + qu$ with eigenvalue η (recall (4.4) and note the operator L (2.13) is just $1/2$ times the operator we are considering here). On the other hand, since we have seen that $\{u_{\text{sub},\mathcal{G}_0,\eta}, u_{\text{dom},\mathcal{G}_0,\eta}\}$ is a basis, η is an eigenvalue in $\mathbb{L}^2(\mathbb{R})$ if and only if (4.13) holds.

Therefore the question we have just raised is about the spectrum of the Schrödinger operator. It is well known – see, for example, [7], Ch. 8, or [23], Ch. 7 – that the spectrum is constituted by an infinite sequence of eigenvalues $\eta_0 < \eta_1 < \cdots$, $\mathtt{Sp} := \{\eta_0, \eta_1, \ldots\}$: the eigenfunction u_k of η_k is $u_{\text{sub},\mathcal{G}_3,\eta}$, or $u_{\text{sub},\mathcal{G}_0,\eta}$. Moreover, as it is explained in detail in [7], Ch. 8, and [23], Ch. 7, for η on the real axis and fixed boundary conditions the number of the zeros is decreasing in η and the location of the zeros is a continuous function of η. In particular $u_{\text{sub},\mathcal{G}_3,\eta}(x)$, and therefore $g_\eta(x)$, has no zero for $\eta < \eta_0$. Note that in our case $\eta_0 > 0$: in fact, (4.3) directly implies that $g_\eta(\infty) = 0$ for η in the spectrum and we have chosen $g_\eta(-\infty) = 1$, so the right-hand side of (4.2) is ∞, but the left-hand side is bounded by 1 for $\eta \le 0$, so $\mathtt{Sp} \subset (0, \infty)$.

Remark 4.2. This tells us in particular that for $\eta < \eta_0$, we can certainly redefine $g_\eta(\cdot)$ so that $g_\eta(0) = 1$. With such a choice of course $g_\eta(\cdot)$ would still be analytic, but it is not defined for all $\eta \in \mathbb{C}$, since for (infinitely many) real values of $\eta < \eta_0$ we have $g_\eta(0) = 0$.

4.3. Analysis of the ODE: The results

We now state the precise estimates for $u_{\text{sub},\mathcal{G}_3,\eta}$, in both asymptotic limits

(1) $|\eta| \to \infty$ (except along an arbitrarily small sector containing the positive real axis), uniformly in x;
(2) $\eta \in \mathbb{C} \setminus \mathtt{Sp}$ is fixed and $x \to \infty$.

Before stating both results, we need to precise that for any $x \in \mathbb{R}$, when $Q_\eta(x) \in \mathbb{C} \setminus (-\infty, 0]$ (which is always the case when η is large, and in a sector of the complex plane that does not contain $(0, \infty)$), we define $Q_\eta^{1/2}(x)$ (resp., $Q_\eta^{1/4}(x)$) as the only square root of $Q_\eta(x)$ (resp., $Q_\eta^{1/2}(x)$) which satisfies $\arg(Q_\eta^{1/2}(x)) \in (-\pi/2, \pi/2]$ (resp., $\arg(Q_\eta^{1/4}(x)) \in (-\pi/4, \pi/4]$). This corresponds of course to choosing what is normally called the *principal branch* of the square root.

Theorem 4.3. *Fix $\mu \in \mathbb{R}$ and $\alpha \in (0, \pi)$. For any $\eta \in \mathbb{C} \setminus (0, \infty)$ there exists a solution u of* (4.4) *satisfying, uniformly in η such that $|\arg(-\eta)| \le \alpha$,*

$$\sup_{x\in\mathbb{R}} \left| u(x) Q_\eta^{1/4}(x) \exp\left(- \int_0^x Q_\eta^{1/2}(y)\,\mathrm{d}y \right) - 1 \right| = \mathrm{O}\big(|\eta|^{-3/4}\big). \tag{4.15}$$

Observe that the above solution is clearly subdominant for $x \to -\infty$ and it is therefore proportional to $u_{\mathrm{sub},\mathcal{G}_3,\eta}(x)$.

We defer the proof of Theorem 4.3 to Section 4.5: from the proof it is not difficult to see that the statement actually remains true in the case of general odd d, actually the exponent in the error term can be replaced by $d/(d+1)$ (see also Remark 4.10). More than that, the statement holds also for even d (Theorem 5.1).

We have similar, albeit more implicit, results when η is fixed and $x \to \infty$.

Theorem 4.4. *For every $\eta \in \mathbb{C}$, the following limit exists*

$$\lim_{x\to+\infty} u_{\mathrm{sub},\mathcal{G}_3}(x) \exp\left(-\frac{x^3}{6} + \mu \frac{x}{2} \right) =: \ell(\eta) \in \mathbb{C}. \tag{4.16}$$

Moreover $\ell(\cdot)$ is entire, $\ell(\eta) = 0$ if and only if $\eta \in \mathtt{Sp}$ and we have $\ell(\eta) > 0$ for $\eta < \eta_0$.

We defer the proof of Theorem 4.4 to Section 4.5. There are several consequences to the theorems we just stated, corresponding to the following three corollaries. The proofs of these three corollaries are deferred to the end of this section, Section 4.6.

Corollary 4.5. *Recall* (2.10) *and* (2.11) *for the definition of $\lambda_0 = \lambda_0(T_{3,\mu})$ and that η_0 is the smallest eigenvalue for* (4.4). *Then*

$$\lambda_0 = \frac{\eta_0}{2}. \tag{4.17}$$

Moreover, $\Phi_{T_{3,\mu}}(\lambda)$ has a simple pole at λ_0 and the value of the residue $-C_{3,\mu}$ (cf. Theorem 2.4*) is*

$$-\frac{c_{3,\mu}}{2\int_{\mathbb{R}} u^2_{\mathrm{sub},\mathcal{G}_3,\eta_0}(x)\,\mathrm{d}x}, \tag{4.18}$$

where $c_{3,\mu} = \lim_{x\to\infty} x^2 \exp(x^3/6 - \mu x/2) u_{\mathrm{sub},\mathcal{G}_3,\eta_0}(x) \in (0,\infty)$.

We are further able to deduce, using Theorem 4.3, the asymptotic behavior of the ratio $g_\eta(-\infty)/g_\eta(\infty)$, which with our normalization choice reduces to $1/g_\eta(\infty)$, as $|\eta| \to \infty$ along any ray that is not the positive real axis.

Corollary 4.6. *Fix $\beta_0 \in (0, \pi)$, uniformly in $\beta \in [-\beta_0, \beta_0]$ we have that*

$$\lim_{\substack{|\eta|\to\infty \\ \arg(-\eta)\to\beta}} \frac{1}{|\eta|^{3/4}} \log\big(\Phi_{T_{3,\mu}}(\eta/2)\big) = - \lim_{\substack{|\eta|\to\infty \\ \arg(-\eta)\to\beta}} \frac{1}{|\eta|^{3/4}} \log g_\eta(\infty) = -c(\beta), \tag{4.19}$$

where

$$c(\beta) := \int_{-\infty}^{\infty} f_0^{(\beta)}(y)\,\mathrm{d}y, \quad \textit{and}$$
$$f_0^{(\beta)}(y) = \frac{1}{\sqrt{2}}\sqrt{\frac{y^4}{4} + \cos(\beta) + \sqrt{1+\cos(\beta)\frac{y^4}{2} + \frac{y^8}{16}}} - \frac{y^2}{2}. \tag{4.20}$$

The fact that $|c(\beta)| < \infty$ follows from $f_0^{(\beta)}(y) = \cos(\beta)/y^2 + \mathrm{O}(1/y^6)$. Note that, for every y, $f_0^{(\beta)}(y)$ increases when $\cos(\beta)$ increases, so the even function $\beta \mapsto c(\beta)$ decreases for $\beta \in [0, \pi)$.

By specializing to the negative real axis $\eta < 0$ (so $\beta = 0$), we are able to give asymptotic results for $g_\eta(\infty)$, that are much sharper than the ones in Corollary 4.6:

Corollary 4.7. *For* $\eta \to -\infty$

$$\Phi_{T_{3,\mu}}(\eta/2) = \frac{1}{g_\eta(\infty)} = \left(1 + \mathrm{O}\left(|\eta|^{-1/4}\right)\right)\exp\left(-(-\eta)^{3/4}F_0 - (-\eta)^{1/4}F_1\right), \tag{4.21}$$

where for $i = 0, 1$, $F_i = \int_{-\infty}^{\infty} f_i(y)\,\mathrm{d}y$, *and*

$$f_0(y) = \sqrt{1 + \frac{y^4}{4}} - \frac{y^2}{2},$$
$$f_1(y) = \frac{\mu f_0(y)}{\sqrt{y^4+4}} = \frac{1}{2}\mu\left(1 - \frac{y^2}{\sqrt{y^4+4}}\right), \tag{4.22}$$

that is, $F_0 = 3\Gamma(-(3/4))^2/(8\sqrt{2\pi}) = 3.496\ldots$ *and* $F_1 = \mu\sqrt{2/\pi}\,\Gamma(3/4)^2 = 1.198\ldots$.

4.4. Proof of Theorems 2.4, 2.5 and of Proposition 2.7 (*d* odd)

The proofs of the two theorems is easily disposed by referring to some of the previous statements. The proof of Proposition 2.7 is instead going to require some work.

Proof of Theorem 2.4. By recalling Remark 3.11, one sees that Theorem 2.4 is just a restatement of Corollary 4.5. □

Proof of Theorem 2.5, formula (2.17). This time what we want is just a restatement of Corollary 4.7. □

Proof of Proposition 2.7. In the whole proof of the proposition, Corollary 4.6, which yields the leading asymptotic behavior of $\Phi_{T_{3,\mu}}(\lambda)$ for $|\lambda| \nearrow \infty$, along any ray in the complex plane, except the positive semi-axis, plays a central role. Let us therefore start by considering the two *standard rays*: the negative semi-axis $\eta < 0$ (Laplace transform with real argument) and $\eta \in \Im$ (characteristic function), that is respectively, $\beta = 0$ and $\beta = \pm\pi/2$. While the first case can be seen just as a warm up (since it is superseded by the sharp results in Corollary 4.7), the second case actually establishes the validity of (2.22) for $d = 3$. In these two cases, we have

$$\begin{aligned} f_0^{(0)}(y) &= \sqrt{1+\frac{y^4}{4}} - \frac{y^2}{2}, \\ f_0^{(\pi/2)}(y) = f_0^{(-\pi/2)}(y) &= \frac{1}{\sqrt{2}}\sqrt{\frac{y^4}{4} + \sqrt{1+\frac{y^8}{16}}} - \frac{y^2}{2}, \end{aligned} \tag{4.23}$$

and note that these two functions are positive: one can actually directly check that $f_0^{(\beta)}(\cdot) > 0$ if and only if $|\beta| \le \pi/2$. Therefore $c(\beta) > 0$ for these values of $\beta \in [-\pi/2, \pi/2]$, and of course also in some open interval containing $[-\pi/2, \pi/2]$. Incidentally we can compute

$$c(0) = \frac{3\Gamma(-(3/4))^2}{8\sqrt{2\pi}} = 3.49607\ldots \tag{4.24}$$

which of course coincides with the quantity F_0 in Corollary 4.7, and

$$c(\pm\pi/2) = \frac{\sqrt{2\pi}\Gamma(1/4)(\cos(\pi/8) - \sin(\pi/8))}{3\Gamma(3/4)} = 1.33789\ldots. \tag{4.25}$$

We are now ready to look at other rays. For this note that Corollary 4.6 implies that for every $\beta_0 \in (0, \pi)$ and every $c < \inf_{|\beta|\le\beta_0} c(\beta)$ there exists $C > 0$ such that for every $\lambda \notin \{z\colon \Re(z) \ge 0, |\arg(z)| \le \pi - \beta_0\}$

$$\left|\Phi_T(\lambda)\right| \le C\exp\bigl(-c|\lambda|^\nu\bigr), \tag{4.26}$$

where $T = T_{3,\mu}$ and $\nu = 3/4$. As we will see in a moment, this estimate is relevant for us as long as $c > 0$: since $c(\cdot)$ is even and (strictly) decreasing for $\beta \in [0, \pi)$ we define $\beta_{\max} \in (0, \pi)$ such that

$$c(\beta_{\max}) = 0. \tag{4.27}$$

Note that (4.25) tells us that $\beta_{\max} > \pi/2$.

The following lemma outlines how the bound (4.26) yields existence and strong regularity result on the density $\mathtt{f}_T(\cdot)$ of the positive random variable T:

Lemma 4.8. *Assume that a random variable T has exponential moment generating function $\Phi_T(\cdot)$ which can be analytically extended beyond the obvious analyticity domain $\{z\colon \Re(z) < 0\}$ to an open domain that contains the complement of the cone $S := \{z\colon \Re(z) > 0, |\arg(z)| < \pi - \beta\}$, for some $\beta \in (\pi/2, \pi)$ and assume that (4.26) holds in $S^{\complement}$ for some positive constants C and c and for some $\nu \in (0, 1)$. Then $\mathtt{f}_T(\cdot)$ exists and it is analytic in the cone $\{z\colon \Re(z) > 0, |\arg(z)| < \beta - \pi/2\}$.*

Lemma 4.8 (proven below) and (4.26) directly yield Proposition 2.7, except for (2.24). For (2.24), we recall that for a positive random variable X the expression $\psi_X(s) := \mathbb{E}[X^{s-1}]$ is called Mellin transform of (the law of) X. The fundamental domain of the Mellin transform is the open strip $\{s \in \mathbb{C}\colon \alpha_- < \Re(s) < \alpha_+\}$, with α_+, respectively α_-, the largest, respectively smallest, value such that $\mathbb{E}[X^{s-1}] < \infty$ for every $s \in (\alpha_-, \alpha_+)$. Of course $\psi_X(\cdot)$ is well defined and analytic in its fundamental domain. Therefore if $X = \exp(T)$ (of course $T = T_{3,\mu}$), then $\Phi_T(\lambda) = \psi_{\exp(T)}(\lambda + 1) =: \psi(\lambda + 1)$ and one easily sees than $\alpha_- = -\infty$ as well as $\alpha_+ = \lambda_0(T)$. We now appeal to a Tauberian Theorem in the realm of Mellin transforms, precisely to [11], Th. 4, part (ii) (paying attention to a misprint in the last formula of the statement: $(\log x)^k$ has to be corrected to $(\log x)^{k-1}$). The condition to apply this statement, that is meromorphic continuation of $\psi(\cdot)$ on a strip larger (to the right) than the fundamental one (in our case $\psi(\cdot)$ can be meromorphically extended to the whole of $\mathbb{C}$) and $\psi(s) = \mathrm{O}(|s|^r)$ for some $r > 1$ and as s that tends to infinity in a suitable strip around α_+ of the form $\{s \in \mathbb{C}\colon \alpha_+ - \delta < \Re(s) < \alpha_+ + \delta'\}$, with δ and δ' positive numbers. In our case such a result is (largely!) achieved by (4.26), and actually with arbitrary δ. What is quantitatively relevant in applying [11], Th. 4, part (ii), is δ' and we choose it to be so that $\{s \in \mathbb{C}\colon \alpha_+ - \delta < \Re(s) < \alpha_+ + \delta'\}$ contains only the pole at $s = \lambda_0(T) + 1$, that, by Theorem 2.4 or, equivalently, by Corollary 4.5, is a simple pole of which we know the residue $C := C_{3,\mu}$. The net result is that for the density $\mathtt{f}_X(\cdot)$ of X we have for $x \to \infty$

$$\mathtt{f}_X(x) = Cx^{-\lambda_0(T)-1} + \mathrm{O}\big(x^{-b-1}\big), \tag{4.28}$$

with $b > \lambda_0(T)$ and smaller than the second smallest eigenvalue of the Schrödinger operator we are dealing with. At this point, we just use the elementary relation

$$\mathtt{f}_T(t) = \exp(t)\mathtt{f}_X\big(\exp(t)\big), \tag{4.29}$$

and we obtain (2.24). This completes the proof of Proposition 2.7. □

Remark 4.9. Th. 4, part (ii) of [11] gives a formula for the asymptotic behavior of the density in terms of the residues of all the poles in the region of meromorphic extension of $\psi(\cdot)$. By generalizing Corollary 4.5 to deal also with more than just the bottom of the spectrum it is certainly possible to get to an asymptotic formula for which each term in the expansion corresponds to a point in the spectrum.

Proof of Lemma 4.8. The existence of $\mathtt{f}_T(\cdot) \in C^\infty$ is a direct consequence of the decay of the characteristic function $\varphi_T(s) = \Phi_T(\mathrm{i}s) = \mathrm{O}(\exp(-c|s|^\nu))$, $|s| \to \infty$: in fact the inversion formula $2\pi \mathtt{f}_T(t) = \int_{-\infty}^{\infty} \exp(-\mathrm{i}ts)\varphi_T(s)\,\mathrm{d}s$ holds as soon as $\varphi_T(\cdot) \in \mathbb{L}^1$ and, since $\varphi_T(\cdot) \in \mathbb{L}^p$ for every $p \geq 1$, we have also that $2\pi f_T^{(n)}(t) = \int_{-\infty}^{\infty} (-\mathrm{i}s)^n \exp(-\mathrm{i}ts)\varphi_T(s)\,\mathrm{d}s$, where we have introduced the obvious notation for the nth-derivative. For the analyticity, we use Cauchy's Theorem: choose $t > 0$, for N positive

$$0 = \int_{\mathrm{C}_N} \lambda^n \exp(-t\lambda)\Phi_T(\lambda)\,\mathrm{d}\lambda = \int_{-\mathrm{i}N}^{\mathrm{i}N} \cdots + \sum_{q=\pm} \int_{\mathrm{R}_N^q} \cdots + \int_{\mathrm{P}_N^{\triangleright}} \cdots, \tag{4.30}$$

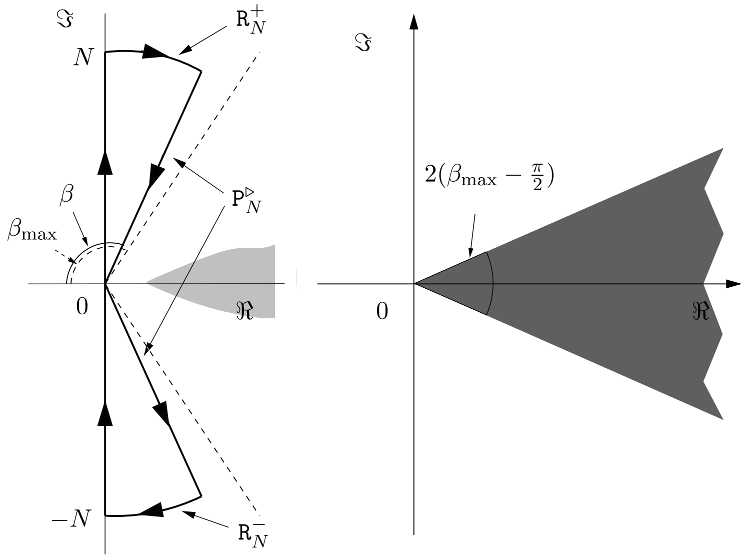

Figure 8. On the left the integration path C_N, which is the union of the straight path going from $-\mathrm{i}N$ to $+\mathrm{i}N$, the two arcs $\mathrm{R}_N^\pm$ and $\mathrm{P}_N^\triangleright$. This path is contained in the domain of analyticity of the analytic continuation of the moment generating function $\Phi_T(\cdot)$, which is the complement of the lightly shadowed region. Actually, in our specific cases $\Phi_T(\cdot)$ can be extended to a meromorphic functions that has (countably many) poles on the positive real axis (that accumulate only at $+\infty$). The angle β that determines the path C_N can be chosen arbitrarily close to $\beta_{\max}$. The rays corresponding to $\beta_{\max}$ in the figure, that is the dashed rays, are the critical rays along which $\Phi_T(\lambda)$ does not have an exponential behavior with the rate $|\lambda|^\nu$: in the region in which the path C_N lies the asymptotic behavior along rays is exponentially vanishing. This makes negligible the contribution of the arcs to the path integral, in the limit $N \to \infty$. On the right instead we draw the cone in which $\mathrm{f}_T(\cdot)$ is analytic, as a consequence of the bounds on $\Phi_T(\cdot)$ underlying the figure on the left.

where the contour C_N and paths $\mathrm{R}_N^\pm$ and $\mathrm{P}_N^\triangleright$ in the complex plane are given in Figure 8. The first term in the right-hand side converges, as $N \to \infty$, to $-\mathrm{i}(-1)^n 2\pi f_T^{(n)}(t)$, whereas the fast decay at infinity of $\Phi_T(\cdot)$ immediately entails that the contributions due to the integration along the arcs $\mathrm{R}_N^\pm$ vanish in the same limits. Therefore

$$f_T^{(n)}(t) = \lim_{N\to\infty} \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{P}_N^\triangleleft} (-\lambda)^n \exp(-t\lambda)\Phi_T(\lambda)\,\mathrm{d}\lambda, \tag{4.31}$$

where $\mathrm{P}_N^\triangleleft$ is the path $\mathrm{P}_N^\triangleright$ with reversed orientation. From (4.31) and the hypotheses, we have

$$\left|f_T^{(n)}(t)\right| \le \frac{C}{\pi}\int_0^\infty r^n \exp\bigl(-rt\sin(\beta)\bigr)\,\mathrm{d}r = \frac{C}{\pi}\frac{n!}{(t\sin(\beta))^{n+1}}. \tag{4.32}$$

Therefore the radius of convergence of the Taylor series at $t > 0$ is (at least) $t\sin(\beta)$, which is the analyticity property claimed in the statement. $\square$

Remark 4.10. As we have already mentioned, all the results we present here can be generalized in a straightforward way to general $d \geq 3$: let us quickly discuss here how the ODE results we just presented generalize for d odd.

- The result of Theorem 4.3 remains true, in fact we get a more precise estimate for greater d, as $\mathrm{O}(|\eta|^{-3/4})$ should be replaced in the general case with $\mathrm{O}(|\eta|^{-d/(d+1)})$.
- The result of Theorem 4.4 remains also valid when replacing $-x^3/6 + \mu x/2$ with the more general $V_\mu(x)$ of (2.3).
- The result of Corollary 4.5 also holds, with the obvious modifications for the value of $C_{d,\mu}$.
- The result of Corollary 4.6 still holds when the exponent of $|\eta|$ is replaced with $\frac{1}{2} + \frac{1}{2(d-1)} = \frac{d}{2(d-1)}$, and $f_0^{(\beta)}(y)$, with respect to the formula in (4.20), is obtained by replacing the exponents 2, 4 and 8 with $(d-1)$, $2(d-1)$, and $4(d-1)$.

It is also interesting to observe that $f_0^{(\beta)}$ does not depend on the parameter μ. In fact, an expression for $c(\beta)$ in terms of special functions can be found for every β, and numerical evidence suggests that $\beta_{\max} = (d-1)\pi/d$.

4.5. Analysis of the ODE: Proof of Theorems 4.3 and 4.4

Proof of Theorem 4.3. We start by making easy observations explaining why $|\eta|$ large enough and $\arg(-\eta) \in [-\alpha, \alpha]$ simplifies our study. Recall $0 \leq \alpha < \pi$, and Q_η, q have been introduced in (4.5). With our two assumptions on η we either have $\Im(\eta) \neq 0$ or $\eta < 0$ and large enough so that $\eta < \min_{x \in \mathbb{R}} q(x) =: q(x_0)$. Either way $Q_\eta(x) \in \mathbb{C} \setminus (-\infty, 0]$ for every $x \in \mathbb{R}$, so that both $Q_\eta^{1/2}$, $Q_\eta^{1/4}$ are analytic.

Moreover, note that if $\arg(q(x_0) - \eta) \in [-(\pi + \alpha)/2, (\alpha + \pi)/2]$ (which holds for $|\eta|$ large enough, in the sector we consider), we have

$$\inf_{x \in \mathbb{R}} \Re\big(Q_\eta^{1/2}(x)\big) = \Re\big(Q_\eta^{1/2}(x_0)\big) \geq \cos\left(\frac{\alpha + \pi}{4}\right)\sqrt{|\eta|}. \tag{4.33}$$

For the imaginary part, we will only need the trivial bound

$$\sup_{x \in \mathbb{R}} \big|\Im\big(Q_\eta^{1/2}(x)\big)\big| = \sqrt{|q(x_0) + \eta|}\,\sin\left(\frac{\alpha + \pi}{4}\right) \leq \sqrt{|\eta|}, \tag{4.34}$$

where we have taken $|\eta|$ large enough that the second inequality above holds.

We now exploit the fact that WKB theory gives us a guess for the asymptotic behavior of u as η tends to infinity away from the positive axis and the next steps is writing an integral equation, that is (4.40), for the ratio between u and the WKB guess. Of course, these steps are applications to our context of ideas taken from the rigorous approach to WKB estimates (see, e.g., [25] or the more recent [10] and references therein).

Let us then set $\xi(x) := \int_0^x Q^{1/2}(y)\,\mathrm{d}y$. If we set $n := Q^{1/4} u$, we then have

$$n''(x) - \frac{Q'}{2Q} n' - Qn - \widetilde{R} n = 0, \tag{4.35}$$

with

$$\widetilde{R} := \frac{Q''}{4Q} - \frac{5(Q')^2}{16Q^2}. \tag{4.36}$$

Let us observe that the equation

$$v''(x) - \frac{Q'}{2Q}v' - Qv = 0, \tag{4.37}$$

admits the linearly independent solutions $v_\pm := \exp(\pm\xi)$ (the Wronskian of this set of solutions is $2Q^{1/2}$). We exploit then the variation of constant formula from which we obtain that if we find a solution $n(\cdot)$ (say, in $\mathbb{L}^2((-\infty, c])$ for every $c \in \mathbb{R}$)

$$n(x) = \exp\bigl(\xi(x)\bigr) + \int_{-\infty}^{x} R(y)n(y)\sinh\bigl(\xi(x) - \xi(y)\bigr)\,\mathrm{d}y, \tag{4.38}$$

where $R := \widetilde{R}/Q^{1/2}$ (remark that R is bounded and $R(x) = \mathrm{O}(1/|x|^4)$ for $|x|$ large), then $n(\cdot)$ solves (4.35). In view of the result, we want to obtain we set also

$$N(x) := n(x)\exp\bigl(-\xi(x)\bigr), \tag{4.39}$$

so that (4.38) becomes

$$N(x) = 1 + \int_{-\infty}^{x} R(y)N(y)\mathcal{K}\bigl(\xi(x) - \xi(y)\bigr)\,\mathrm{d}y =: 1 + \mathcal{T}_\eta N(x), \tag{4.40}$$

where $\mathcal{K}(z) = (1 - \exp(-2z))/2$. We look at $\mathcal{T}_\eta$ as an operator that acts on $\mathbb{L}^\infty$ functions.

Lemma 4.11. *For every* $\alpha \in (0, \pi)$ *there exists* $C > 0$ *such that*, *for any* η *with* $\arg(-\eta) \in [-\alpha, \alpha]$, $\|\mathcal{T}_\eta\|_\infty \le C|\eta|^{-3/4}$.

This lemma, applied to (4.40), tells us that we can write N as the operator $\mathcal{T}_\eta(1-\mathcal{T}_\eta)^{-1}$ applied to the constant function equal to 1 and therefore $\|N\|_\infty \le 2C|\eta|^{-3/4}$ for $|\eta|^{3/4} \ge 2$. Therefore, this completes the proof of Theorem 4.3. □

Proof of Lemma 4.11. In order to bound $\|\mathcal{T}_\eta\|_\infty$, we first need to look more carefully at the argument of $\mathcal{K}$ in the integral. By (4.33), for any $y < x$ we have

$$\Re\bigl(\xi(x) - \xi(y)\bigr) = \int_y^x \Re\bigl(Q_\eta^{1/2}(z)\bigr)\,\mathrm{d}z \ge \cos\left(\frac{\alpha + \pi}{4}\right)\sqrt{|\eta|}(x - y). \tag{4.41}$$

Moreover, by (4.34), we also have for any $y < x$,

$$\bigl|\Im\bigl(\xi(x) - \xi(y)\bigr)\bigr| \le \int_y^x \bigl|\Im\bigl(Q_\eta^{1/2}(z)\bigr)\,\mathrm{d}z\bigr| \le \sqrt{|\eta|}(x - y). \tag{4.42}$$

Combining the last two inequalities, we further deduce that

$$\begin{aligned} \left|\Im\big(\xi(x)-\xi(y)\big)\right| > \pi/4 \quad &\Rightarrow \quad x-y > \frac{\pi}{4\sqrt{|\eta|}} \\ &\Rightarrow \quad \Re\big(\xi(x)-\xi(y)\big) > \frac{\pi}{4}\cos\left(\frac{\alpha+\pi}{4}\right), \end{aligned} \tag{4.43}$$

so $|\Im(\xi(x)-\xi(y))| > \pi/4$, then $|\mathcal{K}(\xi(x)-\xi(y))| \le (1+\exp(-\pi/2)/2)/2 < \sqrt{2}/2$. On the other hand if $|\Im(z)| \le \pi/4$, then $|\mathcal{K}(z)| \le \sqrt{2}/2$. Therefore for any $y < x$,

$$\left|\mathcal{K}\big(\xi(x)-\xi(y)\big)\right| \le \frac{\sqrt{2}}{2}. \tag{4.44}$$

Furthermore, we see by direct inspection that for every $\alpha \in [0,\pi)$ there exists $C_0 > 0$ such that

$$\left|R(y)\right| \le \frac{C_0}{|\eta| + y^4}, \tag{4.45}$$

for every $y \in \mathbb{R}$ and every η such that $\arg(-\eta) \in [-\alpha,\alpha]$. Therefore

$$\|\mathcal{T}_\eta N\|_\infty \le C_0 \frac{\sqrt{2}}{2}\|N\|_\infty \int_{-\infty}^{\infty} \big(|\eta| + y^4\big)^{-1}\,\mathrm{d}y = C\eta^{-3/4}\|N\|_\infty, \tag{4.46}$$

in which the last step is just the definition of C. □

Proof of Theorem 4.4. A good deal of this proof focuses on the asymptotic behavior of $u_{\mathrm{sub},\mathcal{G}_3}$ as $x \to \infty$. Recall in fact (4.11), so that, in view of (4.8), we have that

$$\lim_{x\to+\infty} u_{\mathrm{sub},\mathcal{G}_3,\eta}(x)\exp\left(-\frac{x^3}{6} + \mu\frac{x}{2}\right) = b(\eta) \lim_{x\to+\infty} u_{\mathrm{dom},\mathcal{G}_0,\eta}(x)\exp\left(-\frac{x^3}{6} + \mu\frac{x}{2}\right), \tag{4.47}$$

provided that the limit on the right-hand side exists. We actually aim at proving also the analyticity of the left-hand side in the whole of $\mathbb{C}$, but, since $b(\cdot)$ is entire, this amounts to showing that the limit in the right-hand side yields an entire function.

Furthermore, recall from Section 4.2 that $b(\eta) = 0$ if and only if $\eta \in \mathtt{Sp}$, thus to establish that the left-hand side of (4.47) only vanishes on the spectrum, all we need to establish is that the limit in the right-hand side does not vanish.

As for what concerns η real, we know that $u_{\mathrm{sub},\mathcal{G}_3}(x) > 0$ for every $x \in \mathbb{R}$ if $\eta < \eta_0$ and so the last statement of Theorem 4.4 will again follow if we establish that the limit in the right-hand side is not 0.

We are going to use an approach parallel to that of the last paragraph (we are still doing WKB estimates, even if now $x \to \infty$ instead of $|\eta| \to \infty$), except that now, for $\eta \in \mathbb{R}$, Q_η may take the value zero, making impossible the change of functions of the last paragraph on the whole of $\mathbb{R}$. But now we just working in a neighborhood of $+\infty$ and in fact the first fact to remark is that when x is large $\Re Q_\eta(x)$ is large and $Q_\eta(x)$ is in a small sector containing the positive axis, that is, $\arg Q_\eta(x)$ is small. We will need also more precise information about $Q_\eta(x)$ and we collect them in the following lemma for which we introduce the notation $B_r := \{z \in \mathbb{C}: |z| < r\}$:

Lemma 4.12. *For every $r > 0$ there exists $A_0 > 0$ and $C > 0$ such that for every $A \geq A_0$ we have* (*recall that R is defined below* (4.38))

$$\Re\big(Q_\eta(x)\big) \geq \frac{x^4}{8}, \qquad \big|\arg(Q_\eta)(x)\big| \leq \frac{\pi}{4} \quad \text{and} \quad \big|R(x)\big| \leq \frac{C}{x^4}, \tag{4.48}$$

for $x \geq A$ and $\eta \in \overline{B}_r$. Moreover for $x \to \infty$, we have

$$\begin{aligned} Q_\eta^{1/2}(x) &= \frac{x^2}{2} - \frac{\mu}{2} + \frac{1}{x} + \mathrm{O}\left(\frac{1}{x^2}\right), \\ \frac{1}{Q_\eta^{1/4}(x)} &= \frac{\sqrt{2}}{x} + \mathrm{O}\left(\frac{1}{x^2}\right), \end{aligned} \tag{4.49}$$

uniformly for $\eta \in \overline{B}_r$.

Proof. From (4.5) one sees $Q_\eta(x) \sim x^4/4$ for $x \to \infty$ and that $\Re Q_\eta(x)$ tends to ∞, while $\Im Q_\eta(x)$ stays bounded. Therefore for any $\beta \in (0, \pi)$ and any $r > 0$ there exists $x_0 > 0$ such that $|\arg(Q_\eta)(x)| \leq \beta$ for every $\eta \in \overline{B}_r$ and every $x \geq x_0$. These observations yield (4.48): the first two estimates are immediate, the third requires making (4.38) explicit.

For (4.49) we set $\eta = \eta_1 + \mathrm{i}\eta_2$ and $Q_\eta(x) = r(x)\exp(\mathrm{i}\theta(x))$ so that

$$r(x)^{1/2} = \frac{x^2}{2}\left[\left(1 - \frac{2\mu}{x^2} + \frac{4}{x^3} + \frac{\mu^2 + 4\eta_1}{x^4}\right)^2 + \frac{16\eta_2^2}{x^8}\right]^{1/4}. \tag{4.50}$$

We then observe that for x large (and uniformly in $\eta \in \overline{B}_r$)

$$r(x)^{1/2} = \frac{x^2}{2} - \frac{\mu}{2} + \frac{1}{x} + \mathrm{O}\left(\frac{1}{x^2}\right) \quad \text{and} \quad \theta(x) = \mathrm{O}\left(\frac{1}{x^4}\right), \tag{4.51}$$

and (4.49) follows. □

In view of the estimates, we are after (or, equivalently, in view of what the WKB approach suggests) we now set

$$\xi_A(x) := \int_A^x Q^{1/2}(y)\,\mathrm{d}y, \tag{4.52}$$

and

$$N_{\mathrm{dom},A,\eta}(x) = N_{\mathrm{dom},A}(x) := \frac{Q_\eta^{1/4}(x)\exp(-\xi_A(x))u_{\mathrm{dom},\mathcal{G}_0}(x)}{Q_\eta^{1/4}(A)u_{\mathrm{dom},\mathcal{G}_0}(A)}, \tag{4.53}$$

for $x \geq A$ (of course we have chosen the normalization so that $N_{\mathrm{dom},A}(A) = 1$). Of course this requires

$$Q_\eta^{1/4}(A) u_{\mathrm{dom},\mathcal{G}_0}(A) \neq 0, \tag{4.54}$$

but this is granted, uniformly in $\eta \in \overline{B}_r$, for A sufficiently large because $u_{\mathrm{dom},\mathcal{G}_0}$ is a dominant solution and because of Lemma 4.12. Moreover, by (4.48), $\eta \mapsto Q_\eta^{1/4}(A)$ is entire for A large, while $u_{\mathrm{dom},\mathcal{G}_0}(A)$ for every A is entire in η as we pointed (see (4.9) and paragraph leading to (4.11)), and therefore the expression in (4.54) is entire (in η) for A large.

At this point, we write

$$u_{\mathrm{dom},\mathcal{G}_0,\eta}(x) \exp\left(-\frac{x^3}{6} + \mu \frac{x}{2}\right) = \left(Q_\eta^{1/4}(A) u_{\mathrm{dom},\mathcal{G}_0}(A)\right) N_{\mathrm{dom},A,\eta}(x) h_\eta(x), \tag{4.55}$$

where

$$h_\eta(x) := Q_\eta^{-1/4}(x) \exp\left(\int_A^x Q_\eta^{1/2}(y)\, \mathrm{d}y - \frac{x^3}{6} + \frac{\mu}{2} x\right), \tag{4.56}$$

and we are done if we show that the limits as $x \to \infty$ of $N_{\mathrm{dom},A,\eta}(x)$ and of $h_\eta(x)$ exist, that they are non-zero and that the limit expressions are analytic in $\eta \in B_r$. Let us notice from now that such a statement for $N_{\mathrm{dom},A,\eta}(x)$ involves WKB analysis, while $h_\eta(x)$ is an explicit expression and will be dealt just by applying the Taylor expansion estimates (4.49).

Let us then start with $h_\eta(x)$. For x fixed $h_\cdot(x)$ is now looked upon as a function from $\overline{B}_r$ to $\mathbb{C}$ and it is analytic in B_r. By (4.49), the family of functions $\{h_\cdot(x)\}_{x \geq A}$ possesses a limit as $x \to \infty$. Added to that, (4.49) implies that this family is bounded and bounded away from 0 for every $\eta \in \overline{B}_r$, provided A is sufficiently large. An application of Montel's Theorem [9], page 153, establishes the analyticity of $\eta \mapsto \lim_{x\to\infty} h_\eta(x)$ in B_r. The fact that the family is bounded away from zero implies of course that the limit is bounded away from zero.

Let us then turn to $N_{\mathrm{dom},A,\eta}$. We directly verify exactly like for (4.40) that $N_{\mathrm{dom},A,\eta}$ solves for $x \geq A$

$$N(x) = 1 + \int_A^x R(y) N(y) \mathcal{K}\big(\xi_A(x) - \xi_A(y)\big)\, \mathrm{d}y =: 1 + \mathcal{T}_A N(x). \tag{4.57}$$

$\mathcal{T}_A$ is viewed as an operator acting on $\mathbb{L}^\infty([A,\infty);\mathbb{C})$ and one verifies by exploiting (4.48), exactly like in the proof of Lemma 4.11, that $\|\mathcal{T}_A\| = \mathrm{o}(1/A^3)$ as $A \to \infty$, uniformly in $\eta \in \overline{B}_r$ and therefore that (4.57) has a unique solution $N = N_{\mathrm{dom},A,\eta}$ satisfying

$$\sup_{\eta \in \overline{B}_r} \sup_{x \geq A} \left| N_{\mathrm{dom},A,\eta}(x) - 1 \right| \leq \frac{1}{2}, \tag{4.58}$$

for A sufficiently large. Moreover, by differentiating both sides in (4.57) (the smoothness of $N_{\mathrm{dom},A}$ follows directly from the integral equation) we obtain

$$N'_{\mathrm{dom},A}(x) = \sqrt{Q_\eta(x)} \int_A^x R(y) N_{\mathrm{dom},A}(y) \exp\left(-2\big(\xi_A(x) - \xi_A(y)\big)\right) \mathrm{d}y, \tag{4.59}$$

so that for every $\eta \in \overline{B}_r$

$$\left|N'_{\mathrm{dom},A}(x)\right| \le cx^2 \exp\bigl(-2\Re\xi_A(x)\bigr) \int_A^x \frac{\exp(2\Re\xi_A(y))}{y^4}\,\mathrm{d}y \overset{x\to\infty}{\sim} \frac{c'}{x^3}, \tag{4.60}$$

where c and c' are suitable positive constants: we have of course used the estimates in Lemma 4.12, notably (4.49) and for the asymptotic statement can be obtained by integration by parts (see, e.g., [3], pages 255–256). Therefore for every $\eta \in \overline{B}_r$, $\lim_{x\to\infty} N_{\mathrm{dom},A,\eta}(x)$ exists and (4.60) implies also that the limit is non-zero for A sufficiently large. This can be seen also directly from (4.58). But (4.58) yields analyticity too: since for every $x \ge A$ the function $\eta \mapsto N_{\mathrm{dom},A,\eta}(x)$ is analytic in B_r, by Montel's Theorem [9], page 153, the limit is analytic in B_r.

The proof of Theorem 4.4 is therefore complete. □

4.6. Proof of the corollaries of Section 4.3

Proof of Corollary 4.5. First of all, recall that the change of variable $\eta = 2\lambda$, so $f_\lambda = g_{2\lambda}$, shows that (3.36) in Corollary 3.6 is the same as (4.1). This is spelled out also in (4.2), in a different language. Therefore, by recalling also (4.3) and (4.4), we see that the properties of the subdominant solution $u_{\mathrm{sub},\mathcal{G}_3}$, in particular Remark 4.1, yield the existence result assumed in Corollary 3.6 (and Corollary 3.10). Moreover, Theorem 4.4 guarantees that the right-hand side of (3.46), which is actually simply, $1/f_\lambda(\infty)$, is meromorphic, with poles only on the positive real axis and the first one is $\eta_0/2$. At this point we use the analytic extension argument detailed in Remark 3.11 to get that (3.46) holds for every $\lambda < \lambda_0 = \eta_0/2$. This establishes (4.17).

For the more precise estimate on this pole claimed in Corollary 4.5, it suffices to study the behavior of $g_\eta(\infty)$ near $\eta = \eta_0$, where it vanishes. Therefore, we set $h_\eta(x) := \frac{\mathrm{d}g_\eta(x)}{\mathrm{d}\eta}$, and aim at the existence and evaluation of $h_{\eta_0}(\infty)$. We write $h = h_{\eta_0}$ and introduce

$$v(x) := \frac{\mathrm{d}u_{\mathrm{sub},\mathcal{G}_3}}{\mathrm{d}\eta}\bigg|_{\eta=\eta_0} = \exp\left(\frac{x^3}{6} - \mu\frac{x}{2}\right)h(x), \tag{4.61}$$

so that v is solution to

$$v'' - Q_\eta v = -u_{\mathrm{sub},\mathcal{G}_3}. \tag{4.62}$$

Since we are on the spectrum ($\eta = \eta_0$), that is, $(u_0 :=)\, u_{\mathrm{sub},\mathcal{G}_3} \propto u_{\mathrm{sub},\mathcal{G}_0}$, and thus also

$$u_{\mathrm{dom},\mathcal{G}_3} \propto u_{\mathrm{dom},\mathcal{G}_0}, \tag{4.63}$$

where we have set $u_{\mathrm{dom},\mathcal{G}_3} := u_{\mathrm{sub},\mathcal{G}_2}$ by using exactly the same argument that lead to the definition of $u_{\mathrm{dom},\mathcal{G}_0}$. Note that we have made the precise choice (i.e., we choose the multiplicative constant) of $u_{\mathrm{sub},\mathcal{G}_3}$ given by (4.7), that we recall here

$$u_{\mathrm{sub},\mathcal{G}_3}(x) = u_0(x) \overset{x\to-\infty}{\sim} \exp\left(\frac{x^3}{6} - \mu\frac{x}{2}\right). \tag{4.64}$$

By [23], Th. 6.1 and Th. 7.2, we have also that there exists $c > 0$ (for the positivity recall that the solution is positive for $\eta < \eta_0$) such that

$$u_0(x) \overset{x\to+\infty}{\sim} \frac{c}{x^2} \exp\left(-\frac{x^3}{6} + \mu\frac{x}{2}\right). \tag{4.65}$$

On the other hand, we can also make a precise choice of $U_0 := u_{\mathrm{dom},\mathcal{G}_0}$ by fixing the multiplicative constant in the asymptotic behavior of the subdominant solution in the sector $\mathcal{G}_1$ (that we use to define the dominant solution in $\mathcal{G}_0$, like in Section 4.1). This in turn gives a definite choice of the asymptotic behaviors of U_0

$$U_0(x) \overset{x\to-\infty}{\sim} \frac{c'}{x^2} \exp\left(-\frac{x^3}{6} + \mu\frac{x}{2}\right) \quad \text{and} \quad U_0(x) \overset{x\to\infty}{\sim} \exp\left(\frac{x^3}{6} - \mu\frac{x}{2}\right), \tag{4.66}$$

where c' is a constant that appears as a result of fixing to one the multiplicative constant in the behavior at $+\infty$ of the dominant solution. Three observations are in order:

(1) The asymptotic behaviors (4.66) are obtained precisely like in the proof of Theorem 4.16: the proof for the case $x \to \infty$ is actually completely contained in the proof of Theorem 4.16, the case $x \to -\infty$ requires redoing the asymptotic computation starting from (4.56).
(2) The leading asymptotic behaviors of u_0' and U_0' are obtained by taking the derivative of the asymptotic relations (4.64), (4.65) and (4.66) and by keeping the leading order: this is proven in [23], Th. 6.1, for the subdominant case, while for the dominant case it just requires a straightforward generalization of the argument in the proof of Theorem 4.4, starting from (4.59) (note that it is a matter of refining (4.60)).
(3) We have made a definite choice of U_0 by exploiting the subdominant solution in $\mathcal{G}_1$, which is unique up to a multiplicative constant, and by fixing the constant with the second asymptotic statement in (4.66). Of course we can replace U_0 by adding a multiple of u_0, and this does not change (4.66): one can clearly see that also the final result we obtain, that is, (4.70), is invariant under such a change.

Obviously u_0, U_0 form a system of independent solutions to (4.4). It is well known (and straightforward to check) that the Wronskian $((u_0)'U_0 - (U_0)'u_0)(x)$ is a constant that we call W. W can be computed by using (4.64), (4.65) and (4.66) along with point (2) in the list above. Since it can actually be computed in the two limits $x \to \pm\infty$, one finds $((u_0)'U_0 - (U_0)'u_0)(\infty) = -c$ and $((u_0)'U_0 - (U_0)'u_0)(-\infty) = c'$, so $c' = -c < 0$. We now use the variation of constants method to give a general expression [7], Th. 6.4, for v (by Remark 4.1 $v(-\infty) = 0$)

$$v(x) = \frac{1}{W}\left(U_0(x)\int_{-\infty}^{x} u_0^2(y)\,\mathrm{d}y - u_0(x)\int_{-\infty}^{x} u_0(y)U_0(y)\right), \tag{4.67}$$

and for clarity we will substitute $W = -c$ only at the end of the computation. Obviously $u_0 \in \mathbb{L}^2$ so that $\int_{\mathbb{R}} u_0^2(y)\,\mathrm{d}y \in (0,\infty)$. Also, by (4.64), (4.65) and (4.66) we see that $u_0U_0(x) \overset{x\to\pm\infty}{\sim} \frac{c}{x^2}$, so that also $\int_{\mathbb{R}} u_0U_0(y)\,\mathrm{d}y \in (0,\infty)$. Therefore, since U_0 is dominant at $+\infty$ whereas u_0 is subdominant, it follows that the second term in the sum of (4.67) is negligible when compared to the first. Hence,

$$v(x) \overset{x\to\infty}{\sim} \frac{U_0(x)}{W} \int_{-\infty}^{\infty} u_0^2(y)\,\mathrm{d}y. \tag{4.68}$$

We now deduce from (4.61) and (4.66) that (here we insert also $W = -1/c$)

$$h(x) \overset{x\to\infty}{\longrightarrow} -\frac{1}{c} \int_{-\infty}^{\infty} u_0^2(y)\,\mathrm{d}y, \tag{4.69}$$

and therefore

$$g_\eta(\infty) \overset{\eta\to\eta_0}{\sim} (\eta_0 - \eta) \int_{-\infty}^{\infty} \frac{u_0^2(y)}{c}\,\mathrm{d}y. \tag{4.70}$$

Recall that $\Phi_{T_{3,\mu}}(\lambda) = \frac{1}{g_\eta(\infty)}$ and $\eta_0 - \eta = 2(\lambda_0 - \lambda)$, so we are done with the proof of Corollary 4.5. □

Proof of Corollary 4.6. We recall that the basic formula is (4.2), that is Corollary 3.10, complemented by Remark 3.11 and Remark 4.1, and that we found practical at a certain stage to decide that $g_\eta(-\infty) = 1$ (cf. (4.8)). In reality, in this proof the crucial tool is Theorem 4.3, and that result contains a constant c that we do not determine explicitly. This actually amounts to saying that for this proof it is easier to think in terms of $g_\eta(-\infty)/g_\eta(\infty)$, rather than $1/g_\eta(\infty)$. In any case, by applying Theorem 4.3 one obtains

$$\frac{g_\eta(-\infty)}{g_\eta(\infty)} = \lim_{x\to\infty} \frac{g_\eta(-x)}{g_\eta(x)} = \lim_{x\to\infty} \exp\left(-\int_{-x}^{x} \left(Q_\eta^{1/2}(z) - \frac{1}{2}z^2 + \frac{\mu}{2}\right)\mathrm{d}z\right). \tag{4.71}$$

Let us set $\beta_\eta = \arg(-\eta)$, so by assumption $\beta_\eta \to \beta$. We are aiming at proving that if we fix any $\beta_0 \in (0, \pi)$, for every $\delta > 0$ there exists $\kappa > 0$ such that

$$\begin{aligned} &\left| \frac{1}{|\eta|^{3/4}} \lim_{x\to\infty} \Re\left[\int_{-x}^{x} \left(Q_\eta^{1/2}(z) - \frac{1}{2}z^2 + \frac{\mu}{2}\right)\mathrm{d}z\right] - c(\beta_\eta)\right| \le \delta \\ &\text{for } |\eta| > \kappa \text{ and } |\beta_\eta| \le \beta_0, \end{aligned} \tag{4.72}$$

where $c(\beta_\eta)$ is as in (4.20).

In order to establish this, recall that $Q_\eta(z) = z + \frac{1}{4}(z^2 - \mu)^2 - \eta$, so that the leading order depends on wether $|z|^4$ or $|\eta|$ is the largest. More precisely, there exists a A_0 large enough so that, for any z such that $|z| \ge A|\eta|^{1/4}$ with $A \ge A_0$, we find (similarly to (4.49))

$$\left| Q_\eta^{1/2}(z) - \frac{z^2}{2} + \frac{\mu}{2} - \frac{1}{z} \right| \le C\frac{|\eta|}{z^2}, \tag{4.73}$$

where C is a positive constant which does not depend on A nor η. Using the fact that $x \mapsto 1/x$ is odd, and then (4.73), we find that

$$\begin{aligned} &\lim_{x\to\infty}\left|\int_{x\geq|z|\geq A|\eta|^{1/4}}\left(Q_\eta^{1/2}(z)-\frac{z^2}{2}+\frac{\mu}{2}\right)\mathrm{d}z\right| \\ &\quad=\lim_{x\to\infty}\left|\int_{x\geq|z|\geq A|\eta|^{1/4}}\left(Q_\eta^{1/2}(z)-\frac{z^2}{2}+\frac{\mu}{2}-\frac{1}{z}\right)\mathrm{d}z\right| \\ &\quad\leq\lim_{x\to\infty}\int_{x\geq|z|\geq A|\eta|^{1/4}}\frac{C|\eta|}{z^2}=2\frac{C}{A}|\eta|^{3/4}. \end{aligned} \tag{4.74}$$

On the other hand,

$$\int_{-A|\eta|^{1/4}}^{A|\eta|^{1/4}}\left(Q_\eta^{1/2}(z)-\frac{1}{2}z^2+\frac{\mu}{2}\right)\mathrm{d}z=|\eta|^{1/4}\int_{-A}^{A}\left(Q_\eta^{1/2}\big(y|\eta|^{1/4}\big)-\frac{|\eta|^{1/2}y^2}{2}+\frac{\mu}{2}\right)\mathrm{d}y. \tag{4.75}$$

It remains to estimate the leading order, uniformly for $|\eta|$ large and in the sector $|\beta_\eta|\leq\beta_0$ (of course we can assume $\beta_0>\pi/2$), of $\Re(Q_\eta^{1/2}(y|\eta|^{1/4}))$. Once again we set $Q_\eta(x)=r_\eta(x)\exp(\mathrm{i}\theta(x))$ and

- We find that the norm of $Q_\eta^{1/2}(y|\eta|^{1/4})$ is

$$r_\eta^{1/2}\big(|\eta|^{1/4}y\big)=|\eta|^{1/2}\left[1+\cos(\beta)\frac{y^4}{2}+\frac{y^8}{16}\right]^{1/4}+\mathrm{o}\big(|\eta|^{1/2}\big). \tag{4.76}$$

- The cosine of the argument of $Q_\eta^{1/2}(y|\eta|^{1/4})$ is (again we use $\beta_\eta\to\beta$)

$$\begin{aligned} \cos\left(\frac{1}{2}\theta\big(|\eta|^{1/4}y\big)\right)&=\frac{1}{\sqrt{2}}\sqrt{1+1\Big/\sqrt{1+\frac{\sin^2(\beta)}{\cos^2(\beta)+\cos(\beta)y^4/2+y^8/16}}}+\mathrm{o}(1) \\ &=\frac{1}{\sqrt{2}}\sqrt{1+\frac{\cos(\beta)+y^4/4}{\sqrt{1+\cos(\beta)y^4/2+y^8/16}}}+\mathrm{o}(1), \end{aligned} \tag{4.77}$$

  where $\mathrm{o}(1)$ is as $|\eta|\to\infty$ in the sector $|\beta|\leq\pi/2$ (i.e., $\cos(\beta)\geq0$, see below for the other cases).

The estimate (4.76) above is a simple consequence of the exact expression for $r_\eta^{1/2}(x)$, cf. (4.50), and the fact that $\beta_\eta\to\beta$.

As for (4.77), we of course also use that $\beta_\eta\to\beta$, but then recall we made the extra assumption $\cos(\beta)\geq0$ and write

$$\theta\big(|\eta|^{1/4}y\big)=\arctan\left(\frac{\sin(\beta)}{\cos(\beta)+y^4/4}\right)+\mathrm{o}(1), \tag{4.78}$$

with o(1) as in (4.77), and then simply use the fact that

$$\cos\left(\frac{1}{2}\arctan(x)\right) = \frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{1+x^2}}}. \tag{4.79}$$

Combining (4.76) and (4.77) yields, in the case $\cos(\beta)\geq 0$,

$$\begin{aligned}
&\Re\big(Q_\eta^{1/2}\big(y|\eta|^{1/4}\big)\big)\\
&\quad = |\eta|^{1/2}\left[1+\cos(\beta)\frac{y^4}{2}+\frac{y^8}{16}\right]^{1/4}\sqrt{1+\frac{\cos(\beta)+y^4/4}{\sqrt{1+\cos(\beta)y^4/2+y^8/16}}}+\mathrm{o}\big(|\eta|^{1/2}\big) \qquad (4.80)\\
&\quad = |\eta|^{1/2}\frac{1}{\sqrt{2}}\sqrt{\sqrt{1+\cos(\beta)\frac{y^4}{2}+\frac{y^8}{16}}+\cos(\beta)+\frac{y^4}{4}}+\mathrm{o}\big(|\eta|^{1/2}\big),
\end{aligned}$$

as long as $\cos(\beta)\geq 0$. The cases $\cos(\beta)<0$, $\sin(\beta)>0$ and $\cos(\beta)<0$, $\sin(\beta)<0$ are treated in a similar way, but then in the expression of the argument we need to add or subtract π to $\arctan(\frac{\sin(\beta)}{\cos(\beta)+y^4/4})$, and then use a similar expression for $\sin(\frac{1}{2}\arctan(x))$. In addition for these estimates one needs to use the fact that $|\beta_\eta|\leq\beta_0\in(\pi/2,\pi)$, so that $\cos(\beta_\eta)$ is bounded away from -1. Elementary algebraic manipulation then lead to the exact same expression for $\Re(Q_\eta^{1/2}(y|\eta|^{1/4}))$ as above.

We have therefore found

$$\begin{aligned}
&\Re\left[|\eta|^{1/4}\int_{-A}^{A}\left(Q_\eta^{1/2}\big(y|\eta|^{1/4}\big)-\frac{|\eta|^{1/2}y^2}{2}+\frac{\mu}{2}\right)\mathrm{d}y\right]\\
&\qquad = |\eta|^{3/4}\int_{-A}^{A} f_0^{(\beta)}(y)\,\mathrm{d}y+\mathrm{o}\big(|\eta|^{3/4}\big),
\end{aligned} \tag{4.81}$$

uniformly when $|\beta_\eta|\leq\beta_0$. Along with (4.74), this proves (4.72) and therefore the proof of Corollary 4.6 is complete. □

Proof of Corollary 4.7. Let us now turn to the more detailed estimate of Corollary 4.7 in the case of $\rho:=-\eta>0$. By the same change of variables $z=y\eta^{1/4}$ we get

$$\frac{g_\eta(-\infty)}{g_\eta(\infty)} = \exp\left(-\rho^{1/4}\int_{-\infty}^{\infty}\left(Q_\eta^{1/2}\big(y\rho^{1/4}\big)-\frac{\rho^{1/2}y^2}{2}+\frac{\mu}{2}\right)\mathrm{d}y\right). \tag{4.82}$$

In this case elementary computations lead to

$$\rho^{1/4}\left(\sqrt{Q_\eta\big(\rho^{1/4}y\big)}-\frac{1}{2}\rho^{1/2}y^2+\frac{\mu}{2}\right) = \rho^{3/4}f_0(y)+\rho^{1/4}f_1(y)+\frac{y}{\sqrt{y^4+4}}+r_\eta(y), \tag{4.83}$$

where $|r_\eta(y)|\leq C/(\rho^{1/4}(1+y^2))$. □

5. ODE analysis: The pitchfork case

We consider here the even d case. Like in the previous section we set $\eta = 2\lambda \in \mathbb{C}$, but now we look at even solutions to the equation $g'' - V'_\mu g' + \eta g = 0$, that is, we look for the (unique) solution $g = g_\eta$ to

$$g''(x) + V'_\mu(x)g'(x) + \eta g(x) = 0, \qquad g(0) = 1, g'(0) = 0. \tag{5.1}$$

Again we set $g(x) = u(x)\exp(V_\mu(x))$, $x \in \mathbb{R}$, which leads to the unique solution to

$$u''(x) - Q_\eta(x)u(x) = 0, \qquad u(0) = 1, u'(0) = 0, \tag{5.2}$$

with, like in (4.5), $Q_\eta(x) = (V'_\mu)^2(x) - V''_\mu(x) - \eta$. Since $V_\mu(\cdot)$ is even, the function $Q_\eta(\cdot)$ is even too. Therefore when $v(\cdot)$ is *any* given solution to $u'' - Q_\eta u = 0$, also $v(-\cdot)$ is a solution. In addition when v *does not vanish at the origin*, then $u = (v(\cdot) + v(-\cdot))/2v(0)$ is also a solution to $u'' - Q_\eta u = 0$ and in fact u is the unique solution to (5.2) as it obviously satisfies $u(0) = 1, u'(0) = 0$. Since our solution to (5.1) is $g = u\exp(V_\mu)$ it is also even.

This simple argument can be applied to a solution to $u'' - Q_\eta u = 0$ which is subdominant in a given sector. More precisely Theorem 6.1 in [23] guarantees existence of subdominant (and dominant) solutions, and they are defined in the open sectors $\mathcal{G}_k = \{x \in \mathbb{C}\colon |\arg(x) - \frac{k\pi}{d}| < \frac{\pi}{2d}\}$, $k \in \{-d+1, -d+2, \dots, d-1, d\}$ (see Figure 7). Of course $\mathcal{G}_d$ contains $(-\infty, 0)$ while $\mathcal{G}_0$ contains $(0,\infty)$. More interestingly, because of the symmetry with respect to the origin we know that $u_{\mathrm{sub},\mathcal{G}_d}(\cdot) = u_{\mathrm{sub},\mathcal{G}_0}(-\cdot)$.

Letting

$$u_1 := c_1(u_{\mathrm{sub},\mathcal{G}_0} + u_{\mathrm{sub},\mathcal{G}_d}), \qquad \text{with } c_1 := \begin{cases} 1/\big(2u_{\mathrm{sub},\mathcal{G}_0}(0)\big) & \text{if } u_{\mathrm{sub},\mathcal{G}_0}(0) \neq 0, \\ 1 & \text{otherwise,} \end{cases} \tag{5.3}$$

we can use our previous reasoning to see that when $u_{\mathrm{sub},\mathcal{G}_0}(0) \neq 0$ (as will see this is always the case for η left of the spectrum), then u_1 must be the solution to (5.2).

In the second case $u_{\mathrm{sub},\mathcal{G}_0}(0) = 0$, then u_1 is the solution to $u'' - Q_\eta u = 0, u(0) = u'(0) = 0$ so that in fact $u_1 \equiv 0$.

Similarly, we let

$$u_2 := c_2(u_{\mathrm{dom},\mathcal{G}_0} + u_{\mathrm{dom},\mathcal{G}_d}), \qquad \text{with } c_2 := \begin{cases} 1/\big(2u_{\mathrm{dom},\mathcal{G}_0}(0)\big) & \text{if } u_{\mathrm{dom},\mathcal{G}_0}(0) \neq 0, \\ 1 & \text{otherwise,} \end{cases} \tag{5.4}$$

where, in strict analogy with what we have done in Section 4.1, we made the choice $u_{\mathrm{dom},\mathcal{G}_0} := u_{\mathrm{sub},\mathcal{G}_1}$, and by symmetry again $u_{\mathrm{dom},\mathcal{G}_0}(\cdot) = u_{\mathrm{dom},\mathcal{G}_d}(-\cdot)$. Again by the same reasoning, when $u_{\mathrm{dom},\mathcal{G}_0}(0) \neq 0$ (as we will see this always happens when η is strictly left of the spectrum), then u_2 is the solution to (5.2), whereas in the case $u_{\mathrm{dom},\mathcal{G}_0}(0) = 0$, we find $u_2 \equiv 0$.

Note finally that $u_{\mathrm{sub},\mathcal{G}_0}$ and $u_{\mathrm{dom},\mathcal{G}_0}$ form a basis, thus $u_1 \equiv 0$ and $u_2 \equiv 0$ cannot happen at the same time and $u_\eta = u = (u_1 + u_2)/(u_1(0) + u_2(0))$ always is the solution to (5.2).

In [7], Ch. 9, Problem 1, one can find a proof of the fact that (5.2) admits a $\mathbb{L}^2([0,\infty))$ solution if and only if $\eta \in \mathtt{Sp} = \{\eta_0, \eta_1, \dots\}$, with $\eta_j < \eta_{j+1}$.

Such solution has to be proportional to $u_{\mathrm{sub},\mathcal{G}_0}$ because it has a zero limit at infinity. In particular $u_{\mathrm{sub},\mathcal{G}_0}(0) \neq 0$. By unicity of the solution to (5.2), we conclude (as in the odd case) that when $\eta \in \mathtt{Sp}$, $u_{\mathrm{sub},\mathcal{G}_0} = u_{\mathrm{sub},\mathcal{G}_d}$ and we have in fact a $\mathbb{L}^2((-\infty,\infty))$ solution.

In [7], Ch. 9, Problem 1, there is also a precise characterization of the number and locations of the zeros of the solutions to (5.2) and, in particular, for $\eta \le \eta_0$ the solution to (5.2) does not change sign, that is, $u(\cdot) > 0$. As announced earlier, $u_{\mathrm{sub},\mathcal{G}_0}(0) \neq 0$ as long as $\eta \le \eta_0$.

Because of Theorem 7.1 in [23], also $u_{\mathrm{dom},\mathcal{G}_0}(0) \neq 0$ as long as $\eta < \eta_0$. Moreover, notice that it is exactly when $\eta \in \mathtt{Sp}$ that we have $u_2 \equiv 0$, that is $u_{\mathrm{dom},\mathcal{G}_0}$ is odd.

Regularity properties turn out to be easier than in the odd case. Indeed note that we have chosen to fix $g(0)$, and therefore $u(0)$, to 1, along with the zero slope condition. In the odd d case instead we had to put a boundary condition at $-\infty$, $g(-\infty) = 1$, which contains information both on the slope and the size of the function: dealing with these *non-standard* boundary conditions has required the approach developed in [23], notably for the regularity issues. It is clear that the boundary conditions in the even d case are more standard and results like the analytic dependence of solutions on the parameter of the equations are also standard, see, for example, [7], Ch. 3.

Nevertheless the fact that $u_\eta(x)$ is entire both in η and x can also be directly extracted from the representation formula we have just obtained in terms of subdominant and dominant solutions, of course by exploiting the analyticity properties of (sub)dominant solutions (this provides an approach alternative to [7], Ch. 3).

The considerations we just made directly imply that for the solution $u = u_\eta$ of (5.2) one can write

$$u = a u_{\mathrm{sub},\mathcal{G}_0} + b u_{\mathrm{dom},\mathcal{G}_0}, \tag{5.5}$$

where $a = a(\eta)$, $b = b(\eta)$ and $a(\cdot)$ and $b(\cdot)$ are entire functions. Moreover, as in the odd case, $b(\eta) = 0$ if and only if $\eta \in \mathtt{Sp}$.

Of course the sharp behavior for x large of $u_{\mathrm{sub},\mathcal{G}_0}$ can be taken from [23], Th. 6.1, and the one of $u_{\mathrm{dom},\mathcal{G}_0}$ can be derived by the argument in the proof of Theorem 4.3. As usual, these solutions are defined up to a multiplicative constant.

By the exact same techniques as in Section 4, and similar to Theorems 4.3, 4.4, we are further able to describe precisely (up to a multiplicative constant) the asymptotic behaviour of u_η, both in the limit $\eta \to -\infty$, and $x \to \infty$. As before we define $Q_\eta^{1/2}(x)$ (resp., $Q_\eta^{1/4}(\mathrm{x})$) the square root of $Q_\eta(x)$ (resp., of $Q_\eta^{1/2}(x)$) satisfying $\arg(Q_\eta^{1/2}(x)) \in (-\pi/2, \pi/2]$ (resp., $\arg(Q_\eta^{1/4}(x)) \in (-\pi/4, \pi/4])$.

Theorem 5.1. *Fix $\mu \in \mathbb{R}$ and $\alpha \in (0,\pi)$. For any $\eta \in \mathbb{C} \setminus (0,\infty)$ there exists a solution u to $u'' - Q_\eta u = 0$ satisfying, uniformly in $\eta \in \mathbb{C}$ such that $|\arg(\eta)| < \alpha$,*

$$\sup_{x\in\mathbb{R}} \left| u(x) Q_\eta^{1/4}(x) \exp\left(- \int_0^x Q_\eta^{1/2}(y)\,\mathrm{d}y \right) - 1 \right| = \mathrm{O}\left(|\eta|^{-d/(d+1)}\right). \tag{5.6}$$

Theorem 5.2. *For $\eta \in \mathbb{C}$ the following limit exists*:

$$\lim_{x\to\infty} u_{\mathrm{sub},\mathcal{G}_d}(x) \exp\left(V_\mu(x)\right) =: \ell(\eta). \tag{5.7}$$

Moreover $\ell(\cdot)$ is entire, $\ell(\eta) = 0$ if and only if $\eta \in \mathtt{Sp}$ and $\ell(\eta) > 0$ for $\eta < \eta_0 = \min(\mathtt{Sp})$.

We now turn to the asymptotic analysis of the Laplace transform when $\eta \nearrow \eta_0$, when $|\eta| \to \infty$ along a ray which is not the positive half line, and the more precise result in the particular case when $\eta \to -\infty$ (the latter will only be stated in the case $d = 4$).

Recall that $\lambda_0(T_{d,\mu})$ is defined in (2.10) and (2.11).

Corollary 5.3. *We have that* $2\lambda_0(T_{d,\mu}) = \eta_0$. *Moreover,* $\Phi_{T_{d,\mu}}(\cdot)$ *extends as a meromorphic function to* $\mathbb{C}$ *and it has a simple pole in* λ_0 *with residue*

$$C_{d,\mu} = -\frac{u_{\mathrm{sub},\mathcal{G}_0,\eta_0}(0)}{\int_{\mathbb{R}} u^2_{\mathrm{sub},\mathcal{G}_0,\eta_0}(x)\,\mathrm{d}x}. \tag{5.8}$$

The proof is very similar to the proof of Corollary 4.5. It goes through the differentiation step (4.62) at $\eta = \eta_0$.

In the even case, we have $u_0 := u_{\mathrm{sub},\mathcal{G}_0,\eta_0} = u_{\mathrm{sub},\mathcal{G}_d,\eta_0}$ by symmetry, where we choose the multiplicative factor so that

$$u_0(x) \overset{x\to\pm\infty}{\sim} \frac{1}{|x|^{d-1}} \exp\bigl(V_\mu(x)\bigr). \tag{5.9}$$

Equation (5.9) simplifies the computation of the Wronskian that this time can be chosen equal to one (by properly choosing the dominant solution U_0 so that $U_0 \sim_{x\to-\infty} \frac{1}{2}\exp(-V_\mu(x))$), this yields $h(x) = v(x)\exp(V_\mu(x)) \sim_{x\to\infty} \frac{-1}{2}\int_{\mathbb{R}} u_0^2(x)\,\mathrm{d}x$.

It follows that $g_{\eta_0}(\infty) \sim (\eta_0 - \eta)\int_{\mathbb{R}} u_0^2(x)\,\mathrm{d}x$, but we shall not forget that in the even case $\Phi_{T_{d,\mu}}(\lambda) = \frac{g_\eta(0)}{g_\eta(\infty)}$, and from this (and the fact that $\eta_0 - \eta = 2(\lambda_0 - \lambda)$), (5.8) follows.

When $|\eta| \to \infty$ on a ray that is not the positive half-line, we are able to deduce the following result, analogous to Corollary 4.6. Recall here that $\beta = \arg(-\eta)$.

Corollary 5.4. *Fix* $\beta_0 \in (0, \pi)$, *uniformly in* $\beta \in [-\beta_0, \beta_0]$ *we have that*

$$\lim_{\substack{|\eta|\to\infty \\ \arg(-\eta)\to\beta}} |\eta|^{-d/(2(d-1))} \log g_\eta(\infty) = -c(\beta), \tag{5.10}$$

where $c(\beta) := \int_0^\infty f_0^{(\beta)}(y)\,\mathrm{d}y$, *and*

$$f_0^{(\beta)}(y) = \frac{1}{\sqrt{2}}\sqrt{y^{2(d-1)} + \cos(\beta) + \sqrt{1 + 2\cos(\beta)y^{2(d-1)} + y^{4(d-1)}}} - y^{d-1}. \tag{5.11}$$

We now turn to the particular case $d = 4$ to write the details of the precise asymptotics when $\eta \to -\infty$ (general even d can be treated with a similar method, but the general expressions are cumbersome). Note that in this case we are looking at

$$V_\mu(x) = -\frac{x^4}{4} + \mu\frac{x^2}{2}, \qquad Q_\eta(x) = \bigl(x^3 - \mu x\bigr)^2 + 3x^2 - \mu - \eta. \tag{5.12}$$

Corollary 5.5. *Consider the solution $g = g_\eta$ to (5.1). As $\eta \to -\infty$ we have*

$$\frac{1}{g_\eta(\infty)} = 2|\eta|^{1/4} \exp\bigl(-|\eta|^{2/3} F_0 - |\eta|^{1/3} F_1 - F_2 + \mathrm{O}\bigl(|\eta|^{-1/3}\bigr)\bigr), \tag{5.13}$$

where for $i = 0, 1, 2$, $F_i = \int_0^\infty f_i(y)\,\mathrm{d}y$, and

$$\begin{aligned} f_0(y) &= \sqrt{1+y^6} - y^3, \\ f_1(y) &= \mu\Bigl(y - \frac{y^4}{\sqrt{y^6+1}}\Bigr), \\ f_2(y) &= \frac{1}{2}\Bigl(\frac{(\mu^2+3)y^2}{\sqrt{y^6+1}} - \frac{3}{y+1} - \frac{\mu^2 y^8}{(y^6+1)^{3/2}}\Bigr) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}\Bigl(\bigl(\operatorname{arcsinh}\bigl(y^3\bigr) - 3\mu^2 \log(1+y)\bigr) + \frac{y^3}{3\sqrt{1+y^6}}\Bigr). \end{aligned} \tag{5.14}$$

Note that $F_2 = \frac{1}{2}\log 2 + \frac{\mu^2}{6}$. F_1 and F_2 can be written in terms of *elliptic integrals of the first and second kind*: for $\phi \in (-\pi/2, \pi/2)$ and $m < 1$

$$\mathtt{E}_\pm(\phi|m) := \int_0^\phi \bigl(\sqrt{1 - m\sin^2(\theta)}\bigr)^{\pm 1}\,\mathrm{d}\theta. \tag{5.15}$$

With this definition and by setting $q_\pm = \sqrt{3} \pm 2$, we have

$$F_0 = \frac{3^{3/4}}{8}\mathtt{E}_-\bigl(\arccos(q_-)|q_+/4\bigr), \tag{5.16}$$

and

$$\begin{aligned} F_1 = \frac{\mu}{12}\bigl(&-3 + 3\sqrt{3} \\ &+ 3^{1/4}\bigl(6\mathtt{E}_+\bigl(\arccos(q_-)|q_+/4\bigr) + (-3+\sqrt{3})\mathtt{E}_-\bigl(\arccos(q_-)|q_+/4\bigr)\bigr)\bigr). \end{aligned} \tag{5.17}$$

From this, we derive the constants that appear in the statement of Theorem 2.5:

$$C_{2/3} := 2^{2/3} F_0 = 1.6693\ldots \quad \text{and} \quad C_{1/3} = \frac{2^{1/3} F_1}{\mu} = 0.5432\ldots. \tag{5.18}$$

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References

[1] Arecchi, F.T., Politi, A. and Ulivi, L. (1982). Stochastic time description of transitions in unstable and multistable systems. *Il Nuovo Cimento* **71B** 119–154.

[2] Bakhtin, Y. (2008). Exit asymptotics for small diffusion about an unstable equilibrium. *Stochastic Process*. *Appl*. **118** 839–851. MR2411523

[3] Bender, C.M. and Orszag, S.A. (1978). *Advanced Mathematical Methods for Scientists and Engineers*. *International Series in Pure and Applied Mathematics*. New York: McGraw-Hill. MR0538168

[4] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871

[5] Cáceres, M.O., Fuentes, M.A. and Budde, C.E. (1997). Stochastic escape processes from a non-symmetric potential normal form. II. The marginal case. *J*. *Phys*. *A* **30** 2287–2296. MR1457379

[6] Carinci, G. (2013). Random hysteresis loops. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **49** 307–339. MR3088372

[7] Coddington, E.A. and Levinson, N. (1955). *Theory of Ordinary Differential Equations*. New York: McGraw-Hill. MR0069338

[8] Colet, P., De Pasquale, F., Caceres, M.O. and San Miguel, M. (1990). Theory for relaxation at a subcritical pitchfork bifurcation. *Phys*. *Rev*. *A* (3) **41** 1901–1911.

[9] Conway, J.B. (1978). *Functions of One Complex Variable*, 2nd ed. *Graduate Texts in Mathematics* **11**. New York: Springer. MR0503901

[10] Costin, O. (2009). *Asymptotics and Borel Summability*. *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics* **141**. Boca Raton, FL: CRC Press. MR2474083

[11] Flajolet, P., Gourdon, X. and Dumas, P. (1995). Mellin transforms and asymptotics: Harmonic sums. *Theoret*. *Comput*. *Sci*. **144** 3–58. Special volume on mathematical analysis of algorithms. MR1337752

[12] Freidlin, M.I. and Wentzell, A.D. (1998). *Random Perturbations of Dynamical Systems*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **260**. New York: Springer. Translated from the 1979 Russian original by Joseph Szücs. MR1652127

[13] Giacomin, G. and Merle, M. Scaling limits of transit and exit times from non hyperbolic fixed points. Preprint.

[14] Groisman, P. and Rossi, J.D. (2007). Explosion time in stochastic differential equations with small diffusion. *Electron*. *J*. *Differential Equations* **2007** No. 140, 9. MR2349968

[15] Horsthemke, W. and Lefever, R. (1984). *Noise-Induced Transitions*: *Theory and Applications in Physics*, *Chemistry*, *and Biology*. *Springer Series in Synergetics* **15**. Berlin: Springer. MR0724433

[16] Karatzas, I. and Shreve, S.E. (1988). *Brownian Motion and Stochastic Calculus*. *Graduate Texts in Mathematics* **113**. New York: Springer. MR0917065

[17] Le Gall, J.-F. (2013). *Mouvement Brownien*, *Martingales et Calcul Stochastique*. *Mathématiques & Applications* (*Berlin*) [*Mathematics & Applications*] **71**. Heidelberg: Springer. MR3184878

[18] Lindner, B., García-Ojalvo, J. and Schimansky-Geier, L. (2004). Effects of noise in excitable systems. *Phys*. *Rep*. **392** 321–424.

[19] Lindner, B., Longtin, A. and Bulsara, A. (2003). Analytic expressions for rate and CV of a type I neuron driven by white Gaussian noise. *Neural Comput*. **15** 1760–1787.

[20] McGill, P. (1981). A direct proof of the Ray–Knight theorem. In *Seminar on Probability*, *XV* (*Univ*. *Strasbourg*, *Strasbourg*, 1979/1980) (*French*). *Lecture Notes in Math*. **850** 206–209. Berlin: Springer. MR0622564

[21] Nakagawa, K. (2007). Application of Tauberian theorem to the exponential decay of the tail probability of a random variable. *IEEE Trans*. *Inform*. *Theory* **53** 3239–3249. MR2417689

[22] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Berlin: Springer. MR1725357

[23] Sibuya, Y. (1975). *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient*. *North-Holland Mathematics Studies* **18**. New York: North-Holland. MR0486867

[24] Sigeti, D. and Horsthemke, W. (1989). Pseudo-regular oscillations induced by external noise. *J. Stat. Phys*. **54** 1217–1222. MR0993147

[25] Titchmarsh, E.C. (1962). *Eigenfunction Expansions Associated with Second-Order Differential Equations*. *Part i*, 2nd ed. Oxford: Clarendon Press. MR0176151

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