

Testing monotonicity via local least concave majorants

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We propose a new testing procedure for detecting localized departures from monotonicity of a signal embedded in white noise. In fact, we perform simultaneously several tests that aim at detecting departures from concavity for the integrated signal over various intervals of different sizes and localizations. Each of these local tests relies on estimating the distance between the restriction of the integrated signal to some interval and its least concave majorant. Our test can be easily implemented and is proved to achieve the optimal uniform separation rate simultaneously for a wide range of Hölderian alternatives. Moreover, we show how this test can be extended to a Gaussian regression framework with unknown variance. A simulation study confirms the good performance of our procedure in practice.

Keywords: adaptivity; least concave majorant; monotonicity; multiple test; non-parametric; uniform separation rate

1. Introduction

Suppose that we observe on the interval $[0, 1]$ a stochastic process F_n that is governed by the white noise model

$$F_n(t) = \int_0^t f(x) dx + \frac{\sigma}{\sqrt{n}} W(t), \quad (1)$$

where $n \geq 1$ is a given integer, $\sigma > 0$ is known, $f : [0, 1] \rightarrow \mathbb{R}$ is an unknown function on $[0, 1]$ assumed to be integrable, and W is a standard Brownian motion on $[0, 1]$ starting at 0. We aim to test the null hypothesis that f is non-increasing on $[0, 1]$ against the general alternative that it is not.

Several non-parametric procedures have already been proposed, either in model (1) or, most often, in a regression model. To avoid going back and forth between the two directions of monotonicity, we will only talk about non-increasing monotonicity since the other direction can be treated similarly via a straightforward transformation. Without being exhaustive, we now review some tests that are well suited in the regression framework for detecting global departures from monotonicity. Bowman, Jones and Gijbels [7] propose a procedure based on the smallest bandwidth for which a kernel-type estimator of the regression function is monotone. Their

test rejects monotonicity if this critical bandwidth is too large. Durot [11] exploits the equivalence between monotonicity of a continuous regression curve g defined on $[0, 1]$ and concavity of $G : t \mapsto \int_0^t g(x) dx, t \in [0, 1]$. In the uniform design setting, Durot's test rejects monotonicity when the supremum distance between an empirical estimator of G and its least concave majorant is too large. The test has the correct asymptotic level, and has the advantage of being easy to implement. Domínguez-Menchero, González-Rodríguez and López-Palomo [9] propose a test statistic based on the \mathbb{L}_2 distance of a regression estimator to the set of all monotone curves. Tailored for fixed design regression models, their method allows for repeated measurements at a given design point, and presents also the merit of being easy to implement. Still in the fixed design setting, Baraud, Huet and Laurent [4] construct a multiple test that, roughly speaking, rejects monotonicity if there is at least one partition of $[0, 1]$ into intervals, among a given collection, such that the estimated projection of the regression function on the set of piecewise constant functions on this partition is too far from the set of monotone functions. They show that their test has the correct level for any given sample size and study its uniform separation rate, with respect to an \mathbb{L}_2 criterion, over Hölderian balls of functions. We may also mention some procedures for testing monotonicity in other frameworks. For instance, Durot [12] or Groeneboom and Jongbloed [16,17] test the monotonicity of a hazard rate, whereas Delgado and Escanciano [8] test the monotonicity of a conditional distribution. As Durot [11], they consider a statistic based on some distance between an empirical estimator of the function of interest and its least concave majorant.

Other procedures were considered to detect local departures from monotonicity. Hall and Heckman [18] test negativity of the derivative of the regression function via a statistic based on the slopes of the fitted least-squares regression lines over small blocks of observations. Gijbels, Hall, Jones and Koch [15] propose two statistics based on signs of differences of the response variable. In the case of a uniform fixed design and i.i.d. errors with a bounded density, the authors study the asymptotic power of their procedure against local alternatives under which the regression function departs linearly from the null hypothesis at a certain rate. Ghosal, Sen and van der Vaart [14] test negativity of the first derivative of the regression function via a locally weighted version of Kendall's tau. The asymptotic level of their test is guaranteed and their test is powerful provided that the derivative of the regression function locally departs from the null at a rate that is fast enough. Dümbgen and Spokoiny [10] and Baraud, Huet and Laurent [5] both propose two multiple testing procedures, either in the Gaussian white noise model for [10] or in a regression model for [5]. The former authors consider two procedures based on the supremum, over all bandwidths, of kernel-based estimators for some distance from f to the null hypothesis. The distance they estimate is either the supremum distance from f to the set of non-increasing functions or the supremum of f' . The latter authors propose a procedure based on the difference of local means and another one based on local slopes, a method that is akin to that of Hall and Heckman [18]. In both papers, the uniform separation rate of each test is studied over a range of Hölderian balls of functions. Last, let us mention the (unfortunately non-conservative) alternative approach developed by Hall and Van Keilegom [19] for testing local monotonicity of a hazard rate.

In this paper, we propose a multiple testing procedure that may be seen as a localized version of the test considered by [11]. Based on the observation of F_n in (1), it rejects monotonicity if there is at least one subinterval of $[0, 1]$, among a given collection, such that the local least

concave majorant of F_n on this interval is too far from F_n . Our test has the correct level for any given n , independently of the degree of smoothness of f . Its implementation is easy and does not require bootstrap nor any a priori choice of smoothing parameter. Moreover, we show that it is powerful simultaneously against most of the alternatives considered in [14]. We also study the uniform separation rate of our test, with respect to two different criteria, over a range of Hölderian balls of functions. We recover the uniform separation rates obtained in [5,10], as well as new ones, and check that our test achieves the optimal uniform separation rate simultaneously over the considered range of Hölderian balls. Besides, we are concerned with the more realistic Gaussian regression framework with fixed design and unknown variance. We describe how our test can be adapted to such a framework, and prove that it still enjoys similar properties. Finally, we briefly discuss how our method could be extended to more general models.

The organization of the paper is as follows. In Section 2, we describe our testing procedure and the critical region in the white noise model for a prescribed level. Section 3 contains theoretical results about the power of the test. In Section 4, we turn to the regression framework. Section 5 is devoted to the practical implementation of our test – both in the white noise and regression frameworks – and to its comparison with other procedures via a simulation study. In Section 6, we discuss possible extensions of our test to more general models. All proofs are postponed to Section 7 or to the supplementary material [1].

2. Testing procedure

Let us fix a closed sub-interval I of $[0, 1]$ and denote by \mathcal{D}^I the set of all functions from $[0, 1]$ to \mathbb{R} which are non-increasing on I . We start with describing a procedure for testing the null hypothesis

$$H_0^I : f \in \mathcal{D}^I$$

against $H_1^I : f \notin \mathcal{D}^I$ within the framework (1). Since the cumulative function

$$F(t) = \int_0^t f(x) dx, \quad t \in [0, 1], \quad (2)$$

is concave on I under H_0^I and F_n estimates F , we reject H_0^I when F_n is “too far from being concave on I ”. Thus, we consider a procedure based on a local least concave majorant. For every continuous function $G : [0, 1] \rightarrow \mathbb{R}$, we denote by \widehat{G}^I the least concave majorant of the restriction of G to I , and simply denote $\widehat{G}^{[0,1]}$ by \widehat{G} . Our test statistic is then

$$S_n^I = \sqrt{\frac{n}{\sigma^2|I|}} \sup_{t \in I} (\widehat{F}_n^I(t) - F_n(t)),$$

where $|I|$ denotes the length of I , and we reject H_0^I when S_n^I is too large. From the following lemma, S_n^I is the supremum distance between \widehat{F}_n^I and F_n over I , normalized in such a way that its distribution does not depend on I under the least favorable hypothesis that f is constant on I .

Lemma 2.1. *Let I be a closed sub-interval of $[0, 1]$. If $f \equiv c$ over I for some $c \in \mathbb{R}$, then $S_n^I = \sup_{t \in I} (\widehat{W}^I(t) - W(t)) / \sqrt{|I|}$ and is distributed as*

$$Z := \sup_{t \in [0,1]} (\widehat{W}(t) - W(t)). \tag{3}$$

For a fixed $\alpha \in (0, 1)$, we reject the null hypothesis H_0^I at level α if

$$S_n^I > q(\alpha), \tag{4}$$

where $q(\alpha)$ is calibrated under the hypothesis that f is constant on I , that is, $q(\alpha)$ is the $(1 - \alpha)$ -quantile of Z . As stated in the following theorem, this test is of non-asymptotic level α and the hypothesis that the function f is constant on I is least favorable, under no prior assumption on f .

Theorem 2.1. *For every $\alpha \in (0, 1)$,*

$$\sup_{f \in \mathcal{D}^I} \mathbb{P}_f [S_n^I > q(\alpha)] = \alpha.$$

Moreover, the supremum is achieved at $f \equiv c$ over I , for any fixed $c \in \mathbb{R}$.

We now turn to our main goal of testing the null hypothesis

$$H_0 : f \in \mathcal{D} \tag{5}$$

against the general alternative $H_1 : f \notin \mathcal{D}$, where \mathcal{D} denotes the set of non-increasing functions from $[0, 1]$ to \mathbb{R} . In the case where the monotonicity assumption is rejected, we would also like to detect the places where the monotonicity constraint is not satisfied. Therefore, we consider a finite collection \mathcal{C}_n of sub-intervals of $[0, 1]$, that may depend on n , and propose to combine all the local tests of H_0^I against H_1^I for $I \in \mathcal{C}_n$. In the spirit of the heuristic union-intersection principle of Roy [22] (see Chapter 2), we accept H_0 when we accept H_0^I for all $I \in \mathcal{C}_n$, which leads to $\max_{I \in \mathcal{C}_n} S_n^I$ as a natural test statistic. More precisely, given $\alpha \in (0, 1)$, we reject the null hypothesis H_0 at level α if

$$\max_{I \in \mathcal{C}_n} S_n^I > s_{\alpha,n}, \tag{6}$$

where $s_{\alpha,n}$ is calibrated under the hypothesis that f is a constant function, that is, $s_{\alpha,n}$ is the $(1 - \alpha)$ -quantile of

$$\max_{I \in \mathcal{C}_n} \sqrt{\frac{1}{|I|}} \sup_{t \in I} (\widehat{W}^I(t) - W(t)).$$

If H_0 is rejected, then we are able to identify one or several intervals $I \in \mathcal{C}_n$ where the monotonicity assumption is violated: these are intervals where $S_n^I > s_{\alpha,n}$. Moreover, Theorem 2.2 below shows that this multiple testing procedure has a non-asymptotic level α .

Theorem 2.2. *For every $\alpha \in (0, 1)$,*

$$\sup_{f \in \mathcal{D}} \mathbb{P}_f \left[\max_{I \in \mathcal{C}_n} S_n^I > s_{\alpha,n} \right] = \alpha.$$

Moreover, the above supremum is achieved at $f \equiv c$ on $[0, 1]$ for any fixed $c \in \mathbb{R}$.

Let us recall that the distribution of S_n^I does not depend on I under the least favorable hypothesis. This allows us to perform all the local tests with the same critical threshold $s_{\alpha,n}$, which is of practical interest. On the contrary, the multiple test in [5] involves several critical values, which induces a complication in its practical implementation (see Section 5.1 in [5]).

The least concave majorants involved in our procedure can be computed by first discretizing the intervals $I \in \mathcal{C}_n$ and then using, for example, the Pool Adjacent Violators Algorithm on a finite number of points, see [6], Chapter 1, page 13. Thus, $s_{\alpha,n}$ can be computed using Monte Carlo simulations and the test is easily implementable. However, in the white noise model this requires simulating a large number of (discretized) Brownian motion paths and the computation of a local least concave majorant on each $I \in \mathcal{C}_n$, which can be computationally expensive. An alternative to Monte Carlo simulations is to replace the critical threshold $s_{\alpha,n}$ with an upper-bound that is easier to compute. Two different proposals for an upper-bound are given in the following lemma.

Lemma 2.2. *For every $\gamma \in (0, 1)$, let $q(\gamma)$ denote the $(1 - \gamma)$ -quantile of the variable Z defined in (3). Then, for every $\alpha \in (0, 1)$, we have*

$$s_{\alpha,n} \leq q\left(\frac{\alpha}{|\mathcal{C}_n|}\right) \leq 2\sqrt{2 \log\left(\frac{2|\mathcal{C}_n|}{\alpha}\right)}.$$

Consider the test that rejects H_0 if

$$\max_{I \in \mathcal{C}_n} S_n^I > t_{\alpha,n}, \tag{7}$$

where $t_{\alpha,n}$ denotes either $q(\alpha/|\mathcal{C}_n|)$ or $2\sqrt{2 \log(2|\mathcal{C}_n|/\alpha)}$. Combining Theorem 2.2 with Lemma 2.2 proves that this test is at most of non-asymptotic level α . In practice, the first proposal of definition of $t_{\alpha,n}$ can be computed by using either recent results of Balabdaoui and Pitman [3] about a characterization of the distribution function of Z or fast Monte Carlo simulations. Moreover, it is easy to see that all the theoretical results we obtain in Section 3 about the performance of the test with critical region (6) continue to hold for the test with critical region (7) with both proposals for $t_{\alpha,n}$. In practice however, the test with critical region (7) may have lower power than the test (6), see Section 5.

3. Performance of the test

Keeping in mind the connection between the white noise model and the regression model of Section 4, we study the theoretical performance of our test in model (1) for

$$C_n = \left\{ \left[\frac{i}{n}, \frac{j}{n} \right], i < j \text{ in } \{0, \dots, n\} \right\}, \tag{8}$$

which can be viewed as the collection of all possible subintervals of $[0, 1]$ in model (20). Of course, if we knew in advance the interval I over which f is likely to violate the non-increasing assumption, then the power would be largest for the choice $C_n = \{I\}$, which would be the right collection for testing H_0^I instead of H_0 . However, such a situation is far from being realistic since I is in general unknown. With the choice (8), we expect C_n to contain an interval close enough to I . Therefore, by performing local tests simultaneously on all intervals of C_n , the resulting multiple testing procedure is expected to detect a wide class of alternatives. Other possible choices of C_n will be discussed in Section 6.

In the sequel, we take $n \geq 2$, and α and β are some fixed numbers in $(0, 1)$. We will give a sufficient condition for our test to achieve a prescribed power: we provide a condition on f which ensures that

$$\mathbb{P}_f \left[\max_{I \in C_n} S_n^I > s_{\alpha,n} \right] \geq 1 - \beta. \tag{9}$$

Then, based on this condition, we study the uniform separation rate of our test over Hölderian classes of functions.

3.1. Power of the test

For every $x < y$ in $[0, 1]$, let

$$\bar{f}_{xy} = \frac{1}{y-x} \int_x^y f(u) du. \tag{10}$$

In the following theorem, we prove that our test achieves a prescribed power provided there are $x < t < y$ such that \bar{f}_{xy} is too large as compared to \bar{f}_{xt} .

Theorem 3.1. *There exists $C(\alpha, \beta) > 0$ only depending on α and β such that (9) holds provided there exist $x, y \in [0, 1]$ such that $y - x \geq 2/n$ and*

$$\sup_{t \in [x,y]} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C(\alpha, \beta) \sqrt{\frac{\sigma^2 \log n}{n}}. \tag{11}$$

To illustrate this theorem, let us consider a sequence of alternatives where f , which may depend on n , is assumed to be continuously differentiable. Moreover, we assume that there exist an interval $[t_n - \Delta_n, t_n + \Delta_n] \subset [0, 1]$ and positive numbers M, λ_n and $\delta_n \leq \Delta_n$ such that

$$\begin{cases} f'(t) \geq 0, & \text{on } [t_n - \Delta_n, t_n + \Delta_n], \\ f'(t) \geq M\lambda_n, & \text{on } [t_n - \delta_n, t_n + \delta_n]. \end{cases} \tag{12}$$

Here, t_n, λ_n, δ_n and Δ_n may depend on n while M does not. Under these assumptions, we obtain the following corollary.

Corollary 3.1. *There exists a positive number $C(\alpha, \beta)$ only depending on α and β such that (9) holds provided $\Delta_n \geq 1/n$ and*

$$\frac{M\delta_n^2\lambda_n}{\sqrt{\Delta_n}} \geq C(\alpha, \beta)\sqrt{\frac{\sigma^2 \log n}{n}}. \tag{13}$$

Corollary 3.1 allows us to compare our test to the one constructed in [14] for monotonicity of a mean function f in a regression model with random design and n observations. Indeed, Theorem 5.2 in [14], when translated to the case of non-increasing monotonicity as in (5) provides sufficient conditions on $(\lambda_n, \delta_n, \Delta_n)$ for the test in [14] to be powerful against an alternative of the form (12) with a large enough M . One can check that if $(\lambda_n, \delta_n, \Delta_n)$ satisfies one of the sufficient conditions given in Theorem 5.2 in [14], then it also satisfies

$$\frac{\delta_n^2\lambda_n}{\sqrt{\Delta_n}} \geq C\sqrt{\frac{\sigma^2 \log(1/\Delta_n)}{n}} \tag{14}$$

for some constant $C > 0$ which does not depend on n . Hence, it also satisfies (13) provided that

$$C\sqrt{\log(1/\Delta_n)} \geq \frac{C(\alpha, \beta)}{M}\sqrt{\log n}. \tag{15}$$

It follows that Corollary 3.1 shows that our test is powerful simultaneously against all the alternatives considered in Theorem 5.2 in [14] for which $\Delta_n \geq 1/n$ and (15) holds. Noticing that (15) holds, for instance, when $\Delta_n \leq n^{-\epsilon}$ for some $\epsilon > 0$ and M large enough, we conclude that our test is powerful *simultaneously* for most of the alternatives against which the test in [14] is shown to be powerful. However, as opposed to our test the test in [14] is *not* simultaneously powerful against those alternatives: a given alternative is detected only if a parameter h_n is chosen in manner that depends on that particular alternative.

As a second illustration of Theorem 3.1, consider the alternative that f is U-shaped, that is f is convex and there exists $x_0 \in (0, 1)$ such that f is non-increasing on $[0, x_0]$ and increasing on $[x_0, 1]$. Thus, f violates the null hypothesis on $[x_0, 1]$ and deviation from non-increasing monotonicity is related to $f(1) - f(x_0) > 0$.

Corollary 3.2. *Assume f is U-shaped, denote by x_0 the greatest location of the minimum of f , and let $\rho = f(1) - f(x_0)$ and*

$$R = \lim_{x \uparrow 1} \frac{f(x) - f(1)}{x - 1} < \infty.$$

Then, there exist a constant $C_0 > 0$ and a real number $C(\alpha, \beta) > 0$ only depending on α and β such that (9) holds provided $\rho > C_0 R/n$ and

$$\rho > C(\alpha, \beta) \left(\frac{\sigma^2 R \log n}{n} \right)^{1/3}.$$

3.2. Uniform separation rates

In this section, we focus on Hölderian alternatives: we study the performance of our test assuming that for some $R > 0$ and $s \in (0, 2]$ both unknown, f belongs to the Hölderian class $\mathcal{F}(s, R)$ defined as follows. For all $R > 0$ and $s \in (0, 1]$, $\mathcal{F}(s, R)$ is the set of functions $f : [0, 1] \rightarrow \mathbb{R}$ satisfying

$$|f(u) - f(v)| \leq R|u - v|^s \quad \text{for all } u, v \in [0, 1],$$

while for all $R > 0$ and $s \in (1, 2]$, $\mathcal{F}(s, R)$ is defined as the set of differentiable functions $f : [0, 1] \rightarrow \mathbb{R}$ with derivative f' satisfying $f' \in \mathcal{F}(s - 1, R)$.

To evaluate the performance of our test, we consider a criterion that measures the discrepancy of f from \mathcal{D} . In the case where we only assume that $f \in \mathcal{F}(s, R)$ for some $R > 0$ and $s \in (0, 2]$, we consider the criterion

$$\Delta_1(f) = \inf_{g \in \mathcal{D}} \sup_{t \in [0, 1]} |f(t) - g(t)|,$$

which is the supremum distance from f to \mathcal{D} . Then, we restrict our attention to the case where $f \in \mathcal{F}(s, R)$ for some $R > 0$ and $s \in (1, 2]$ and consider the criterion

$$\Delta_2(f) = \sup_{t \in [0, 1]} f'(t).$$

We recall that, for a given class of functions \mathcal{F} and a given criterion Δ , the uniform separation rate of an α -level test Φ over \mathcal{F} with respect to Δ is defined by

$$\rho(\Phi, \mathcal{F}, \Delta) = \inf\{\rho > 0, \mathbb{P}_f(\Phi \text{ rejects } H_0) \geq 1 - \beta \text{ for all } f \in \mathcal{F} \text{ s.t. } \Delta(f) \geq \rho\}$$

and allows us to compare the performance of α -level tests: the smaller the better. The following theorem provides an upper-bound for the uniform separation rate of our test with respect to the criteria introduced above.

Theorem 3.2. *Let \mathcal{C}_n be the collection (8), T_n be the test with critical region (6), and let $R > 0$ and $s \in (0, 2]$. Assume $n^s \sqrt{\log n} \geq R/\sigma$ and, in the case where $s \in (1, 2]$, assume moreover that $R \geq 2^{1+2s} \sigma \sqrt{(\log n)/n}$. Then, there exists a positive real $C(s, \alpha, \beta)$ only depending on s, α and β such that*

$$\rho(T_n, \mathcal{F}(s, R), \Delta_1) \leq C(s, \alpha, \beta) R^{1/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{s/(1+2s)} \quad (16)$$

and, in case $s \in (1, 2]$,

$$\rho(T_n, \mathcal{F}(s, R), \Delta_2) \leq C(s, \alpha, \beta) R^{3/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{(s-1)/(1+2s)}. \quad (17)$$

It should be noticed that the conditions

$$n^s \sqrt{\log n} \geq R/\sigma \quad \text{and} \quad R \geq 2^{1+2s} \sigma \sqrt{(\log n)/n}$$

simply mean that n is sufficiently large when compared to R/σ .

Let us now discuss the optimality of the upper-bounds given in Theorem 3.2. Indeed, Proposition 3.1 below proves that, for each criterion Δ_1 and Δ_2 and any choice of the smoothness parameter s considered in Theorem 3.2, no test can achieve a better rate over $\mathcal{F}(s, R)$, up to a multiplicative constant that does not depend on n .

Proposition 3.1. *Assume $f \in \mathcal{F}(s, R)$ for some $R > 0$ and $s \in (0, 2]$. Let \mathcal{T}_α be the set of all α -level tests for testing that $f \in \mathcal{D}$. Assume moreover that $R \geq \sigma \sqrt{(\log n)/n^{1-\varepsilon}}$ for some $\varepsilon \in (0, 1)$ and n is large enough. Then, there exists $\kappa(s, \alpha, \beta, \varepsilon) > 0$ only depending on s, α, β and ε such that*

$$\inf_{\Phi_n \in \mathcal{T}_\alpha} \rho(\Phi_n, \mathcal{F}(s, R), \Delta_1) \geq \kappa(s, \alpha, \beta, \varepsilon) R^{1/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{s/(1+2s)} \tag{18}$$

and, in case $s \in (1, 2]$,

$$\inf_{\Phi_n \in \mathcal{T}_\alpha} \rho(\Phi_n, \mathcal{F}(s, R), \Delta_2) \geq \kappa(s, \alpha, \beta, \varepsilon) R^{3/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{(s-1)/(1+2s)}. \tag{19}$$

The lower-bound (18) is proved in [5] (Proposition 2), and the other one in [1]. According to (16) and (18), our test thus achieves the optimal uniform separation rate (up to a multiplicative constant) with respect to Δ_1 , simultaneously over all classes $\mathcal{F}(s, R)$ for $s \in (0, 2]$ and a wide range of values of R . A testing procedure enjoying similar performance in a Gaussian regression model, at least for $s \in (0, 1]$, can be found in [5], Section 2. In the white noise model we consider here, [10] (Section 3.1) propose a procedure that achieves the precise optimal uniform separation rate (we mean, with the optimal constant) with respect to Δ_1 , simultaneously over all classes $\mathcal{F}(1, R)$ with $R > 0$. But, to our knowledge, our test is the first one to achieve the rate (16) in the case $s \in (1, 2]$. On the other hand, according to (17) and (19), our test also achieves the optimal uniform separation rate (up to a multiplicative constant) with respect to Δ_2 , simultaneously over all classes $\mathcal{F}(s, R)$ for $s \in (1, 2]$ and a wide range of values of R . The second procedure proposed by [5] (Section 3) achieves this rate in a Gaussian regression model, and the second procedure proposed by [10] (Section 3.2) in the white noise model is proved to achieve the optimal rate for $f \in \mathcal{F}(2, R)$.

4. Testing procedure in a regression framework

In this section, we explain how our testing procedure can be extended, with similar performance, to the more realistic regression model

$$Y_i = f(i/n) + \sigma \epsilon_i, \quad i = 1, \dots, n, \tag{20}$$

where $f : [0, 1] \rightarrow \mathbb{R}$ and $\sigma > 0$ are unknown, and $(\epsilon_i)_{1 \leq i \leq n}$ are independent standard Gaussian variables. Based on the observations $(Y_i)_{1 \leq i \leq n}$, we would like to test $H_0 : f \in \mathcal{D}$ against $H_1 : f \notin \mathcal{D}$ where as above, \mathcal{D} denotes the set of all non-increasing functions from $[0, 1]$ to \mathbb{R} .

If σ^2 were known, then going from the white noise model (1) to the regression model (20) would amount to replace $F(t) = \int_0^t f(x) dx$ and the rescaled Brownian motion $n^{-1/2}W(t)$, $0 \leq t \leq 1$, by the approximations $(1/n) \sum_{1 \leq i \leq j} f(i/n)$ and $(1/n) \sum_{1 \leq i \leq j} \epsilon_i$, $1 \leq j \leq n$, respectively, so that a counterpart for F_n in the regression model is the continuous piecewise linear function F_n^{reg} on $[0, 1]$ interpolating between the points

$$\left(\frac{j}{n}, \frac{1}{n} \sum_{0 \leq i \leq j} Y_i \right), \quad j = 0, \dots, n,$$

where $Y_0 = 0$. Note that F_n^{reg} is nothing but the cumulative sum diagram of the data Y_i , $1 \leq i \leq n$, with equal weights $w_i = 1/n$ (see also Barlow *et al.* [6]). Precisely, if σ^2 were known, then we would consider a finite collection \mathcal{C}_n of sub-intervals of $[0, 1]$ and we would reject H_0 if

$$\max_{I \in \mathcal{C}_n} S_n^{\text{reg}, I} > r_{\alpha, n},$$

where $r_{\alpha, n}$ is calibrated under the hypothesis that $f \equiv 0$ and where for all $I \in \mathcal{C}_n$,

$$S_n^{\text{reg}, I} = \sqrt{\frac{n}{\sigma^2 |I|}} \sup_{t \in I} (\widehat{F}_n^{\text{reg}, I}(t) - F_n^{\text{reg}}(t)). \tag{21}$$

Since σ^2 is unknown, we need to estimate it. For ease of notation, we assume that n is even and we consider the estimator

$$\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n/2} (Y_{2i} - Y_{2i-1})^2.$$

We introduce $\bar{n} = n/2$, $\sigma_0 = \sigma/\sqrt{2}$,

$$\bar{Y}_i = (Y_{2i-1} + Y_{2i})/2 \quad \text{and} \quad \bar{\epsilon}_i = (\epsilon_{2i-1} + \epsilon_{2i})/\sqrt{2}$$

for all $i = 1, \dots, \bar{n}$, and we define on $[0, 1]$ the function

$$\bar{f}_n(t) = (f(t - 1/n) + f(t))/2,$$

where $f(t)$ is defined in an arbitrary way for all $t \leq 0$. Thus, from the original model (20), we deduce the regression model

$$\bar{Y}_i = \bar{f}_n(i/\bar{n}) + \sigma_0 \bar{\epsilon}_i, \quad i = 1, \dots, \bar{n}, \tag{22}$$

with the advantage that the observations $(\bar{Y}_i)_{1 \leq i \leq \bar{n}}$ are independent of $\widehat{\sigma}^2$ (see the proof of Theorem 4.1). Note that $(\bar{\epsilon}_i)_{1 \leq i \leq \bar{n}}$ are independent standard Gaussian variables, and a natural estimator for σ_0^2 is $\widehat{\sigma}_0^2 = \widehat{\sigma}^2/2$. Thus, the regression model (22) has similar features as model (20), so we proceed as described above, just replacing the unknown variance by its estimator. Precisely,

we choose some finite collection $\mathcal{C}_{\bar{n}}$ of closed sub-intervals of $[0, 1]$ with endpoints on the grid $\{i/\bar{n}; i = 0, \dots, \bar{n}\}$ and for all $I \in \mathcal{C}_{\bar{n}}$, we define

$$\widehat{S}_{\bar{n}}^{\text{reg}, I} = \sqrt{\frac{\bar{n}}{\widehat{\sigma}_0^2 |I|}} \sup_{t \in I} (\widehat{F}_{\bar{n}}^{\text{reg}, I}(t) - F_{\bar{n}}^{\text{reg}}(t)),$$

where $F_{\bar{n}}^{\text{reg}}$ is the cumulative sum diagram of the data $\bar{Y}_i, 1 \leq i \leq \bar{n}$, with equal weights $w_i = 1/\bar{n}$. For a given $\alpha \in (0, 1)$, we reject $H_0: f \in \mathcal{D}$ at level α when

$$\max_{I \in \mathcal{C}_{\bar{n}}} \widehat{S}_{\bar{n}}^{\text{reg}, I} > r_{\alpha, n}, \tag{23}$$

where $r_{\alpha, n}$ is calibrated under the hypothesis that $f \equiv 0$. In order to describe more precisely $r_{\alpha, n}$, let us define

$$Z_n^{\text{reg}} = \max_{I \in \mathcal{C}_{\bar{n}}} \sqrt{\frac{\bar{n}}{|I|}} \sup_{t \in I} (\widehat{G}_{\bar{n}}^I(t) - G_{\bar{n}}(t)), \tag{24}$$

where $G_{\bar{n}}$ is the cumulative sum diagram of the variables $\bar{\epsilon}_i, 1 \leq i \leq \bar{n}$, with equal weights $w_i = 1/\bar{n}$. Although $G_{\bar{n}}$ is not observed, its distribution is entirely known, and so is the distribution of Z_n^{reg} . We define $r_{\alpha, n}$ as the $(1 - \alpha)$ -quantile of $Z_n^{\text{reg}}/\sqrt{\chi^2(\bar{n})/\bar{n}}$, where $\chi^2(\bar{n})$ is a random variable independent of Z_n^{reg} and having chi-square distribution with \bar{n} degrees of freedom. Approximated values for the quantiles $r_{\alpha, n}$ can be obtained via Monte Carlo simulations, and the test with critical region (23) is of non-asymptotic level α , as stated in the following theorem.

Theorem 4.1. *For every $\alpha \in (0, 1)$,*

$$\sup_{f \in \mathcal{D}} \mathbb{P}_f \left[\max_{I \in \mathcal{C}_{\bar{n}}} \widehat{S}_{\bar{n}}^{\text{reg}, I} > r_{\alpha, n} \right] = \alpha.$$

Moreover, the above supremum is achieved at $f \equiv c$, for any fixed $c \in \mathbb{R}$.

As in Section 3, we study the performance of the test in the case where

$$\mathcal{C}_{\bar{n}} = \left\{ \left[\frac{i}{\bar{n}}, \frac{j}{\bar{n}} \right], i < j \text{ in } \{0, \dots, \bar{n}\} \right\}. \tag{25}$$

We obtain uniform separation rates that are comparable with the optimal rates we have obtained in the white noise model, see Theorem 3.2 and Proposition 3.1. For all $s \in (1, 2]$, $R > 0$ and $L > 0$, denote

$$\mathcal{F}(s, R, L) = \mathcal{F}(s, R) \cap \{f : [0, 1] \rightarrow \mathbb{R} \text{ s.t. } \|f'\|_{\infty} \leq L\}.$$

Theorem 4.2. *Let α, β in $(0, 1)$, $\mathcal{C}_{\bar{n}}$ be the collection (25) and T_n^{reg} be the test with critical region (23). Let $L > 0, s \in (0, 2]$ and $R > 0$ and assume $n \geq 18 \log(2/\alpha)$. In case $s \in (0, 1]$, we assume that $R/\sigma \leq n^s$ whereas in case $s \in (1, 2]$, we assume that $L/\sigma \leq n, \bar{n}^s \sqrt{\log \bar{n}} \geq 3^{s+1/2} R/\sigma$*

and $R/\sigma \geq 2^{1+2s} \sqrt{(\log \bar{n})/\bar{n}}$. Then, there exists a positive real $C(s, \alpha, \beta)$ only depending on s , α and β such that in case $s \in (0, 1]$,

$$\rho(T_n^{\text{reg}}, \mathcal{F}(s, R), \Delta_1) \leq C(s, \alpha, \beta) R^{1/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{s/(1+2s)} \tag{26}$$

and, in case $s \in (1, 2]$,

$$\rho(T_n^{\text{reg}}, \mathcal{F}(s, R, L), \Delta_1) \leq C(s, \alpha, \beta) R^{1/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{s/(1+2s)} \tag{27}$$

and

$$\rho(T_n^{\text{reg}}, \mathcal{F}(s, R, L), \Delta_2) \leq C(s, \alpha, \beta) R^{3/(1+2s)} \left(\frac{\sigma^2 \log n}{n} \right)^{(s-1)/(1+2s)}. \tag{28}$$

5. Simulation study and power comparison

In this section, we are first concerned with some algorithmic aspects as well as with power comparisons. First, we explain how to compute approximate values of our test statistics

$$S_n := \max_{I \in \mathcal{C}_n} S_n^I \quad \text{or} \quad S_n^{\text{reg}} := \max_{I \in \mathcal{C}_n} \widehat{S}_n^{\text{reg}, I}.$$

We give also a brief description of how the critical thresholds and the power are calculated. For $n = 100$ we study the power of our test under various alternatives in the white noise and Gaussian regression models so as to provide a comparison with other monotonicity tests such as those proposed by Gijbels *et al.* [15] and Baraud *et al.* [5]. Although the tests are expected to behave better for large sample sizes, we give below power results in the regression model for $n = 50$ and some of the examples considered in the aforementioned papers to give an idea of the performance of the corresponding test for moderate sample sizes.

5.1. Implementing the test

Computing S_n

We describe here the numerical procedures used to compute our statistic for the white noise model. Some of these numerical procedures take a simpler form for the Gaussian regression model, and hence they will be only briefly described below. In the white noise model, the observed process F_n can be only computed on a discrete grid of $[0, 1]$. For each subinterval I of $[0, 1]$, let \tilde{F}_n^I be the approximation of the restriction of F_n to I obtained via a linear interpolation between the values of F_n at the points of the chosen grid. Also, let \widehat{F}_n^I be its least concave majorant. For a given pair of integers (i, j) such that $0 \leq i < j \leq n$, we use the Pool Adjacent Violators Algorithm (see, e.g., [6]) to compute the slopes and vertices of $\widehat{F}_n^{I_{ij}}$, where $I_{ij} = [i/n, j/n]$.

In order to gain in numerical efficiency, we compute the least concave majorants progressively, taking advantage of previous computations on smaller intervals using a concatenation procedure. More precisely, we first compute $\widehat{F}_n^{I_j^{(j+1)}}$, for all $j \in \{0, \dots, n - 1\}$, and store the corresponding sets of vertices. Then, for $l \in \{j + 2, \dots, n\}$, the least concave majorant $\widehat{F}_n^{I_{jl}}$ is also equal to the least concave majorant of the function resulting from the concatenation of $\widehat{F}_n^{I_j^{(j+1)}}$ and $\widehat{F}_n^{I_{(l-1)l}}$, whose sets of vertices were previously stored. This merging step reduces the computation time substantially, because the number of vertices of a least concave majorant on I_{ij} is often much smaller than the number of grid points in I_{ij} . At each step of the algorithm, the maximum of $\widehat{F}_n^{I_{jl}} - \widetilde{F}_n^{I_{jl}}$ on I_{jl} is multiplied by $n/\sqrt{(l-j)}$ and stored. Last, we obtain the (approximated) value of the test statistic S_n by taking the largest value of those rescaled maxima and then dividing by σ .

Computing the statistic S_n^{reg}

The discretized nature of this setting makes the computations faster than in the white noise model. The least concave majorants based on the independent data $\bar{Y}_i, i = 1, \dots, \bar{n}$ are progressively computed on $I_{ij} = \{i/\bar{n}, \dots, j/\bar{n}\}, 0 \leq i < j \leq \bar{n}$, using the concatenation technique as described above. Note that each data point \bar{Y}_i is assigned to the design point $x_i = i/\bar{n}, i = 1, \dots, \bar{n}$, so that only half of the original grid is exploited. This is the price to be paid for not knowing the variance of the noise. The maximum deviation between the cumulative sum diagrams and their corresponding least concave majorants yields the value of the test statistic S_n^{reg} after division by the estimate $\widehat{\sigma}_0 = (\bar{n}^{-1} \sum_{j=1}^{\bar{n}} (Y_{2i-1} - Y_{2i}))^{1/2}$.

In Figure 1, we illustrate the computation of Z_n^{reg} . Independent replications of the above calculations under the hypothesis $f \equiv 0$ enable us to compute the empirical quantiles of S_n^{reg} . For a

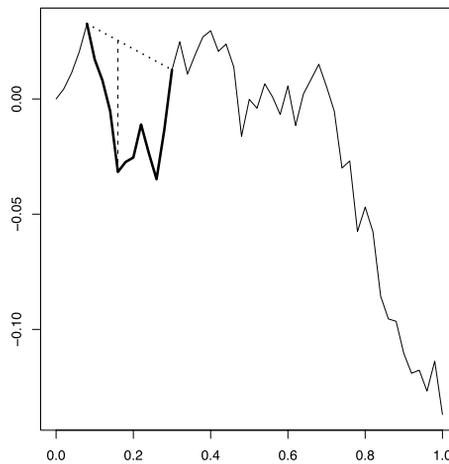


Figure 1. The plot of the cumulative sum diagram of $\bar{Y}_i, i = 1, \dots, \bar{n} = 50$ based on 100 independent realizations of standard Gaussians Y_1, \dots, Y_{100} , and the least concave majorant on $I = \{4/50, \dots, 15/50\}$ yielding the maximal value of deviation Z_n^{reg} .

given $\alpha \in (0, 1)$, the empirical quantile of order $1 - \alpha$ will be taken as an approximation for the critical threshold for the asymptotic level α , which will be denoted by $r_{\alpha,n}$.

Computing the critical thresholds and the power

We now describe how we determine the critical region of our tests for a given level $\alpha \in (0, 1)$. The calculation of the power under the alternative hypothesis is performed along the same lines, hence its details are skipped. Determining the critical region relies on computing Monte Carlo estimates of $s_{\alpha,n}$ and $r_{\alpha,n}$, the $(1 - \alpha)$ -quantiles of the statistic S_n and S_n^{reg} under the least favorable hypothesis $f \equiv 0$. The calculations in the regression setting have been described above and are much simpler than for the white noise model. Thus, we only provide some details of how the approximation of the critical threshold $s_{\alpha,n}$ is performed. Approximation of $s_{\alpha,n}$ requires simulation of C independent copies of Brownian motion on $[0, 1]$. For a chosen $r \in \mathbb{N}^*$, we simulate $m = n \times r$ independent standard Gaussians Y_1, \dots, Y_m . The rescaled partial sums $m^{-1/2} \sum_{i=1}^k Y_i$, for $k = 0, \dots, m$, provide approximate values for Brownian motion at the points of the regular grid $\{k/m; k = 0, \dots, m\}$ of $[0, 1]$. We then proceed as explained in the previous paragraph to obtain the approximations $\tilde{W}^{I_{ij}}, 0 \leq i < j \leq n$, of the restrictions of Brownian motion to the intervals I_{ij} and their respective least concave majorants $\hat{W}^{I_{ij}}$. In the sequel, we fix $n = 100$. Figure 2 shows an example where approximated value of S_n for $f \equiv 0$ is found to be equal to $0.611 \times \sqrt{100/11} \approx 1.842$, where 0.611 is the length of the vertical dashed line representing the maximal difference between \tilde{W}^I and its least concave majorant. For the approximation of Brownian motion, we have taken $r = 1000$.

Based on $C = 5000$ runs, we found that $r = 1000$ and 10000 yield close results for the distribution of the approximated value of the test statistic S_n . On the other hand, larger values of r make the computations prohibitively slow. Thus, we chose $r = 1000$ as a good compromise for

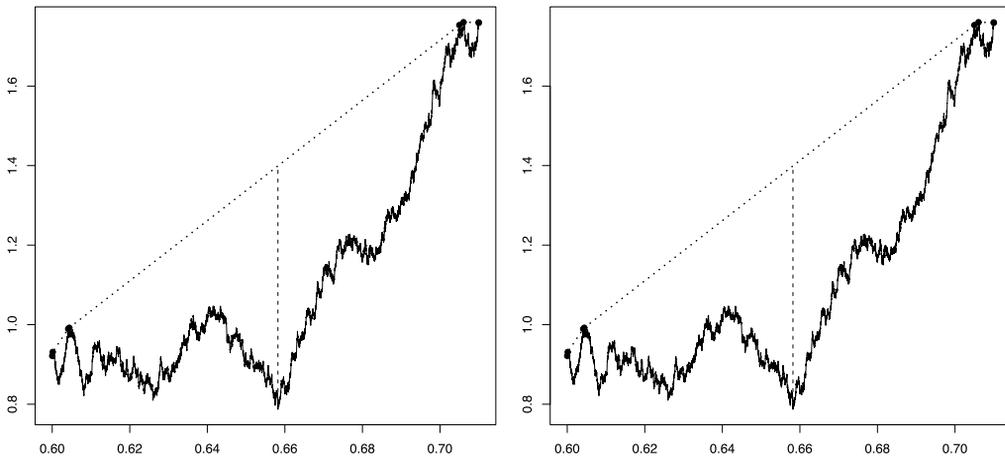


Figure 2. Left plot: Brownian approximation on the unit interval and the least concave majorant on the subinterval $I = [61/100, 72/100]$ yielding the approximate value of S_n for $f \equiv 0$. Right plot: magnified plot of the least concave majorant on I . The vertices are shown in bullets.

Table 1. Monte Carlo estimates of $s_{\alpha,n}$, the $(1 - \alpha)$ -quantiles of S_n for $f \equiv 0$. The estimation is based on $C = 5000$ runs, $n = 100$ and $r = 1000$

α	0.01	0.02	0.03	0.04	0.05
$s_{\alpha,n}$	2.451860	2.384279	2.343395	2.308095	2.278482
α	0.06	0.07	0.08	0.09	0.10
$s_{\alpha,n}$	2.254443	2.235377	2.217335	2.205137	2.191502

Monte Carlo estimation of the quantiles as well as for power calculations. Based on $C = 5000$ runs, Monte Carlo estimates of $s_{\alpha,n}$, the $(1 - \alpha)$ -quantiles of S_n for $f \equiv 0$, were computed for $\alpha \in \{0.01, 0.02, \dots, 0.1\}$ and are gathered in Table 1. In particular, the Monte Carlo estimate of the quantile of order $1 - \alpha = 0.95$ is found to be equal to 2.278482. Note that the true quantile should be comprised between $q(\alpha)$ and $q(2\alpha[n(n + 1)]^{-1})$, the $1 - \alpha$ and $1 - 2\alpha[n(n + 1)]^{-1}$ quantiles of Z (cf. Section 2). A numerical method for finding very precise approximations of upper quantiles of Z was developed by Balabdaoui and Filali [2] using a Gaver–Stehfest algorithm. For $\alpha = 0.05$, the Gaver–Stehfest approximation of $q(\alpha)$ and $q(2\alpha[n(n + 1)]^{-1})$ computed yields the values 1.46279052 and 2.60451660, respectively. Hence, the obtained Monte Carlo estimate seems to be consistent with the theory.

For testing monotonicity in the regression model in (22), Table 2 give values of $r_{\alpha,n}$ for $n = 100$ and $\alpha \in \{0.01, 0.02, \dots, 0.1\}$. The approximated quantiles are obtained with $C = 5000$ independent draws from a standard Gaussian. The Monte Carlo estimate of the quantile of order $1 - \alpha = 0.95$ is found to be equal to $r_{0.05,100} = 1.970304$, and hence smaller than its counterpart in the white noise model. Recall that the statistic S_n^{reg} has the same distribution as $Z_n^{\text{reg}} / \sqrt{\chi^2(\bar{n})/\bar{n}}$. Thus, as n gets larger, we expect the distribution of S_n^{reg} to be closer to that of S_n . We have added Table 3 where we give the obtained values of $r_{\alpha,n}$ for $n = 1000$, and which are clearly close to the approximated quantiles obtained in Table 1 in the white noise model.

5.2. Power study

In this subsection, we shall determine the power of our tests when the true signal deviates either globally or locally from monotonicity. The functions we consider here have already been used by Gijbels, Hall, Jones and Koch [15] and Baraud, Huet and Laurent [5]. Our goal is then two-fold:

Table 2. Monte Carlo estimates of $r_{\alpha,n}$, the $(1 - \alpha)$ -quantiles of S_n^{reg} for $f \equiv 0$. The estimation is based on $C = 5000$ runs and $n = 100$

α	0.01	0.02	0.03	0.04	0.05
$r_{\alpha,n}$	2.150903	2.090423	2.013185	1.998276	1.970304
α	0.06	0.07	0.08	0.09	0.10
$r_{\alpha,n}$	1.950475	1.938510	1.906807	1.892049	1.870080

Table 3. Monte Carlo estimates of $r_{\alpha,n}$, the $(1 - \alpha)$ -quantiles of S_n^{reg} for $f \equiv 0$. The estimation is based on $C = 5000$ runs and $n = 1000$

α	0.01	0.02	0.03	0.04	0.05
$r_{\alpha,n}$	2.475129	2.405715	2.351605	2.320332	2.295896
α	0.06	0.07	0.08	0.09	0.10
$r_{\alpha,n}$	2.277919	2.249911	2.232569	2.214476	2.202449

compute the power of our test, and compare its performance to that of the tests considered in these papers.

In Tables 4 and 5 below, the columns labeled S_n and S_n^{reg} give the power of the tests of level $\alpha = 0.05$ based on the statistics S_n and S_n^{reg} in the white noise and regression model (20), respectively, that is, the proportion of $S_n > s_{\alpha,n}$ and $S_n^{\text{reg}} > r_{\alpha,n}$ among the total number of runs.

Functions considered by Baraud et al. [5]

Baraud, Huet and Laurent [5] consider a regression model with deterministic design points $0 \leq x_1 \leq \dots \leq x_n \leq 1$ and Gaussian noise whose variance is finite and equal to σ^2 . Their monotonicity test of the true regression function is based on partitioning the set $\{1, \dots, n\}$ into $l \in \{2, \dots, l_n\}$ subintervals for a given integer $2 \leq l_n \leq n$. Below, we use T_B as a shorthand notation for their local mean test where the maximal number of subsets in the partition is $l_n = 25$ and $x_i = i/n, i = 1, \dots, n$. The basis of power comparison consists of the following functions

$$\begin{aligned}
 f_1(x) &= -15(x - 0.5)^3 1_{x \leq 0.5} - 0.3(x - 0.5) + \exp(-250(x - 0.25)^2), \\
 f_2(x) &= 1.5\sigma x, \\
 f_3(x) &= 0.2 \exp(-50(x - 0.5)^2), \\
 f_4(x) &= -0.1 \cos(6\pi x), \\
 f_5(x) &= -0.2x + f_3(x), \\
 f_6(x) &= -0.2x + f_4(x), \\
 f_7(x) &= -(1 + x) + 0.25 \exp(-50(x - 0.5)^2).
 \end{aligned}$$

Note that f_7 is a special case of Model III of Gijbels *et al.* [15], which we consider below.

Table 4 gathers power results for the functions $f_1/\sigma_1, \dots, f_7/\sigma_7$ where σ_i^2 is the variance of the noise considered by the authors when the true function is f_i (see the second column in Table 4). Calculation of the power of our local least concave majorant test was based on 1000 independent runs in both white noise and regression models. We see that in both models our tests perform as well as the local mean test of Baraud *et al.* [5], except for the function f_7 , which is a special case of Model III considered by Gijbels *et al.* [15] ($a = 0.45$ and $\sigma = 0.1$), and where our test in the regression model seems to be doing a bit worse than local mean test. We would like to note that in the white noise model the power obtained for the functions f_1, \dots, f_6 is not much altered when we replace the quantile $s_{\alpha,n}$ by the upper-bound $q(2\alpha[n(n + 1)]^{-1}) \approx 2.60451660$

Table 4. Power of the tests based on S_n , S_n^{reg} and T_B for $n = 100$ (see text for details)

Function	σ^2	S_n	S_n^{reg}	T_B
f_1	0.01	1.00	0.99	0.99
f_2	0.01	0.99	1.00	0.99
f_3	0.01	1.00	0.98	1.00
f_4	0.01	0.99	0.99	0.99
f_5	0.004	1.00	0.99	0.99
f_6	0.006	1.00	0.99	0.98
f_7	0.01	0.79	0.68	0.76

(see Lemma 2.2): in this case, the power is slightly smaller and we find a minimum difference of order -0.070 .

Functions considered by Gijbels et al. [15]

Gijbels *et al.* [15] consider a regression model where the deviation from monotonicity of the true regression function f defined on $[0, 1]$ depends on a parameter $a > 0$ in the following manner

$$f_a(x) = -(1 + x) + a \exp(-50(x - 0.5)^2), \quad x \in [0, 1].$$

(Note that we have multiplied their function by (-1) to obtain a perturbation of a decreasing function, here $-1 - x$.) We refer to [15] for a description of their tests. Here, we compare the performance of our test to their test based on the statistic T_{run} (runs of equal signs).

For $a = 0$, where the true function $-1 - x$ satisfies H_0 , all three tests have a rejection probability which is much smaller than $\alpha = 0.05$. Our test in the white noise model seems to be however less conservative than our test in the regression model and the test of Gijbels *et al.* [15]. Our power values in both models (1) and (20) suggest that our tests are exhibiting comparable performance, except for the configuration $a = 0.45$ and $\sigma = 0.1$ where our test based on S_n^{reg} seems to be doing a bit worse. This fact was also noted above in our comparison with the local mean test of Baraud *et al.* [5]. In the white noise model, we would like to note that replacing the quantile $s_{\alpha,n}$ by $q(2\alpha[n(n + 1)]^{-1})$ implies now a strong decrease in the power. We conclude that the upper-bound of Lemma 2.2, although interesting in its own right, should be used with caution.

Table 5. Power of the tests based on S_n , S_n^{reg} and T_{run} for $n = 100$ (see text for details)

σ	$a = 0$			$a = 0.25$			$a = 0.45$		
	0.025	0.05	0.1	0.025	0.05	0.1	0.025	0.05	0.1
S_n	0.010	0.018	0.014	0.246	0.043	0.031	1.000	1.000	0.796
S_n^{reg}	0.000	0.002	0.013	0.404	0.053	0.007	1.000	1.000	0.683
T_{run}	0.000	0.000	0.000	0.106	0.037	0.014	1.000	1.000	0.805

Table 6. Power of the test based on S_n^{reg} for $n = 50$ (see text for details)

Function	σ^2	Power
f_1	0.01	0.44
f_2	0.01	0.68
f_3	0.01	0.84
f_4	0.01	0.36
f_5	0.004	0.90
f_6	0.006	0.44
f_7	0.01	0.34

Finally, we have carried out further investigation of the performance of our test in the regression model for the moderate sample size $n = 50$. The Monte Carlo estimate of the 95% quantile of S_n^{reg} is found to be approximately equal to 1.837931. Table 6 gives the obtained values of the power for the alternatives considered by Baraud *et al.* [5]. As expected, the test loses from its performance for this smaller sample size but is still able to detect deviation from monotonicity for some of those alternatives with large power (functions f_3 and f_5). The alternatives considered by Gijbels *et al.* [15] present a much bigger challenge and the power of our test is found to be small with values between 0.003 and 0.01 for $a = 0.25$. For $a = 0.45$, our test is found to be powerful with power values equal to 1 and 0.954 for $\sigma = 0.025$ and 0.05, respectively.

6. Discussion

In this section, we compare our test method with some of its (existing) competitors and also discuss possible generalizations to other settings.

6.1. Comparison with competing tests

In this article, we have proposed a new procedure for testing monotonicity which is able to detect localized departures from monotonicity, with exact prescribed level, either in the white noise model or in the Gaussian regression framework with unknown variance. Firstly, as explained in Section 5, our test statistic, which is a maximum over a finite number of intervals – increasing with n – of local least concave majorants, can be computed exactly and efficiently. This is a big advantage when our method is compared with the two test statistics proposed in [10], as they rely on a family of kernel estimators indexed by a continuous set of bandwidths which must be in practice discretized. Secondly, the distribution of our test statistic under the least favorable hypothesis is known and its quantiles can be evaluated via Monte Carlo simulations. We insist on the fact that, as opposed to [5], only one quantile has to be evaluated for a given level, and that no bootstrap is required as opposed to [18]. Moreover, our test statistic does not rely on any smoothness assumption, because no smoothing parameter is involved in the construction of the test as opposed to [14]. In terms of power, an interesting property of our

procedure is its adaptivity. Indeed, our procedure attains the same rates of separation as [14] (without having to play with some smoothness parameter), as well as the optimal rates obtained in the four procedures in [5,10]. Lastly, the detailed comparative study above shows that the power values we attain are generally similar to those obtained by [5] and better than the ones obtained by [15]. Note also that in practice our procedure reaches at most the prescribed level, which may not be the case for [18], even for Gaussian errors. We conclude that our approach seems to enjoys all the qualities of the existing methods.

6.2. About the choice of \mathcal{C}_n

As explained at the beginning of Section 3, our choice of \mathcal{C}_n is motivated by our wish to detect as many alternatives as possible. However, if one knows in advance properties of the subinterval where f is likely to violate the non-increasing assumption, then one can incorporate this knowledge to the choice of \mathcal{C}_n , that is, one can choose a reduced collection of subintervals in such a way that for the largest interval $I \subset [0, 1]$ where f is likely to violate the non-increasing assumption, there is an interval close to I in the chosen collection \mathcal{C}_n . By “reduced collection”, we mean a collection that is included in the one defined in (8), and by “an interval close to I ”, we mean, for instance, an interval whose intersection with I has a length of the same order of magnitude (up to a multiplicative constant that does not depend on n) as the length of I , and where the increment of f has the same order as on I . It can be seen from our proofs that for an arbitrary choice of \mathcal{C}_n , we obtain that there exists $C(\alpha, \beta) > 0$ only depending on α and β such that (9) holds provided there exists $[x, y] \in \mathcal{C}_n$ such that

$$\sup_{t \in [x, y]} \frac{t - x}{\sqrt{y - x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C(\alpha, \beta) \sqrt{\frac{\sigma^2 \log |\mathcal{C}_n|}{n}}. \tag{29}$$

Note that in the case where \mathcal{C}_n is given by (8), $|\mathcal{C}_n| = n(n + 1)/2$ so that $\log |\mathcal{C}_n|$ is of the order $\log n$, hence the $\log n$ term in Theorem 3.1. In view of condition (29), good power properties are obtained if one chooses \mathcal{C}_n in such a way that $|\mathcal{C}_n|$ is not too large, but \mathcal{C}_n contains an interval $[x, y]$ close to the largest interval where f is likely to violate the non-increasing assumption: \mathcal{C}_n must have good approximation properties while having a moderate cardinality. For instance, to test that f is non-increasing on $[0, 1]$ against the alternative that f is U-shaped on $[0, 1]$, one can consider a collection of intervals of the form $[x, 1]$. However, in such a case, only alternatives that are increasing on the right boundary of $[0, 1]$ could be detected. More generally, considering a reduced collection \mathcal{C}_n may cause a loss of adaptivity so we do not pursue the study of our test in the case where it is defined with a reduced collection \mathcal{C}_n .

6.3. Possible extensions to more general models

Recall that in the case where we observe Y_1, \dots, Y_n according to (20) with a known $\sigma > 0$ and i.i.d. standard Gaussian ϵ_i 's, we reject $H_0 : f \in \mathcal{D}$ against the alternative $H_1 : f \notin \mathcal{D}$ if

$$\max_{I \in \mathcal{C}_n} S_n^{\text{reg}, I} > r_{\alpha, n}, \tag{30}$$

where $S_n^{\text{reg},I}$ is given by (21) and $r_{\alpha,n}$ is the $(1 - \alpha)$ -quantile of $\max_{I \in \mathcal{C}_n} S_n^{\text{reg},I}$ under the hypothesis that $f \equiv 0$.

The hypothesis that the ϵ_i 's are standard Gaussian can easily be relaxed provided that the common distribution remains known. Indeed, provided both σ and the common distribution of the ϵ_i 's are known, the law of $\max_{I \in \mathcal{C}_n} S_n^{\text{reg},I}$ is entirely known (at least theoretically) under the least favorable hypothesis that $f \equiv 0$, so we can obtain approximate value of the $(1 - \alpha)$ -quantile $r_{\alpha,n}$ of this law via Monte Carlo simulations. The critical region (30) then defines a test with level α . If there exists $C > 0$ and $C' > 0$ such that for all $x > 0$ and $I = [i/n, j/n] \in \mathcal{C}_n$,

$$\mathbb{P}\left(\max_{i < k \leq j} \sum_{l=i+1}^k \delta_l > x\sqrt{j-i}\right) \leq C \exp(-C'x^2),$$

where δ_i denotes either ϵ_i or $-\epsilon_i$, then this test has similar power properties as in the Gaussian case (the rates in Theorem 4.2 still hold, with possibly different constants).

In the case where the distribution of the ϵ_i 's is known but σ is unknown, it is tempting to replace σ with an estimator $\widehat{\sigma}_n$ in the definition of the test statistic, and to consider the critical region $T_n/\widehat{\sigma}_n > \widehat{r}_{\alpha,n}$ where

$$T_n = \sigma \max_{I \in \mathcal{C}_n} S_n^{\text{reg},I}$$

is observable and $\widehat{r}_{\alpha,n}$ is the $(1 - \alpha)$ -quantile of $T_n/\widehat{\sigma}_n$ under the hypothesis that $f \equiv 0$. However, the distribution of $T_n/\widehat{\sigma}_n$ is not known in the general case even if $f \equiv 0$. In the particular case where the ϵ_i 's are standard Gaussian, we suggest in Section 4 to rather consider

$$\frac{\sigma_0}{\widehat{\sigma}_0} \max_{I \in \mathcal{C}_n} S_n^{\text{reg},I}$$

as a test statistic, where $\widehat{\sigma}_0$ and $S_n^{\text{reg},I}$ are defined in such a way that the distribution of $\widehat{\sigma}_0/\sigma_0$ is known if $f \equiv 0$, and $\widehat{\sigma}_0$ is independent of $\max_{I \in \mathcal{C}_n} S_n^{\text{reg},I}$. This way, the distribution of the test statistic is known under the hypothesis that $f \equiv 0$ and we are able to calibrate the test. Except in the Gaussian case, the distribution of the test statistic is not known even under the hypothesis that $f \equiv 0$. In such situations, it is tempting to argue asymptotically, as $n \rightarrow \infty$. This requires computation of the limit distribution of the test statistic $T_n/\widehat{\sigma}_n$ under the least favorable hypothesis. More precisely, suppose there exist sequences a_n and b_n such that if $f \equiv 0$,

$$a_n(T_n/\sigma - b_n)$$

converges in distribution to T as $n \rightarrow \infty$, where T is a random variable with a continuous distribution function. Suppose, moreover, that $\widehat{\sigma}_n$ converges in probability to σ as $n \rightarrow \infty$, and either $a_n b_n = O(1)$ or $\widehat{\sigma}_n = \sigma + o_P(1/(a_n b_n))$. Denoting by s_α the $(1 - \alpha)$ -quantile of T , the test defined by the critical region

$$T_n/\widehat{\sigma}_n > b_n + a_n^{-1} s_\alpha \tag{31}$$

has asymptotic level α . This means that one can extend our method (even in the case where the distribution of the ϵ_i 's is unknown) by considering a critical region of the form (31).

Other extensions of the method are conceivable to test, for instance, that a density function or a failure rate is non-increasing on a given interval, or that a regression function is non-increasing in a heteroscedastic model. In such cases, the exact distribution of the test statistic is not known even under the least favorable hypothesis since it depends on unknown parameters. Similar to the regression case with non-Gaussian errors above, it is then tempting to replace the unknown parameters with consistent estimators and argue asymptotically. Such asymptotic arguments, with the computation of the limit distribution of the test statistic, is beyond the scope of the present paper and is left for future research.

7. Proofs

Without loss of generality (see [1]), we assume for simplicity that the noise level is $\sigma = 1$.

7.1. Proofs for Section 2

Proof of Lemma 2.1. Define $B(t) = (W(t(b - a) + a) - W(a))/\sqrt{|I|}$ on $[0, 1]$. From Lemma 2.1 in [13], it follows that for all $t \in I$,

$$\widehat{W}^I(t) - W(t) = \sqrt{|I|} \left[\widehat{B} \left(\frac{t - a}{b - a} \right) - B \left(\frac{t - a}{b - a} \right) \right].$$

If $f \equiv c$ on I then F is linear on I , so Lemma 2.1 in [13] shows that $\widehat{F}_n^I = F + \widehat{W}^I/\sqrt{n}$ and

$$S_n^I = \frac{1}{\sqrt{|I|}} \sup_{t \in I} (\widehat{W}^I(t) - W(t)) = \sup_{t \in [0,1]} (\widehat{B}(t) - B(t)).$$

But B is distributed like W , so S_n^I is distributed like Z . □

Proof of Theorem 2.1. For every $f \in \mathcal{D}^I$, F is concave on I , so the process $F + \widehat{W}^I/\sqrt{n}$ is concave and above F_n on I . Thus, it is also above \widehat{F}_n^I , and hence

$$S_n^I \leq \sqrt{\frac{n}{|I|}} \sup_{t \in I} \left(F(t) + \frac{1}{\sqrt{n}} \widehat{W}^I(t) - F_n(t) \right) \leq \frac{1}{\sqrt{|I|}} \sup_{t \in I} (\widehat{W}^I(t) - W(t)).$$

Since Z is continuously distributed (see [11], Lemma 1), Lemma 2.1 yields

$$\sup_{f \in \mathcal{D}^I} \mathbb{P}_f [S_n^I > q(\alpha)] \leq \mathbb{P}[Z > q(\alpha)] = \alpha.$$

Now, suppose $f \equiv c$ over I for a fixed $c \in \mathbb{R}$. Then Lemma 2.1 shows that

$$\mathbb{P}_f [S_n^I > q(\alpha)] = \mathbb{P}[Z > q(\alpha)] = \alpha. \quad \square$$

Proof of Theorem 2.2. Suppose $f \in \mathcal{D}$. We deduce as in the proof of Theorem 2.1 that

$$\max_{I \in \mathcal{C}_n} S_n^I \leq \max_{I \in \mathcal{C}_n} \sqrt{\frac{1}{|I|}} \sup_{t \in I} (\widehat{W}^I(t) - W(t))$$

with equality when $f \equiv c$ for some fixed $c \in \mathbb{R}$, hence Theorem 2.2. \square

Proof of Lemma 2.2. It follows from Lemma 2.1 that for every $x \geq 0$,

$$\mathbb{P} \left[\max_{I \in \mathcal{C}_n} \sqrt{\frac{1}{|I|}} \sup_{t \in I} (\widehat{W}^I(t) - W(t)) > x \right] \leq |\mathcal{C}_n| \mathbb{P}[Z > x].$$

Thus, with $x = q(\alpha/|\mathcal{C}_n|)$, we obtain the first inequality. Now, from the definition (3) of Z , we have

$$Z \leq \sup_{t \in [0,1]} W(t) + \sup_{t \in [0,1]} (-W(t)).$$

Since W has the same distribution as $-W$ and its supremum over $[0, 1]$ satisfies an exponential inequality (see, e.g., [21], Chapter II, Proposition 1.8), it follows that for every $x \geq 0$,

$$\mathbb{P}(Z > x) \leq 2\mathbb{P} \left[\sup_{t \in [0,1]} W(t) > \frac{x}{2} \right] \leq 2 \exp\left(-\frac{x^2}{8}\right). \tag{32}$$

In particular, for every $\gamma \in (0, 1)$, applying (32) with $x = 2\sqrt{2 \log(2/\gamma)}$ implies that $q(\gamma) \leq 2\sqrt{2 \log(2/\gamma)}$ and completes the proof. \square

7.2. Proof of Theorem 3.1

We first prove the following lemma.

Lemma 7.1. Assume \mathcal{C}_n is any finite collection of subintervals of $[0, 1]$ and

$$\max_{I \in \mathcal{C}_n} \sqrt{\frac{n}{|I|}} \sup_{t \in I} (\widehat{F}^I(t) - F(t)) \geq 2\sqrt{2} \left(\sqrt{\log\left(\frac{2|\mathcal{C}_n|}{\alpha}\right)} + \sqrt{\log\left(\frac{2}{\beta}\right)} \right) \tag{33}$$

for some α and β in $(0, 1)$. Then, (9) holds.

Proof. Let $I \in \mathcal{C}_n$ achieving the maximum in (33) and $\epsilon = \sqrt{4 \log(2/\beta) / \log(2|\mathcal{C}_n|/\alpha)}$. It follows from the definition of S_n^I , Lemma 2.2 and (33) that

$$\mathbb{P}_f \left[\max_{I \in \mathcal{C}_n} S_n^I > s_{\alpha,n} \right] \geq \mathbb{P}_f \left[\sqrt{\frac{n}{|I|}} \sup_{t \in I} (\widehat{F}_n^I(t) - F_n(t)) > 2\sqrt{2 \log\left(\frac{2|\mathcal{C}_n|}{\alpha}\right)} \right].$$

Since $\widehat{F}_n^I \geq F_n$ and $\widehat{F}^I \geq F$ on I , the triangle inequality yields

$$\sup_{t \in I} (\widehat{F}_n^I(t) - F_n(t)) \geq \sup_{t \in I} (\widehat{F}^I(t) - F(t)) - \sup_{t \in I} \left| \widehat{F}_n^I(t) - \widehat{F}^I(t) - \frac{1}{\sqrt{n}} W(t) \right|.$$

Besides, with $I = [a, b]$, we have for every $t \in I$:

$$\left| \widehat{F}_n^I(t) - \widehat{F}^I(t) - \frac{1}{\sqrt{n}} W(t) \right| \leq \left| \widehat{F}_n^I(t) - \widehat{F}^I(t) - \frac{1}{\sqrt{n}} W(a) \right| + \frac{1}{\sqrt{n}} |W(t) - W(a)|.$$

But $\widehat{F}_n^I(t) - W(a)/\sqrt{n}$ is the least concave majorant at time t of the process $\{F_n(u) - W(a)/\sqrt{n}\}_{u \in I}$, so it follows from Lemma 2.2 in [13] that

$$\begin{aligned} \sup_{t \in I} \left| \widehat{F}_n^I(t) - \widehat{F}^I(t) - \frac{1}{\sqrt{n}} W(a) \right| &\leq \sup_{t \in I} \left| F_n(t) - F(t) - \frac{1}{\sqrt{n}} W(a) \right| \\ &\leq \frac{1}{\sqrt{n}} \sup_{t \in I} |W(t) - W(a)|. \end{aligned}$$

Hence,

$$\sup_{t \in I} \left| \widehat{F}_n^I(t) - \widehat{F}^I(t) - \frac{1}{\sqrt{n}} W(t) \right| \leq \frac{2}{\sqrt{n}} \sup_{t \in I} |W(t) - W(a)|.$$

By scaling, the right-hand side is distributed like $2\sqrt{|I|/n} \sup_{t \in [0,1]} |W(t)|$, so combining all previous inequalities leads to

$$\mathbb{P}_f \left[\max_{I \in \mathcal{C}_n} S_n^I > s_{\alpha,n} \right] \geq \mathbb{P}_f \left[2 \sup_{t \in [0,1]} |W(t)| < \epsilon \sqrt{2 \log \left(\frac{2|\mathcal{C}_n|}{\alpha} \right)} \right].$$

We conclude with the same arguments as in (32) that

$$\mathbb{P}_f \left[\max_{I \in \mathcal{C}_n} S_n^I > s_{\alpha,n} \right] \geq 1 - 2 \exp \left(-\frac{\epsilon^2}{4} \log \left(\frac{2|\mathcal{C}_n|}{\alpha} \right) \right).$$

By definition of ϵ , the right-hand side is $1 - \beta$, hence inequality (9). □

Let us turn now to the proof of Theorem 3.1 and recall that we restrict ourselves without loss of generality to the case $\sigma = 1$. Assume (11) for some $x, y \in [0, 1]$ such that $y - x \geq 2/n$, and write $I_0 = [x, y]$. The linear function

$$t \mapsto F(x) + (t - x) \frac{F(y) - F(x)}{y - x} = F(t) + (t - x)(\bar{f}_{xy} - \bar{f}_{xt}),$$

where \bar{f}_{xy} is defined by (10), coincides with F at the boundaries x and y of the interval I_0 . So this function is below \widehat{F}^{I_0} on I_0 and we obtain

$$\frac{1}{\sqrt{|I_0|}} \sup_{t \in I_0} (\widehat{F}^{I_0}(t) - F(t)) \geq \sup_{t \in [x,y]} \frac{t - x}{\sqrt{y - x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C(\alpha, \beta) \sqrt{\frac{\log n}{n}}.$$

Now, let I be the smallest interval in \mathcal{C}_n containing I_0 . Since $|I_0| \geq 2/n$, we have $|I| \leq |I_0| + 2/n \leq 2|I_0|$. Moreover, $\widehat{F}^I \geq \widehat{F}^{I_0}$ on I_0 , so

$$\frac{1}{\sqrt{|I|}} \sup_{t \in I} (\widehat{F}^I(t) - F(t)) \geq \frac{1}{\sqrt{2|I_0|}} \sup_{t \in I_0} (\widehat{F}^{I_0}(t) - F(t)) \geq C(\alpha, \beta) \sqrt{\frac{\log n}{2n}}. \quad (34)$$

Since $|\mathcal{C}_n| = n(n + 1)/2$, it follows that (33) holds provided that $C(\alpha, \beta)$ is large enough, so Theorem 3.1 follows from Lemma 7.1.

7.3. A useful lemma

In the case f is not non-decreasing such that f is assumed to be smooth enough, then Lemma 7.2 below may serve as a tool to prove that condition (11) in Theorem 3.1 is fulfilled.

Lemma 7.2. *Assume $f \notin \mathcal{D}$ and $f(u) - f(v) \leq R(u - v)^s$ for all $u \geq v$, for some $R > 0$ and $s \in (0, 1]$. Let $x_0 < y_0$ in $[0, 1]$ such that $\rho := f(y_0) - f(x_0) > 0$. Then, there exist an interval $[x, y] \subset [x_0, y_0]$ and a real $C(s) > 0$ that only depends on s such that*

$$\sup_{t \in [x, y]} \frac{t - x}{\sqrt{y - x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C(s) R^{-1/(2s)} \rho^{1+1/(2s)}. \quad (35)$$

Proof. Let $s \in (0, 1]$ and $L \geq 1$ be fixed, and let $\mathcal{G}(s, L)$ be the set of integrable functions $g : [0, 1] \rightarrow \mathbb{R}$ such that $g(0) = 0$, $g(1) = 1$, and

$$g(u) - g(v) \leq L(u - v)^s \quad \text{for all } u \geq v.$$

Let $\mathcal{G}_0(s, L)$ be the set of functions $g \in \mathcal{G}(s, L)$ such that $\bar{g}_{01} \geq 1/2$, where for every $x < y$, \bar{g}_{xy} is defined as in (10). We first prove that for all $g \in \mathcal{G}(s, L)$,

$$\sup_{0 \leq x < t < y \leq 1} \frac{t - x}{\sqrt{y - x}} (\bar{g}_{xy} - \bar{g}_{xt}) \geq C(s, L) := \frac{s}{7 \times 2^{1+s+1/s}} L^{-1/(2s)}. \quad (36)$$

We first consider the case where $g \in \mathcal{G}_0(s, L)$ and argue by contradiction. Assume there exists $g \in \mathcal{G}_0(s, L)$ such that inequality (36) is not satisfied. For every integer k , let $m_k = \bar{g}_{02^{-k}}$. Setting $x_k = 0$, $t_k = 2^{-k-1}$ and $y_k = 2^{-k}$, it follows from our assumption on g that

$$m_k - m_{k+1} = 2^{1+k/2} \frac{t_k - x_k}{\sqrt{y_k - x_k}} (\bar{g}_{x_k y_k} - \bar{g}_{x_k t_k}) < 2^{1+k/2} C(s, L).$$

But for all integers $k_0 \geq 0$,

$$\bar{g}_{01} = m_0 = \sum_{k=0}^{k_0} (m_k - m_{k+1}) + m_{k_0+1}.$$

By assumption, $\bar{g}_{01} \geq 1/2$ whence

$$\frac{1}{2} < C(s, L) \sum_{k=0}^{k_0} 2^{1+k/2} + m_{k_0+1} < 7C(s, L)2^{k_0/2} + m_{k_0+1}. \tag{37}$$

Since $g(0) = 0$, for all integers $k_0 \geq 0$ we have

$$m_{k_0+1} = 2^{k_0+1} \int_0^{2^{-k_0-1}} (g(u) - g(0)) du < L2^{-s(k_0+1)}.$$

From (37) and the definition of $C(s, L)$, we obtain that for all integers $k_0 \geq 0$,

$$\frac{1}{2} < \frac{s}{2^{1+s+1/s}} L^{-1/(2s)} 2^{k_0/2} + L2^{-s(k_0+1)}. \tag{38}$$

In particular, consider $k_0 = \sup\{k \in \mathbb{N} : 2^{k/2} \leq 2^{1/s} L^{1/(2s)}\}$, which is well defined since $s \in (0, 1]$ and $L \geq 1$. By definition of k_0 , $2^{k_0/2} \leq 2^{1/s} L^{1/(2s)}$ and $2^{(k_0+1)/2} > 2^{1/s} L^{1/(2s)}$, so (38) implies $s2^{-s} > 1/2$. This is a contradiction because $s2^{-s} \leq 1/2$ for all $s \in (0, 1]$. Hence, (36) holds for all $s \in (0, 1]$, $L \geq 1$ and $g \in \mathcal{G}_0(s, L)$. Now, for every $g \in \mathcal{G}(s, L)$ we set $\tilde{g} = g$ if $\bar{g}_{01} \geq 1/2$, and $\tilde{g}(u) := 1 - g(1 - u)$ otherwise, so that $\tilde{g} \in \mathcal{G}_0(s, L)$. Noting that

$$\frac{t-x}{\sqrt{y-x}} (\bar{g}_{xy} - \bar{g}_{xt}) = \frac{y-t}{\sqrt{y-x}} (\bar{g}_{ty} - \bar{g}_{xy})$$

for all $x < t < y$, we obtain

$$\sup_{0 \leq x < t < y \leq 1} \frac{t-x}{\sqrt{y-x}} (\bar{g}_{xy} - \bar{g}_{xt}) = \sup_{0 \leq x < t < y \leq 1} \frac{t-x}{\sqrt{y-x}} (\bar{\tilde{g}}_{xy} - \bar{\tilde{g}}_{xt}) \geq C(s, L),$$

since (36) holds for all $s \in (0, 1]$, $L \geq 1$ and $g \in \mathcal{G}_0(s, L)$. Hence, (36) holds for all $s \in (0, 1]$, $L \geq 1$ and $g \in \mathcal{G}(s, L)$.

Finally, under the assumptions of Lemma 7.2, the function

$$g(u) = \frac{1}{\rho} (f(x_0 + (y_0 - x_0)u) - f(x_0)), \quad u \in [0, 1],$$

belongs to $\mathcal{G}(s, L)$ with $L = R(y_0 - x_0)^s / \rho \geq 1$. Thus, it follows from (36) that there exists $C(s) > 0$ only depending on s such that

$$\begin{aligned} \sup_{x_0 \leq x < t < y \leq y_0} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) &= \rho \sqrt{y_0 - x_0} \sup_{0 \leq x < t < y \leq 1} \frac{t-x}{\sqrt{y-x}} (\bar{g}_{xy} - \bar{g}_{xt}) \\ &\geq C(s) R^{-1/(2s)} \rho^{1+1/(2s)}. \end{aligned} \quad \square$$

7.4. Remaining proofs for Section 3

Proof of Corollary 3.1. Setting $t = t_n$, $x = t_n - \Delta_n$, $y = t_n + \Delta_n$, we obtain

$$\begin{aligned} \frac{t-x}{\sqrt{y-x}}(\bar{f}_{xy} - \bar{f}_{xt}) &= \frac{1}{2\sqrt{2\Delta_n}} \left(\int_t^y f(u) \, du - \int_x^t f(u) \, du \right) \\ &\geq \frac{1}{2\sqrt{2\Delta_n}} \left(\int_{t-\delta_n}^t (f(u + \delta_n) - f(u)) \, du \right) \geq \frac{M\delta_n^2\lambda_n}{2\sqrt{2\Delta_n}}, \end{aligned}$$

so Corollary 3.1 follows from Theorem 3.1. □

Proof of Corollary 3.2. Since f is convex, we can apply Lemma 7.2 with $s = 1$ and R defined in Corollary 3.2. Therefore, there exist $[x, y] \subset [x_0, 1]$ and $C > 0$ such that

$$\sup_{t \in [x, y]} \frac{t-x}{\sqrt{y-x}}(\bar{f}_{xy} - \bar{f}_{xt}) \geq CR^{-1/2}\rho^{3/2}. \tag{39}$$

By change of variable, we have for all $t \in [x, y]$,

$$\bar{f}_{xy} - \bar{f}_{xt} = \frac{1}{y-x} \int_x^y \left(f(v) - f\left(\frac{v-x}{y-x}(t-x) + x\right) \right) \, dv. \tag{40}$$

By convexity of f and definition of R , this implies that for all $t \in [x, y]$,

$$\bar{f}_{xy} - \bar{f}_{xt} \leq \frac{R}{y-x} \int_x^y \left(\frac{v-x}{y-x}(y-t) \right) \, dv \leq \frac{R}{2}(y-x),$$

hence

$$\sup_{t \in [x, y]} \frac{t-x}{\sqrt{y-x}}(\bar{f}_{xy} - \bar{f}_{xt}) \leq \frac{R}{2}(y-x)^{3/2}. \tag{41}$$

Combining inequalities (39) and (41) proves that $y-x \geq (2C)^{2/3}\rho/R$. Therefore, $y-x \geq 2/n$ provided $\rho > C_0R/n$ for a large enough C_0 . From Theorem 3.1, it follows that (33) holds provided

$$CR^{-1/2}\rho^{3/2} > C(\alpha, \beta) \sqrt{\frac{\sigma^2 \log n}{n}}$$

for a large enough $C(\alpha, \beta)$, which completes the proof of Corollary 3.2. □

Proof of Theorem 3.2. Inequality (17) easily follows from Corollary 3.1. Yet, a detailed proof may be found in [1].

We now turn to the proof of inequality (16). Let $f \in \mathcal{F}(s, R)$ for some $s \in (0, 2]$ and $R > 0$ and define

$$\rho_n = C'(s, \alpha, \beta)R^{1/(1+2s)}\left(\frac{\log n}{n}\right)^{s/(1+2s)}, \tag{42}$$

where $C'(s, \alpha, \beta)$ is a positive number to be chosen later, that only depends on s, α and β . Assume $n^s \sqrt{\log n} \geq R, R \geq 2^{1+2s} \sqrt{(\log n)/n}$ in case $s > 1$, and $\Delta_1(f) \geq \rho_n$. The function f^* defined on $[0, 1]$ by

$$f^*(y) = \inf_{x \in [0, y]} f(x)$$

is non-increasing with $f^* \leq f$, so

$$\Delta_1(f) \leq \sup_{t \in [0, 1]} (f(t) - f^*(t)) \leq \sup_{0 \leq x < y \leq 1} (f(y) - f(x)).$$

Since f is continuous and $\Delta_1(f) \geq \rho_n$, this shows that there are $x_0 < y_0$ (that may depend on n) such that $f(y_0) = f(x_0) + \rho_n$.

Consider first the case $s \in (0, 1]$. From Lemma 7.2, there exist an interval $[x, y] \subset [x_0, y_0]$ and a positive number $C(s)$ only depending on s such that

$$\sup_{t \in [x, y]} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C(s)(C'(s, \alpha, \beta))^{1+1/(2s)} \sqrt{\frac{\log n}{n}}. \tag{43}$$

Since $f \in \mathcal{F}(s, R)$, formula (40) implies

$$\sup_{t \in [x, y]} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) \leq \frac{R}{s+1} (y-x)^{s+1/2}.$$

Combining this with (43) proves that

$$y-x \geq (C(s)(s+1))^{2/(1+2s)} (C'(s, \alpha, \beta))^{1/s} \left(\frac{\log n}{nR^2}\right)^{1/(1+2s)}, \tag{44}$$

so in particular, $y-x \geq 2/n$ provided $n^s \sqrt{\log n} \geq R$ and $C'(s, \alpha, \beta)$ is sufficiently large. Thanks to (43) and Theorem 3.1, we obtain that (9) holds in the case $s \in (0, 1]$, provided $C'(s, \alpha, \beta)$ is large enough, hence $\rho(T_n, \mathcal{F}(s, R), \Delta_1) \leq \rho_n$.

Consider now the case $s \in (1, 2]$. Assume that, for a given $C(s, \alpha, \beta) > 0$,

$$\sup_{t \in [0, 1]} f'(t) \leq C(s, \alpha, \beta)R^{3/(1+2s)}\left(\frac{\log n}{n}\right)^{(s-1)/(1+2s)}, \tag{45}$$

since otherwise (17) immediately allows to conclude. As $f \in \mathcal{F}(s, R)$ with some $s > 1$, we also have

$$f(u) - f(v) \leq (u-v) \sup_{t \in [0, 1]} f'(t) \quad \text{for all } u \geq v,$$

where $\sup_t f'(t) > 0$. Therefore, it follows from the definition of x_0, y_0 and Lemma 7.2 that there exist $C > 0$ and $[x, y] \subset [x_0, y_0]$ such that

$$\sup_{x < t < y} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq C \left(\sup_{t \in [0,1]} f'(t) \right)^{-1/2} \rho_n^{3/2}.$$

From (45) and the definition of ρ_n , we get

$$\sup_{t \in [x,y]} \frac{t-x}{\sqrt{y-x}} (\bar{f}_{xy} - \bar{f}_{xt}) \geq \frac{C(C'(s, \alpha, \beta))^{3/2}}{\sqrt{C(s, \alpha, \beta)}} \sqrt{\frac{\log n}{n}}.$$

Similar to (44), and using then (45), one obtains

$$\begin{aligned} y-x &\geq \left(\frac{2C}{\sup_t f'(t) \sqrt{C(s, \alpha, \beta)}} \right)^{2/3} C'(s, \alpha, \beta) \left(\frac{\log n}{n} \right)^{1/3} \\ &\geq (2C)^{2/3} \frac{C'(s, \alpha, \beta)}{C(s, \alpha, \beta)} \left(\frac{\log n}{nR^2} \right)^{1/(1+2s)}. \end{aligned}$$

Then, we conclude with the same arguments as in the case $s \in (0, 1]$. □

7.5. Proofs for Section 4

In the sequel, we denote by $H_{\bar{n}}$ the function $F_{\bar{n}}^{\text{reg}} - \sigma_0 G_{\bar{n}}$, which is continuous and piecewise linear on $[0, 1]$: $H_{\bar{n}}$ is the cumulative sum diagram of the points $f_n(i/\bar{n})$, $1 \leq i \leq \bar{n}$, with equal weights $1/\bar{n}$.

Proof of Theorem 4.1. Assume $f \in \mathcal{D}$. Then, $H_{\bar{n}}$ has decreasing slopes, so it is concave on $[0, 1]$, and even linear when f is constant over $[0, 1]$. Since $\sigma_0/\hat{\sigma}_0 = \sigma/\hat{\sigma}$, we deduce as in the proof of Theorem 2.2 that

$$\max_{I \in \mathcal{C}_{\bar{n}}} \widehat{S}_{\bar{n}}^{\text{reg}, I} \leq (\sigma/\hat{\sigma}) Z_n^{\text{reg}} \tag{46}$$

with equality when f is constant over $[0, 1]$. According to Cochran's theorem, $\bar{n}\hat{\sigma}^2/\sigma^2$ is independent of Z_n^{reg} and distributed as a non-central chi-square variable with \bar{n} degrees of freedom and non-centrality parameter

$$\delta_f^2 = (2\sigma^2)^{-1} \sum_{i=1}^{\bar{n}} (f(2i/n) - f((2i-1)/n))^2. \tag{47}$$

In particular, when f is constant over $[0, 1]$,

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n/2} (\epsilon_{2i} - \epsilon_{2i-1})^2 := \hat{\sigma}_\epsilon^2,$$

where $\bar{n}\widehat{\sigma}_\epsilon^2/\sigma^2$ is a $\chi^2(\bar{n})$ variable independent of Z_n^{reg} . Thus, we deduce from (46) and the stochastic order between non-central chi-square variables with same degrees of freedom (see, e.g., [20], Chapter 29) that

$$\begin{aligned} \mathbb{P}_f\left(\max_{I \in \mathcal{C}_{\bar{n}}} \widehat{S}_n^{\text{reg}, I} > r_{\alpha, n}\right) &\leq \mathbb{E}_f\left[\mathbb{P}_f\left(\widehat{\sigma}^2/\sigma^2 < (Z_n^{\text{reg}}/r_{\alpha, n})^2 \mid Z_n^{\text{reg}}\right)\right] \\ &\leq \mathbb{E}\left[\mathbb{P}\left(\widehat{\sigma}_\epsilon^2/\sigma^2 < (Z_n^{\text{reg}}/r_{\alpha, n})^2 \mid Z_n^{\text{reg}}\right)\right] \\ &\leq \mathbb{P}\left((\sigma/\widehat{\sigma}_\epsilon)Z_n^{\text{reg}} > r_{\alpha, n}\right) \end{aligned}$$

with equalities in the case where f is constant over $[0, 1]$. Both $\widehat{\sigma}_\epsilon$ and Z_n^{reg} are continuously distributed, so their independence, together with the definition of $r_{\alpha, n}$, shows that the latter probability is equal to α . □

Proof of Theorem 4.2. As in Sections 7.1 to 7.4, we may assume that $\sigma = 1$. The line of proof of Theorem 4.2 is close to that of Theorem 3.2. Indeed, it relies on the discrete versions of Lemmas 2.2 and 7.1 stated below. □

Lemma 7.3. *For all $\alpha \in (0, 1)$ and $n \geq 18 \log(2/\alpha)$,*

$$r_{\alpha, n} \leq 2\sqrt{6 \log\left(\frac{2|\mathcal{C}_{\bar{n}}|(2-\alpha)}{\alpha}\right)}.$$

Lemma 7.4. *Let α, β in $(0, 1)$, $\mathcal{C}_{\bar{n}}$ be the collection (25), $L > 0$, $s \in (0, 2]$, and $R > 0$. Assume that $n \geq 18 \log(2/\alpha)$ and either that $f \in \mathcal{F}(s, R)$ for some $s \in (0, 1]$ and $R \leq n^s$, or that $f \in \mathcal{F}(s, R, L)$ for some $s \in (1, 2]$ and $L \leq n$. Then, there exists a positive number $C(\alpha, \beta)$ depending only on α and β such that, for all f satisfying*

$$\max_{I \in \mathcal{C}_{\bar{n}}} \sqrt{\frac{\bar{n}}{|I|}} \sup_{t \in I} (\widehat{H}_n^I(t) - H_n(t)) \geq C(\alpha, \beta) \sqrt{\log \bar{n}}, \tag{48}$$

it holds that

$$\mathbb{P}_f\left(\max_{I \in \mathcal{C}_{\bar{n}}} \widehat{S}_n^{\text{reg}, I} > r_{\alpha, n}\right) \geq 1 - \beta.$$

Detailed proofs of both lemmas and Theorem 4.2 are given in [1].

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Supplementary Material

Supplement to “Testing monotonicity via local least concave majorants” (DOI: [10.3150/12-BEJ496SUPP](https://doi.org/10.3150/12-BEJ496SUPP); .pdf). We collect in the supplement [1] the most technical proofs. Specifically, we prove how to reduce to the case $\sigma = 1$, we prove (19) and we provide a detailed proof for (17) and all results in Section 4.

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