

On the local approximation of mean densities of random closed sets

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Mean density of lower dimensional random closed sets, as well as the mean boundary density of full dimensional random sets, and their estimation are of great interest in many real applications. Only partial results are available so far in current literature, under the assumption that the random set is either stationary, or it is a Boolean model, or it has convex grains. We consider here non-stationary random closed sets (not necessarily Boolean models), whose grains have to satisfy some general regularity conditions, extending previous results. We address the open problem posed in (Bernoulli 15 (2009) 1222–1242) about the approximation of the mean density of lower dimensional random sets by a pointwise limit, and to the open problem posed by Matheron in (Random Sets and Integral Geometry (1975) Wiley) about the existence (and its value) of the so-called specific area of full dimensional random closed sets. The relationship with the spherical contact distribution function, as well as some examples and applications are also discussed.

Keywords: mean density; Minkowski content; random measure; specific area; stochastic geometry

1. Introduction

We remind that a *random closed set* Θ in \mathbb{R}^d is a measurable map

$$\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \longrightarrow (\mathbb{F}, \sigma_{\mathbb{F}}),$$

where \mathbb{F} denotes the class of the closed subsets in \mathbb{R}^d , and $\sigma_{\mathbb{F}}$ is the σ -algebra generated by the so called *Fell topology*, or *hit-or-miss topology*, that is the topology generated by the set system

$$\{\mathcal{F}_G : G \in \mathcal{G}\} \cup \{\mathcal{F}^C : C \in \mathcal{C}\},$$

where \mathcal{G} and \mathcal{C} are the system of the open and compact subsets of \mathbb{R}^d , respectively (e.g., see [22]). We say that a random closed set $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \rightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$ satisfies a certain property (e.g., Θ has Hausdorff dimension n) if Θ satisfies that property \mathbb{P} -a.s.; throughout the paper we shall deal with countably \mathcal{H}^n -rectifiable random closed sets. For a discussion about measurability of $\mathcal{H}^n(\Theta)$, we refer to [7,28].

Let Θ_n be a set of locally finite \mathcal{H}^n -measure; then it induces a random measure μ_{Θ_n} defined by

$$\mu_{\Theta_n}(A) := \mathcal{H}^n(\Theta_n \cap A), \quad A \in \mathcal{B}_{\mathbb{R}^d},$$

and the corresponding expected measure

$$\mathbb{E}[\mu_{\Theta_n}](A) := \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)], \quad A \in \mathcal{B}_{\mathbb{R}^d}.$$

Whenever $\mathbb{E}[\mu_{\Theta_n}]$ is absolutely continuous with respect to \mathcal{H}^d , its density (or Radon–Nikodym derivative) with respect to \mathcal{H}^d is called *mean density of Θ_n* , and it is denoted by λ_{Θ_n} .

The problem of the evaluation and the estimation of the mean density of lower dimensional random closed sets (i.e., with Hausdorff dimension less than d), and in particular of the mean surface density $\lambda_{\partial\Theta}$ for full dimensional random sets, is of great interest in several real applications. We mention, for instance, applications in image analysis (e.g., [17] and reference therein), in medicine (e.g., in studying tumor growth [4]), and in material science in phase-transition models (e.g., [27]). (See also [1,8,10] and references therein.)

In particular, we recall that in the well-known seminal book by Matheron on random closed sets [22], page 50, the so-called *specific area* σ_{Θ} is defined by

$$\sigma_{\Theta}(x) := \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r}, \quad (1)$$

where $\Theta_{\oplus r}$ is the parallel set of Θ at distance $r > 0$, that is, $\Theta_{\oplus r} := \{s \in \mathbb{R}^d : \text{dist}(x, \Theta) \leq r\}$; it is introduced as a probabilistic version of the derivative at 0 of the volume function $V(r) := \mathcal{H}^d(\Theta_{\oplus r})$, and so, whenever the limit exists, as a possible approximation of what we denote by $\lambda_{\partial\Theta}$, the mean boundary density of Θ . The problem of the existence of σ_{Θ} is left as an open problem in [22] (apart from particular cases as stationary random closed sets).

More recently, in [1] the problem of the approximation of the mean density λ_{Θ_n} of lower dimensional non-stationary random closed sets is faced under quite general regularity assumptions on the rectifiability of Θ_n . More precisely, an approximation of λ_{Θ_n} in weak form is proved in [1], Theorem 4; namely

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}} dx = \int_A \lambda_{\Theta_n}(x) dx. \quad (2)$$

The possibility of exchanging limit and integral in the above expression when Θ_n is not stationary with $n > 0$, was left as open problem in [1], Remark 8. (The stationary and the 0-dimensional cases are trivial.)

A first attempt to solve the above mentioned open problems (the one for σ_{Θ} posed by Matheron, and the one for λ_{Θ_n} with $n < d$ posed in [1]), is given in [26], where explicit results are proven for inhomogeneous Boolean models.

The aim of the present paper is to address such open problems for more general random closed sets. Indeed, even if Boolean models are widely studied in stochastic geometry (e.g., see [6]), it is clear that they cannot be taken as model for many real situations in applications. Thus, we revisit here some results in [26], addressing the two mentioned open problems; we provide sufficient conditions on lower dimensional random sets Θ_n so that

$$\lambda_{\Theta_n}(x) = \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n} r^{d-n}}, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d, \quad (3)$$

and so that the specific area σ_{Θ} defined as limit in (1) exists, in the case of random sets Θ with non-negligible \mathcal{H}^d -measure. Such results might allow to face a wider class of possible applications; indeed, for instance, the statistical estimator $\widehat{\lambda}_{\Theta_n}^N(x)$ of the mean density $\lambda_{\Theta_n}(x)$,

introduced in [26] and which we recall here in Corollary 13, can now be applied to very general lower dimensional random sets Θ_n , not only in stationary settings or to Boolean models, and so also to non-stationary germ-grains model whose grains are not assumed to be independent. We also mention here that the estimation of λ_{Θ_n} and σ_{Θ} might be considered as the stochastic analogous to the estimation of a non-random unknown support, and the stochastic counterpart of boundary estimation for a given support, respectively (see, e.g., [5,11]); this might lead to possible further research on this topics.

The plan of the paper is the following: preliminary notions and known results on the so-called Minkowski content of sets and on point processes and germ-grain models are briefly recalled in Section 2. In Section 3, we answer to the open problem posed in [1] mentioned above, that is we prove equation (3); we also provide an explicit expression for $\lambda_{\Theta_n}(x)$. A natural estimator follows as a corollary. Further results and remarks are discussed in the final part of the section; known results on the special case of Boolean models follow here as particular case. In Section 4, random sets with non-negligible \mathcal{H}^d -measure are considered; by recalling recent results on the outer Minkowski content notion we answer to the open problem posed by Matheron in [22] about the existence of the specific area σ_{Θ} of random sets Θ which can be represented as *one-grain* random sets. The relationship between σ_{Θ} , the mean boundary density $\lambda_{\partial\Theta}$ of Θ , and its spherical contact distribution function is studied. Some explicit formulas for the derivative of the contact distribution are also proved.

2. Preliminaries and notation

In this section, we recall basic definitions, notation and results on point processes and geometric measure theory which we shall use in the following.

2.1. The Minkowski content notion and related results

Throughout the paper, \mathcal{H}^n is the n -dimensional Hausdorff measure, dx stands for $\mathcal{H}^d(dx)$, and $\mathcal{B}_{\mathcal{X}}$ is the Borel σ -algebra of any space \mathcal{X} . $B_r(x)$, b_n and \mathbf{S}^{d-1} will denote the closed ball with centre x and radius $r \geq 0$, the volume of the unit ball in \mathbb{R}^n and the unit sphere in \mathbb{R}^d , respectively. We remind that a compact set $A \subset \mathbb{R}^d$ is called *n -rectifiable* ($0 \leq n \leq d-1$ integer) if it can be written as the image of a compact subset of \mathbb{R}^n by a Lipschitz map from \mathbb{R}^n to \mathbb{R}^d ; more in general, a closed subset A of \mathbb{R}^d is said to be *countably \mathcal{H}^n -rectifiable* if there exist countably many n -dimensional Lipschitz graphs $\Gamma_i \subset \mathbb{R}^d$ such that $A \setminus \bigcup_i \Gamma_i$ is \mathcal{H}^n -negligible. (For definitions and basic properties of Hausdorff measure and rectifiable sets see, e.g., [3,13, 15].)

The notion of *n -dimensional Minkowski content* will play a fundamental role throughout the paper. We recall that, given a subset A of \mathbb{R}^d and an integer n with $0 \leq n \leq d$, the *n -dimensional Minkowski content of A* is defined as

$$\mathcal{M}^n(A) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A \oplus r)}{b_{d-n} r^{d-n}}, \quad (4)$$

whenever the limit exists finite. Well known general results about the existence of the Minkowski content of closed sets in \mathbb{R}^d are related to rectifiability properties of the involved sets. In particular, the following theorem is proved in [3], page 110. (We call Radon measure in \mathbb{R}^d any non-negative and σ -additive set function defined on $\mathcal{B}_{\mathbb{R}^d}$ which is finite on bounded sets.)

Theorem 1. *Let $A \subset \mathbb{R}^d$ be a countably \mathcal{H}^n -rectifiable compact set, and assume that*

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in A, \forall r \in (0, 1) \quad (5)$$

holds for some $\gamma > 0$ and some Radon measure $\eta \ll \mathcal{H}^n$ in \mathbb{R}^d . Then $\mathcal{M}^n(A) = \mathcal{H}^n(A)$.

Condition (5) is a kind of quantitative non-degeneracy condition which prevents A from being too sparse; simple examples show that $\mathcal{M}^n(A)$ can be infinite, and $\mathcal{H}^n(A)$ arbitrarily small, when this condition fails [2,3]. The above theorem extends (see [3], Theorem 2.106) the well-known Federer's result [15], page 275, to countably \mathcal{H}^n -rectifiable compact sets; in particular for any n -rectifiable compact set $A \subset \mathbb{R}^d$ there exists a suitable measure η satisfying (5) (see [2], Remark 1). As a consequence, for instance in the case $n = d - 1$, the boundary of any convex body or, more in general, of a set with positive reach, and the boundary of a set with Lipschitz boundary satisfy condition (5). Note also that if a Radon measure η as in Theorem 1 exists, then it can be assumed to be a probability measure, without loss of generality (e.g., see [26]); the next theorem is proved in [26], and provides a result on the existence of the limit in (4) when the measure \mathcal{H}^d is replaced by a measure having density f with respect to \mathcal{H}^d , and so it may be seen as a generalization of the theorem above. $\text{disc } f$ denotes the set of all the points of discontinuity of f .

Theorem 2. *Let $\mu \ll \mathcal{H}^d$ be a positive measure in \mathbb{R}^d , admitting a locally bounded density f , and $A \subset \mathbb{R}^d$ be a countably \mathcal{H}^n -rectifiable compact set such that condition (5) holds for some $\gamma > 0$ and some probability measure $\eta \ll \mathcal{H}^n$ in \mathbb{R}^d . If $\mathcal{H}^n(\text{disc } f) = 0$, then*

$$\lim_{r \downarrow 0} \frac{\mu(A \oplus r)}{b_{d-n} r^{d-n}} = \int_A f(x) \mathcal{H}^n(dx).$$

2.2. Point processes

Here we report some known facts from the theory of point processes just for establishing notation which will be used later. For a more complete exposition of the theory of point processes, see, for example, [12]. Roughly speaking a point process $\tilde{\Phi}$ in \mathbb{R}^d is a locally finite collection $\{\xi_i\}_{i \in \mathbb{N}}$ of random points in \mathbb{R}^d . Formally, $\tilde{\Phi}$ can be seen as a random counting measure, that is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into the space of locally finite counting measures on \mathbb{R}^d . $\tilde{\Phi}$ is called simple if $\tilde{\Phi}(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$; we shall always consider simple point processes.

The measure $\tilde{\Lambda}(A) := \mathbb{E}[\tilde{\Phi}(A)]$ on $\mathcal{B}_{\mathbb{R}^d}$ is called *intensity measure* of $\tilde{\Phi}$; whenever it is absolutely continuous with respect to \mathcal{H}^d , its density is called *intensity* of $\tilde{\Phi}$. It is well known the so-called Campbell's formula (e.g., see [6], page 28), which states that for any measurable

function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ the following holds

$$\mathbb{E} \left[\sum_{x \in \tilde{\Phi}} f(x) \right] = \int_{\mathbb{R}^d} f(x) \tilde{\Lambda}(dx).$$

Another important measure associated to a point process $\tilde{\Phi}$ is the so-called *second factorial moment measure* $\tilde{v}_{[2]}$ of $\tilde{\Phi}$; it is the measure on $\mathcal{B}_{\mathbb{R}^{2d}}$ defined by (e.g., see [6,24])

$$\int f(x, y) \tilde{v}_{[2]}(d(x, y)) = \mathbb{E} \left[\sum_{x, y \in \tilde{\Phi}, x \neq y} f(x, y) \right]$$

for any non-negative measurable function f on \mathbb{R}^{2d} . Moreover, $\tilde{\Phi}$ is said to have *second moment density* \tilde{g} if $\tilde{v}_{[2]} = \tilde{g}v^{2d}$, that is

$$\tilde{v}_{[2]}(C) = \int_C \tilde{g}(x, y) dx dy$$

for any compact $C \subset \mathbb{R}^{2d}$. Informally, $\tilde{g}(x, y)$ represents the joint probability that there are points at two specific locations x and y :

$$\tilde{g}(x, y) dx dy \sim \mathbb{P}(\tilde{\Phi}(dx) > 0, \tilde{\Phi}(dy) > 0).$$

A generalization of the above notion is the so-called *marked point process*. We recall that a marked point process $\Phi = \{\xi_i, K_i\}_{i \in \mathbb{N}}$ on \mathbb{R}^d with marks in a complete separable metric space (c.s.m.s.) \mathbf{K} is a point process on $\mathbb{R}^d \times \mathbf{K}$ with the property that the unmarked process $\{\tilde{\Phi}(B) : B \in \mathcal{B}_{\mathbb{R}^d}\} := \{\Phi(B \times \mathbf{K}) : B \in \mathcal{B}_{\mathbb{R}^d}\}$ is a point process in \mathbb{R}^d . \mathbf{K} is called *mark space*, while the random element K_i of \mathbf{K} is the *mark associated to the point* ξ_i . Φ is said to be *stationary* if the distribution of $\{\xi_i + x, K_i\}_i$ is independent of $x \in \mathbb{R}^d$.

If the marks are independent and identically distributed, and independent of the unmarked point process $\tilde{\Phi}$, then Φ is said to be an *independent marking* of $\tilde{\Phi}$.

The intensity measure of Φ , say Λ , is a σ -finite measure on $\mathcal{B}_{\mathbb{R}^d \times \mathbf{K}}$ defined as $\Lambda(B \times L) := \mathbb{E}[\Phi(B \times L)]$, the mean number of points of Φ in B with marks in L . We recall that a Campbell's formula for marked point processes holds as well [6]:

$$\mathbb{E} \left[\sum_{(x, K) \in \Phi} f(x, K) \right] = \int_{\mathbb{R}^d \times \mathbf{K}} f(x, K) \Lambda(d(x, K)). \quad (6)$$

Since \mathbf{K} is a c.s.m.s. and $\tilde{\Lambda}$ is a σ -finite measure, it is possible to factorize Λ in the following way [21]:

$$\Lambda(d(x, K)) = \kappa(x, dK) \tilde{\Lambda}(dx),$$

where $\tilde{\Lambda}$ is the intensity measure of the unmarked process $\tilde{\Phi}$, and $\kappa(x, \cdot)$ is a probability measure on \mathbf{K} for all $x \in \mathbb{R}^d$, called the *mark distribution at point* x . A common assumption (e.g., see [19])

is that there exist a measurable function $\lambda: \mathbb{R}^d \times \mathbf{K} \rightarrow \mathbb{R}_+$ and a probability measure Q on \mathbf{K} such that

$$\Lambda(d(x, K)) = \lambda(x, K) dx Q(dK), \quad (7)$$

this happens if and only if $\kappa(x, \cdot)$ is absolutely continuous with respect to Q for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$.

If Φ is stationary, then its intensity measure is of the type $\Lambda = \lambda \nu^d \otimes Q$ for some $\lambda > 0$ and Q probability measure on \mathbf{K} . If Φ is an independent marking of $\tilde{\Phi}$, then $\Lambda(d(x, K)) = \tilde{\Lambda}(dx)Q(dK)$, where Q is a probability measure on \mathbf{K} , called distribution of the marks.

Let $(\mathbb{R}^d \times \mathbf{K})^2 := \mathbb{R}^d \times \mathbf{K} \times \mathbb{R}^d \times \mathbf{K}$; the *second factorial moment measure* $\nu_{[2]}$ of Φ is the measure on $\mathcal{B}_{(\mathbb{R}^d \times \mathbf{K})^2}$ so defined [24]

$$\int f(x_1, K_1, x_2, K_2) \nu_{[2]}(d(x_1, K_1, x_2, K_2)) = \mathbb{E} \left[\sum_{\substack{(x_i, K_i), (x_j, K_j) \in \Phi, \\ x_i \neq x_j}} f(x_i, K_i, x_j, K_j) \right] \quad (8)$$

for any non-negative measurable function f on $(\mathbb{R}^d \times \mathbf{K})^2$. By denoting $\tilde{\nu}_{[2]}$ the second factorial moment measure of the unmarked process $\tilde{\Phi}$, for any $B_1, B_2 \in \mathbf{K}$ the measure $\nu_{[2]}(\cdot \times B_1 \times \cdot \times B_2)$ is absolutely continuous with respect to $\tilde{\nu}_{[2]}$; moreover, if $\tilde{\nu}_{[2]}$ is σ -finite then

$$\nu_{[2]}(d(x_1, K_1, x_2, K_2)) = M_{x_1, x_2}(d(K_1, K_2)) \tilde{\nu}_{[2]}(d(x_1, x_2)), \quad (9)$$

where M_{x_1, x_2} is a measure on \mathbf{K}^2 for any fixed x_1 and x_2 , called *two-point mark distribution*. Informally, $\nu_{[2]}(d(x_1, K_1, x_2, K_2))$ represents the joint probability that there are points at two specific locations x_1 and x_2 with marks K_1 and K_2 , respectively.

Similarly to Λ , we shall assume that there exist a measurable function $g: (\mathbb{R}^d \times \mathbf{K})^2 \rightarrow \mathbb{R}_+$ and a probability measure $Q_{[2]}$ on \mathbf{K}^2 such that

$$\nu_{[2]}(d(x_1, K_1, x_2, K_2)) = g(x_1, K_1, x_2, K_2) dx_1 dx_2 Q_{[2]}(d(K_1, K_2)). \quad (10)$$

We remind that if Φ is a marked Poisson point process with intensity measure $\Lambda(d(x, K)) = \kappa(x, dK) \tilde{\Lambda}(dx)$, then $\tilde{\nu}_{[2]} = \tilde{\Lambda} \otimes \tilde{\Lambda}$ and $\nu_{[2]} = \Lambda \otimes \Lambda$, and so

$$M_{x, y}(d(s, t)) = \kappa(x, ds) \kappa(y, dt);$$

in particular, by the assumptions (7) and (10) it follows

$$\begin{aligned} g(x_1, K_1, x_2, K_2) &= \lambda(x_1, K_1) \lambda(x_2, K_2), \\ Q_{[2]}(d(K_1, K_2)) &= Q(dK_1) Q(dK_2). \end{aligned} \quad (11)$$

We also recall that point processes can be considered on quite general metric spaces. In particular, a point process in \mathcal{C}^d , the class of compact subsets of \mathbb{R}^d , is called *particle process* (e.g., see [6] and references therein). It is well known that, by a *center map*, a particle process can be transformed into a marked point process Φ on \mathbb{R}^d with marks in \mathcal{C}^d , by representing any compact set C as a pair (x, Z) , where x may be interpreted as the ‘‘location’’ of C and $Z := C - x$ the

“shape” (or “form”) of C (e.g., see [6], page 192 and [20]). In this case the marked point process $\Phi = \{(X_i, Z_i)\}$ is also called *germ-grain model*. In case of independent marking, the grains Z_i 's are i.i.d. as a typical grain Z_0 with mark distribution Q , which is also called, in this case, *grain distribution* or *distribution of the typical grain*.

Every random closed set in \mathbb{R}^d can be represented as a germ-grain model, and so by a suitable marked point process $\Phi = \{X_i, Z_i\}$. In many examples and applications the random sets Z_i are uniquely determined by suitable random parameters $S \in \mathbf{K}$. For instance, in the very simple case of random balls, $\mathbf{K} = \mathbb{R}_+$ and S is the radius of a ball centred in the origin; in applications to birth-and-growth processes, in some models $\mathbf{K} = \mathbb{R}^d$ and S is the spatial location of the nucleus (e.g., [1], Example 2); in segment processes in \mathbb{R}^2 , $\mathbf{K} = \mathbb{R}_+ \times [0, 2\pi]$ and $S = (L, \alpha)$ where L and α are the random length and orientation of the segment through the origin, respectively (e.g., [26], Example 2); etc. So, in order to use similar notation to previous works (e.g., [26,27]), we shall consider random sets Θ described by marked point processes $\Phi = \{(X_i, S_i)\}$ in \mathbb{R}^d with marks in a suitable mark space \mathbf{K} so that $Z_i = Z(S_i)$ is a random set containing the origin:

$$\Theta(\omega) = \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + Z(s_i), \quad \omega \in \Omega. \quad (12)$$

We also recall that whenever Φ is a marked Poisson point process, Θ is said to be a *Boolean model*.

The intensity measure Λ of Φ is commonly assumed to be such that the mean number of grains hitting any compact subset of \mathbb{R}^d is finite, which is equivalent to say that the mean number of grains hitting the ball $B_R(0)$ is finite for any $R > 0$:

$$\mathbb{E} \left[\sum_{(x_i, s_i) \in \Phi} \mathbf{1}_{(-Z(s_i)) \oplus R}(x_i) \right] \stackrel{(6)}{=} \int_{\mathbb{R}^d \times \mathbf{K}} \mathbf{1}_{(-Z(s)) \oplus R}(x) \Lambda(d(x, s)) < \infty \quad \forall R > 0. \quad (13)$$

3. Mean densities of lower dimensional random closed sets

3.1. Assumptions

Let Θ_n be a random closed set in \mathbb{R}^d with integer Hausdorff dimension $0 < n < d$ as in (12), where Φ has intensity measure $\Lambda(d(x, s)) = \lambda(x, s) dx Q(ds)$ and second factorial moment measure $\nu_{[2]}(d(x, s, y, t)) = g(x, s, y, t) dx dy Q_{[2]}(d(s, t))$ such that the following assumptions are fulfilled:

(A1) for any $(y, s) \in \mathbb{R}^d \times \mathbf{K}$, $y + Z(s)$ is a countably \mathcal{H}^n -rectifiable and compact subset of \mathbb{R}^d , such that there exists a closed set $\Xi(s) \supseteq Z(s)$ such that $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s)) Q(ds) < \infty$ and

$$\mathcal{H}^n(\Xi(s) \cap B_r(x)) \geq \gamma r^n \quad \forall x \in Z(s), \forall r \in (0, 1) \quad (14)$$

for some $\gamma > 0$ independent of y and s ;

(A2) for any $s \in \mathbf{K}$, $\mathcal{H}^n(\text{disc}(\lambda(\cdot, s))) = 0$ and $\lambda(\cdot, s)$ is locally bounded such that for any compact $K \subset \mathbb{R}^d$

$$\sup_{x \in K_{\oplus \text{diam}(Z(s))}} \lambda(x, s) \leq \tilde{\xi}_K(s) \quad (15)$$

for some $\tilde{\xi}_K(s)$ with $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s)) \tilde{\xi}_K(s) \mathcal{Q}(ds) < \infty$;

(A3) for any $(s, y, t) \in \mathbf{K} \times \mathbb{R}^d \times \mathbf{K}$, $\mathcal{H}^n(\text{disc}(g(\cdot, s, y, t))) = 0$ and $g(\cdot, s, y, t)$ is locally bounded such that for any compact $K \subset \mathbb{R}^d$ and $a \in \mathbb{R}^d$,

$$\mathbf{1}_{(a-Z(t))_{\oplus 1}}(y) \sup_{x \in K_{\oplus \text{diam}(Z(s))}} g(x, s, y, t) \leq \xi_{a,K}(s, y, t) \quad (16)$$

for some $\xi_{a,K}(s, y, t)$ with $\int_{\mathbb{R}^d \times \mathbf{K}^2} \mathcal{H}^n(\Xi(s)) \xi_{a,K}(s, y, t) dy \mathcal{Q}_{[2]}(ds, dt) < \infty$.

Before stating our main results, we briefly discuss the above assumptions. As mentioned in the [Introduction](#), we want to find sufficient conditions such that equation (3) holds for a general class of random closed sets Θ_n , so answering to the open problem stated in [1], Remark 8. We point out that such a result has been proved recently in [26] for Boolean models with position-independent grains, and so only in the case in which Φ is a Poisson point process with intensity measure Λ of the type $\Lambda(d((x, s))) = \lambda(x) dx \mathcal{Q}(ds)$. In that work, the assumption that Φ was a marked Poisson point process allowed to apply the explicit expression of the capacity functional of Θ_n , both in proving the exchange between limit and integral in (2), and in providing an explicit formula for the mean density λ_{Θ_n} of Θ_n in terms of its intensity measure Λ . Actually, in order to prove equation (3), the knowledge of the capacity functional of Θ_n is not necessary, by making use of Campbell's formula. Nevertheless, for a general random set Θ_n as in the above assumptions, and so without the further assumption that Φ is a marked Poisson process, we need to introduce also the second factorial moment measure of Φ , and the related assumption (A3). Of course, considering here a generic random set Θ_n (point process Φ), it obvious that the above assumptions are similar to (actually, they generalize) those which appear in [26]; as a matter of fact (A1') and (A2') in [26] coincide with (A1) and (A2) above in the case of independent marking. We also point out that in the particular case of Boolean models, the second factorial moment measure $\nu_{[2]}$ is given in terms of the intensity measure Λ , and so the function g in terms of λ by (11); this is the reason why here assumption (A3) appears, whereas it is already contained in (A1') and (A2') in [26], Theorem 3.13 (see also Corollary 8 below).

We mention also that taking $\nu_{[2]}$ of the type $\nu_{[2]}(d(x, s, y, t)) = g(x, s, y, t) dx dy \mathcal{Q}_{[2]}(d(s, t))$ is in accordance to the assumption in [19], Proposition 4.9, where contact distributions of general germ-grain models with compact convex grains are considered; in that paper $\nu_{[2]}$ is assumed to be absolutely continuous with respect to the product measure $\mathcal{H}^d \otimes \mu$, where μ is σ -finite measure on $\mathbf{K} \times \mathbb{R}^d \times \mathbf{K}$.

Moreover, note that the measure $\mathcal{H}^n(\Xi(s) \cap \cdot)$ in (A1) plays the same role as the measure η of Theorem 1; indeed (A1) might be seen as the stochastic version of (5). (See also [26], Remark 3.6, and the examples discussed in [1].) Roughly speaking, such an assumption tells us that each possible grain associated to any point x of the underlying point process $\tilde{\Phi}$ is sufficiently regular, so that it admits n -dimensional Minkowski content; this explains also why requiring the existence of a constant γ as in (A1) independent on y and s is not too restrictive (see also the example

below about this). Note that the condition $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s)) Q(ds) < \infty$ means that the \mathcal{H}^n -measure of the grains is finite in mean. In order to clarify better the meaning of assumption (A1), let us consider the following simple example.

Example 1. Let Θ_1 be a germ grain model with segments as grains, with random length. (As it will be clear, the orientation of the segments does not take part to the validity of (A1).) Let us only assume that the mean length of the grain is finite. We may notice that the introduction of the suitable random set Ξ is needed only if the length of the segments could be indefinitely close to 0. Indeed, let us first consider the case in which the length is bounded from below by a positive constant, for instance $\mathcal{H}^1(Z(s)) \geq l > 0$ for any $s \in \mathbf{K}$; then

$$\mathcal{H}^1(Z(s) \cap B_r(x)) \geq \min\{l, 1\}r \quad \forall x \in Z(s), \forall r \in (0, 1),$$

and so there exists $\gamma := \min\{l, 1\} > 0$, clearly independent of the position and of the length of the particular grain considered.

Now let us consider the case in which the length is not bounded from below by a positive constant (e.g., the length is uniformly distributed in $[0, L]$). In this case, $l = 0$ and so we have to introduce a suitable random set Ξ satisfying (14); a possible solution is to extend all the segments having length less than 2 (the extension can be done homothetically from the center of the segment, so that measurability of the process is preserved). In particular, for any $s \in \mathbf{K}$, let

$$\Xi(s) = \begin{cases} Z(s), & \text{if } \mathcal{H}^1(Z(s)) \geq 2, \\ Z(s) \text{ extended to length } 2, & \text{if } \mathcal{H}^1(Z(s)) < 2; \end{cases}$$

it follows that (14) holds now with $\gamma = 1$. Since we have assumed that the mean length of the segments is finite, it follows that $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s)) Q(ds) < \infty$, and so (A1) is fulfilled.

Note that we have chosen segments as grains in order to make the example simpler, but it is now clear that the same argument may be applied to fibre processes (in order to provide another example of a random closed set of dimension 1), or even more complicated random sets in \mathbb{R}^d with any integer dimension n .

The role of assumption (A2) and (A3) is more technical, and it will be clearer later in the proofs of the next statements. Finally, it is clear that if λ and g are bounded, the above assumptions (A2) and (A3) simplify (see also Remark 9).

3.2. Main theorem and related results

In this section, we state and prove our main theorem (Theorem 7), which provides a pointwise limit representation of the mean density λ_{Θ_n} of Θ_n . To this aim we need to prove some other related results, before. We start with the following lemma, which tells us that the grains of the random set Θ_n overlap only on a set having negligible \mathcal{H}^n -measure in mean.

Lemma 3. Let Θ_n be a random closed set in \mathbb{R}^d with integer Hausdorff dimension $0 < n < d$ as in (12), where Φ has intensity measure $\Lambda(d(x, s)) = \lambda(x, s) dx Q(ds)$ and second factorial moment measure $\nu_{[2]}(d(x, s, y, t)) = g(x, s, y, t) dx dy Q_{[2]}(d(s, t))$. Then

$$\mathbb{E} \left[\sum_{\substack{(y_i, s_i), (y_j, s_j) \in \Phi, \\ y_i \neq y_j}} \mathcal{H}^n((y_i + Z(s_i)) \cap (y_j + Z(s_j))) \right] = 0.$$

Proof. The following chain of equalities hold:

$$\begin{aligned} & \mathbb{E} \left[\sum_{\substack{(y_i, s_i), (y_j, s_j) \in \Phi, \\ y_i \neq y_j}} \mathcal{H}^n((y_i + Z(s_i)) \cap (y_j + Z(s_j))) \right] \\ & \stackrel{(8)}{=} \int_{(\mathbb{R}^d \times \mathbf{K})^2} \mathcal{H}^n((x + Z(s)) \cap (y + Z(t))) \nu_{[2]}(d(x, s, y, t)) \\ & = \int_{(\mathbb{R}^d \times \mathbf{K})^2} \left(\int_{\mathbb{R}^d} \mathbf{1}_{x+Z(s)}(u) \mathbf{1}_{y+Z(t)}(u) \mathcal{H}^n(du) \right) \nu_{[2]}(d(x, s, y, t)) \\ & = \int_{(\mathbb{R}^d \times \mathbf{K})^2} \left(\int_{\mathbb{R}^d} \mathbf{1}_{u-Z(s)}(x) \mathbf{1}_{u-Z(t)}(y) \mathcal{H}^n(du) \right) g(x, s, y, t) dx dy Q_{[2]}(ds, dt) \\ & = \int_{\mathbb{R}^d} \left(\int_{\mathbf{K}} \int_{\mathbb{R}^d} \int_{\mathbf{K}} \mathbf{1}_{u-Z(t)}(y) \int_{u-Z(s)} g(x, s, y, t) dx dy Q_{[2]}(ds, dt) \right) \mathcal{H}^n(du), \end{aligned}$$

where the last equality is implied by Fubini's theorem. The assertion follows by observing that $\int_{u-Z(s)} g(x, s, y, t) dx = 0$, because $\mathcal{H}^d(Z(s)) = 0$, being lower dimensional. \square

In order to prove our next results, we recall that in [1] it is proved that if $S \subset \mathbb{R}^d$ is a countably \mathcal{H}^n -rectifiable compact set such that

$$\eta(B_r(x)) \geq \gamma r^n \quad \forall x \in S, \forall r \in (0, 1)$$

holds for some $\gamma > 0$ and some finite measure $\eta \ll \mathcal{H}^n$ in \mathbb{R}^d , then

$$\frac{\mathcal{H}^d(S_{\oplus r})}{b_{d-n} r^{d-n}} \leq \frac{\eta(\mathbb{R}^d)}{\gamma} 2^n 4^d \frac{b_d}{b_{d-n}} \quad \forall r < 2. \quad (17)$$

Remark 4. By (17), and the proof of Lemma 3.14 in [26], we know that

$$\mathcal{H}^d(Z(s)_{\oplus R}) \leq \begin{cases} \mathcal{H}^n(\Xi(s)) \gamma^{-1} 2^n 4^d b_d R^{d-n}, & \text{if } R < 2, \\ \mathcal{H}^n(\Xi(s)) \gamma^{-1} 2^n 4^d b_d R^n, & \text{if } R \geq 2, \end{cases}$$

and so condition (13), which guarantees that the mean number of grains intersecting any compact subset of \mathbb{R}^d is finite, is fulfilled:

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbf{K}} \mathbf{1}_{(-Z(s))_{\oplus R}}(x) \Lambda(d(x, s)) \\ & \leq 2^n 4^d b_d \max\{R^{d-n}; R^d\} \int_{\mathbf{K}} \tilde{\xi}_{B_R(0)} \mathcal{H}^n(\Xi(s)) Q(ds) \stackrel{(A2)}{<} \infty \quad \forall R > 0. \end{aligned}$$

As a consequence, together with assumption (A1) which tells us that each grain has finite \mathcal{H}^n -measure in mean, it is easy to see that $\mathbb{E}[\mu_{\Theta_n}]$ is locally bounded. Moreover, by proceeding along the same lines of the proof of Proposition 3.8 in [26], we get that $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] = 0$ for any $A \subset \mathbb{R}^d$ with $\mathcal{H}^d(A) = 0$, that is $\mathbb{E}[\mu_{\Theta_n}]$ is absolutely continuous with respect to \mathcal{H}^d .

By following the hint given in [26], page 494 (there given for Boolean models, but here applied to more general Θ_n), the following proposition, which provides an explicit formula of the mean density λ_{Θ_n} of Θ_n in terms of its intensity measure, is easily proved by means of the above lemma and Campbell's formula. (See also [18] for a similar application.)

Proposition 5. *Under the hypotheses of Lemma 3,*

$$\lambda_{\Theta_n}(y) = \int_{\mathbf{K}} \int_{y-Z(s)} \lambda(x, s) \mathcal{H}^n(dx) Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } y \in \mathbb{R}^d. \quad (18)$$

Proof. By Lemma 3, we know that the event that different grains of Θ_n overlap in a subset of \mathbb{R}^d of positive \mathcal{H}^n -measure has null probability; then the following chain of equalities holds for any $A \in \mathcal{B}_{\mathbb{R}^d}$:

$$\begin{aligned} \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] &= \mathbb{E}\left[\sum_{(y_i, s_i) \in \Phi} \mathcal{H}^n((y_i + Z(s_i)) \cap A)\right] \\ &\stackrel{(6)}{=} \int_{\mathbb{R}^d \times \mathbf{K}} \mathcal{H}^n(y + Z(s) \cap A) \Lambda(d(y, s)) \\ &= \int_{\mathbb{R}^d \times \mathbf{K}} \int_{\mathbb{R}^d} \mathbf{1}_{y+Z(s)}(x) \mathbf{1}_A(x) \mathcal{H}^n(dx) \Lambda(d(y, s)) \\ &= \int_{\mathbb{R}^d \times \mathbf{K}} \int_{\mathbb{R}^d} \mathbf{1}_A(\xi) \mathbf{1}_{Z(s)}(u) \lambda(\xi - u, s) \mathcal{H}^n(du) Q(ds) d\xi \\ &= \int_A \underbrace{\left(\int_{\mathbf{K}} \int_{\mathbb{R}^d} \mathbf{1}_{Z(s)}(\xi - v) \lambda(v, s) \mathcal{H}^n(dv) Q(ds)\right)}_{=:\lambda_{\Theta_n}(\xi)} d\xi \end{aligned}$$

and so the assertion. □

In [1], Proposition 9, it has been proved that for a class of germ-grain models in \mathbb{R}^d with independent and identically distributed grains with finite \mathcal{H}^n -measure, $n < d$, the probability

that a point x belongs to the intersection of two or more enlarged grains is infinitesimally faster than r^{d-n} . The i.i.d. assumption on the grains seems to be too restrictive; we now extend it to more general germ-grain models as in above assumptions. To this end, we shall make use of the assumption (A3), which provides an integrability condition on the second factorial moment measure $\nu_{[2]}$ of Φ , similar to the condition given on the intensity measure Λ in (A2). Such a result will be fundamental in the proof of the main theorem about the validity of equation (3).

Proposition 6. *Under the assumptions in Section 3.1, the probability that a point $x \in \mathbb{R}^d$ belongs to the intersection of two or more enlarged grains $(y + Z(s))_{\oplus r}$ is infinitesimally faster than r^{d-n} .*

Proof. Let us observe that

$$\begin{aligned} & \mathbb{E} \left[\sum_{\substack{(y_i, s_i), (y_j, s_j) \in \Phi, \\ y_i \neq y_j}} \mathbf{1}_{(y_i + Z(s_i))_{\oplus r} \cap (y_j + Z(s_j))_{\oplus r}}(x) \right] \\ & \stackrel{(8)}{=} \int_{(\mathbb{R}^d \times \mathbf{K})^2} \mathbf{1}_{(x - Z(s_1))_{\oplus r}}(y_1) \mathbf{1}_{(x - Z(s_2))_{\oplus r}}(y_2) \nu_{[2]}(dy_1, ds_1, dy_2, ds_2) \\ & = \int_{\mathbb{R}^d \times \mathbf{K}^2} \left(\mathbf{1}_{(x - Z(s_2))_{\oplus r}}(y_2) \int_{(x - Z(s_1))_{\oplus r}} g(y_1, s_1, y_2, s_2) dy_1 \right) dy_2 Q_{[2]}(ds, dy). \end{aligned}$$

By Theorem 2 with $\mu = g(\cdot, s, y, t) \mathcal{H}^d$, together with (A1) and (A3), it follows

$$\begin{aligned} & \lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \int_{(x - Z(s_1))_{\oplus r}} g(y_1, s_1, y_2, s_2) dy_1 \\ & = \int_{x - Z(s_1)} g(y_1, s_1, y_2, s_2) \mathcal{H}^n(dy_1) \quad \forall s_1, s_2 \in \mathbf{K}, \forall y_2 \in \mathbb{R}^d, \end{aligned}$$

and the limit is finite being $g(\cdot, s_1, y_2, s_2)$ locally bounded by (A3), and $\mathcal{H}^n(Z(s)) < \infty$ for any $s \in \mathbf{K}$ by (A1). As $Z(s)$ is lower dimensional for any $s \in \mathbf{K}$, it is clear that

$$\lim_{r \downarrow 0} \mathbf{1}_{(x - Z(s_2))_{\oplus r}}(y_2) = 0 \quad \text{for } \mathcal{H}^d\text{-a.e. } y_2 \in \mathbb{R}^d \forall s_2 \in \mathbf{K},$$

thus

$$\lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \mathbf{1}_{(x - Z(s_2))_{\oplus r}}(y_2) \int_{(x - Z(s_1))_{\oplus r}} g(y_1, s_1, y_2, s_2) dy_1 = 0$$

for \mathcal{H}^d -a.e. $y_2 \in \mathbb{R}^d$, $\forall s_1, s_2 \in \mathbf{K}$. Furthermore, by (17), (A1) and (A3) it follows that for any $r \leq 1$

$$\begin{aligned} & \mathbf{1}_{(x - Z(s_2))_{\oplus r}}(y_2) \frac{1}{b_{d-n} r^{d-n}} \int_{(x - Z(s_1))_{\oplus r}} g(y_1, s_1, y_2, s_2) dy_1 \\ & \leq \mathbf{1}_{(x - Z(s_2))_{\oplus 1}}(y_2) \frac{\mathcal{H}^d(\Xi(s_1)_{\oplus r})}{b_{d-n} r^{d-n}} \sup_{y_1 \in (x - Z(s_1))_{\oplus r}} g(y_1, s_1, y_2, s_2) \\ & \leq \frac{2^n 4^d b_d}{\gamma b_{d-n}} \mathcal{H}^n(\Xi(s_1)) \xi_{x, B_1(x)}(s_1, y_2, s_2). \end{aligned}$$

By assumption (A3), we have that

$$\int_{\mathbb{R}^d \times \mathbf{K}^2} \frac{2^n 4^d b_d}{\gamma b_{d-n}} \mathcal{H}^n(\Xi(s_1) \xi_{x, B_1(x)}(s_1, y_2, s_2)) \, dy_2 \, \mathcal{Q}_{[2]}(ds, dr) < \infty,$$

so the dominated convergence theorem implies

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\sum_{(y_i, s_i), (y_j, s_j) \in \Phi, y_i \neq y_j} \mathbf{1}_{(y_i + Z(s_i))_{\oplus r} \cap (y_j + Z(s_j))_{\oplus r}}(x)]}{b_{d-n} r^{d-n}} = 0. \quad (19)$$

Let W_r be the random variable counting the number of pairs of different enlarged grains of Θ_n which cover the point x :

$$W_r := \#\{(i, j), i < j: x \in (y_i + Z(s_i))_{\oplus r} \cap (y_j + Z(s_j))_{\oplus r}\}; \quad (20)$$

then

$$W_r \leq \sum_{\substack{(y_i, s_i), (y_j, s_j) \in \Phi, \\ y_i \neq y_j}} \mathbf{1}_{(y_i + Z(s_i))_{\oplus r} \cap (y_j + Z(s_j))_{\oplus r}}(x),$$

and so

$$0 \leq \lim_{r \downarrow 0} \frac{\mathbb{P}(W_r > 0)}{b_{d-n} r^{d-n}} \leq \lim_{r \downarrow 0} \frac{\sum_{k=1}^{\infty} k \mathbb{P}(W_r = k)}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\mathbb{E}[W_r]}{b_{d-n} r^{d-n}} \stackrel{(19)}{\leq} 0,$$

and so the assertion. \square

We are ready now to state and prove the main result of the section.

Theorem 7. *Under the assumptions in Section 3.1,*

$$\lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n_{\oplus r}})}{b_{d-n} r^{d-n}} = \lambda_{\Theta_n}(x), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (21)$$

Proof. Let Y_r be the random variable counting the number of enlarged grains which cover the point x :

$$Y_r := \sum_{(y_i, s_i) \in \Phi} \mathbf{1}_{(y_i + Z(s_i))_{\oplus r}}(x),$$

and W_r be the random variable defined in (20). By the proof of Proposition 6, we know that

$$\mathbb{P}(W_r > 0) = o(r^{d-n}) \quad \text{and} \quad \mathbb{E}[W_r] = o(r^{d-n});$$

thus, noticing now that

$$W_r = \begin{cases} 0, & \text{if } Y = 0, 1, \\ 1, & \text{if } Y = 2, \\ \binom{Y_r}{2}, & \text{if } Y \geq 3, \end{cases}$$

we get

$$\mathbb{P}(Y_r = 2) = \mathbb{P}(W_r = 1) \leq \mathbb{P}(W_r > 0) = o(r^{d-n})$$

and

$$0 \leq \mathbb{E}[Y_r; Y_r \geq 3] \leq \mathbb{E}[W_r; Y_r \geq 3] \leq \mathbb{E}[W_r] = o(r^{d-n}),$$

which imply

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n \oplus r})}{b_{d-n} r^{d-n}} &= \lim_{r \downarrow 0} \frac{\mathbb{P}(Y_r > 0)}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\mathbb{P}(Y_r = 1) + o(r^{d-n})}{b_{d-n} r^{d-n}} = \lim_{r \downarrow 0} \frac{\mathbb{E}[Y_r]}{b_{d-n} r^{d-n}} \\ &\stackrel{(6)}{=} \lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \int_{\mathbf{K}} \int_{(x-Z(s))_{\oplus r}} \lambda(y, s) \, dy \, Q(ds). \end{aligned} \quad (22)$$

By Theorem 2 with $\mu(dy) = \lambda(y, s) \, dy$, it follows that

$$\lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \int_{(x-Z(s))_{\oplus r}} \lambda(y, s) \, dy = \int_{x-Z(s)} \lambda(y, s) \mathcal{H}^n(dy),$$

besides, by observing that

$$\begin{aligned} &\frac{1}{b_{d-n} r^{d-n}} \int_{(x-Z(s))_{\oplus r}} \lambda(y, s) \, dy \\ &\leq \frac{\mathcal{H}^d((Z(s))_{\oplus r})}{b_{d-n} r^{d-n}} \sup_{y \in (x-Z(s))_{\oplus r}} \lambda(y, s) \stackrel{(17),(15)}{\leq} \frac{2^n 4^d b_d}{\gamma b_{d-n}} \mathcal{H}^n(\Xi(s)) \tilde{\xi}_{B_2(x)}(s) \quad \forall r < 2, \end{aligned}$$

assumption (A2) and the dominated convergence theorem imply

$$\lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \int_{\mathbf{K}} \int_{(x-Z(s))_{\oplus r}} \lambda(y, s) \, dy \, Q(ds) = \int_{\mathbf{K}} \int_{x-Z(s)} \lambda(y, s) \mathcal{H}^n(dy) \, Q(ds),$$

and so, by (18),

$$\lambda_{\Theta_n}(x) = \lim_{r \downarrow 0} \frac{1}{b_{d-n} r^{d-n}} \int_{\mathbf{K}} \int_{(x-Z(s))_{\oplus r}} \lambda(y, s) \, dy \, Q(ds) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (23)$$

Finally, the assertion follows:

$$\lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n \oplus r})}{b_{d-n} r^{d-n}} \stackrel{(22),(23)}{=} \lambda_{\Theta_n}(x) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad \square$$

3.3. Corollaries and remarks

We point out that equations (18) and (21) have been proved in [26], Theorem 3.13, for a general class of Boolean models Θ_n with intensity measure Λ of the type $\Lambda(d(x, s)) = f(x) \, dx \, Q(ds)$,

and so with position-independent grains and typical grain Z_0 , by using the explicit form of the capacity functional of Θ_n . Actually, Proposition 5 and Theorem 7 generalize to Boolean models with position-dependent grains, as stated in the following corollary, under the assumptions (A1) and (A2) only, in accordance with the above mentioned result in [26].

Corollary 8 (Particular case: Boolean models). *If Θ_n is a Boolean model with intensity measure $\Lambda(d(x, s)) = \lambda(x, s) dx Q(ds)$, then all the results stated in the above section hold under assumptions (A1) and (A2).*

Proof. It is enough to note that assumption (A3) is implied by (A1) and (A2). Indeed, by (11) $g(\cdot, s, y, t) = \lambda(\cdot, s)\lambda(y, t)$, so that $g(\cdot, s, y, t)$ is locally bounded and $\mathcal{H}^n(\text{disc}(g(\cdot, s, y, t))) = 0$ by (A2), whereas (16) holds with $\xi_{a,K} := \xi_K(s)\mathbf{1}_{(a-Z(t))_{\oplus 1}}(y)\lambda(y, t)$, by observing that

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbf{K}^2} \mathcal{H}^n(\Xi(s))\xi_{a,K}(s, y, t) dy Q_{[2]}(ds, dt) \\ &= \int_{\mathbb{R}^d \times \mathbf{K}} \mathbf{1}_{(a-Z(t))_{\oplus 1}}(y)\lambda(y, t) dy Q(dt) \int_{\mathbf{K}} \mathcal{H}^n(\Xi(s))\tilde{\xi}_K(s) Q(ds), \end{aligned}$$

with $\int_{\mathbf{K}} \mathcal{H}^n(\Xi(s))\tilde{\xi}_K(s) Q(ds) < \infty$ by (A2), and

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbf{K}} \mathbf{1}_{(a-Z(t))_{\oplus 1}}(y)\lambda(y, t) dy Q(dt) &\leq \int_{\mathbf{K}} \mathcal{H}^d((a-Z(t))_{\oplus 1}) \sup_{y \in (a-Z(t))_{\oplus 1}} \lambda(y, t) Q(dt) \\ &\stackrel{(5), (A1)}{\leq} \int_{\mathbf{K}} \frac{\mathcal{H}^n(\Xi(t))}{\gamma} 2^n 4^d b_d \sup_{y \in (a-Z(t))_{\oplus 1}} \lambda(y, t) Q(dt) \\ &\stackrel{(A2)}{\leq} \int_{\mathbf{K}} \frac{\mathcal{H}^n(\Xi(t))}{\gamma} 2^n 4^d b_d \tilde{\xi}_{B_1(a)}(t) Q(dt) \stackrel{(A2)}{<} \infty. \quad \square \end{aligned}$$

Remark 9 (Independent marking). If the point process Φ is an independent marking of $\tilde{\Phi}$, then the two-point mark distribution $M_{x,y}(ds, dt)$ in (9) is independent of x and y , so that $M_{x,y}(ds, dt) = Q_{[2]}(ds, dt) = Q(ds)Q(dt)$; accordingly, $g(x, s, y, t) = \tilde{g}(x, y)$. As a consequence, assumption (A3) simplifies by replacing $g(x, s, y, t)$ with $\tilde{g}(x, y)$. We also recall that $\tilde{g}(x, y)$ can be written in terms of the so-called *pair-correlation function* $\rho(x, y)$ in this way:

$$\tilde{g}(x, y) = \rho(x, y)\lambda(x)\lambda(y).$$

Moreover, if in particular λ and g are bounded, say by c_1 and c_2 in \mathbb{R} , respectively, then the finiteness of the integral in assumptions (A2) and (A3) is trivially satisfied by (A1), by taking $\tilde{\xi}_K(s) \equiv c_1$ in (15) and $\xi_{a,K}(s, y, t) := c_2\mathbf{1}_{(a-Z(t))_{\oplus 1}}(y)$ in (16), and noticing that

$$\begin{aligned} & \int_{\mathbb{R}^d \times \mathbf{K}^2} \mathcal{H}^n(\Xi(s))\xi_{a,K}(s, y, t) dy Q_{[2]}(ds, dt) \\ &\leq c_2 \int_{\mathbf{K}} \frac{\mathcal{H}^n(\Xi(t))}{\gamma} 2^n 4^d b_d Q(dt) \int_{\mathbb{K}} \mathcal{H}^n(\Xi(s)) Q(ds) \stackrel{(A1)}{<} \infty. \end{aligned}$$

Example 2. Simple examples of point processes $\tilde{\Phi}$ having bounded intensity λ and second moment density \tilde{g} , are, for instance, the binomial process of m points in a compact region $W \subset \mathbb{R}^d$ with $\mathcal{H}^d(W) > 0$, and the Matèrn cluster process (e.g., see [6]). We remind that for the binomial process we have $\lambda(x) = m/\mathcal{H}^d(W)$ and $\tilde{g}(x, y) = m(m-1)/(\mathcal{H}^d(W))^2$; whereas for a Matèrn cluster process in \mathbb{R}^2 in which the parent process is a uniform Poisson process with intensity α , and each cluster consists of $N \sim \text{Poisson}(m)$ points independently and uniformly distributed in the ball $B_r(x)$, where x is the centre of the cluster, we have $\lambda = m\alpha$, and $\tilde{g}(x, y) = \alpha^2 m^2 + \alpha m^2 \mathcal{H}^2(B_r(x) \cap B_r(y))/(\pi^2 r^4) \leq \alpha^2 m^2 + \alpha m^2/(\pi r^2)$. Other examples of processes with bounded intensity and second moment density are considered for instance in [23]. These, together with Example 1, which gives an insight into the validity of assumption (A1), provide simple examples where all the assumptions (A1)–(A3) hold.

Example 3. We mention that an important case of random sets of dimension 1 is given by the so-called *fibre processes* (e.g., see [8]); they can taken as models in different fields, as Biology (e.g., fibre systems in soils [8], Section 3.2.3) and Medicine (e.g., modelling vessels in certain angiogenesis processes [9,10]), and it is clear that assuming stationarity or that the fibres are the grains of a Boolean model might be too restrictive in applications. Now, we have results for studying also the more general case in which the fibres are not independent of each other; note that assumptions (A1) and (A2) are generally satisfied in applications: (A1) is trivial, since fibres are usually assumed to be rectifiable (see also Example 1), while (A2) and (A3) hold whenever λ are g are, for instance, bounded and continuous, as observed in the remark above.

Moreover, Proposition 5 applies and an explicit expression for λ_{Θ_1} can be obtained in terms of the intensity measure of the process. In particular, in order to provide an explicit example, let us notice that in [26], Example 2, an explicit formula for λ_{Θ_1} is given for an inhomogeneous segment Boolean model in \mathbb{R}^2 , whose segments have random length and orientation; the same assumptions on the intensity measure also apply now to more general segment processes, not necessarily Boolean models (e.g., with $\tilde{\Phi}$ as in Example 2).

Remark 10 (“one-grain” random set). It is worth noting that, as a very particular case of point process $\tilde{\Phi}$, we may consider the case in which $\tilde{\Phi} = \{X\}$, that is it is given by only one random point X in \mathbb{R}^d . Obviously, in this case $g \equiv 0$, and only assumptions (A1) and (A2) have to be satisfied for the validity of all the results stated above. Even if this case might seem trivial, actually it can be taken as a model for several real applications, and it is of great interest, because it emerges that whenever a random closed set Θ_n can be described by a random point $X \in \mathbb{R}^d$ (not necessarily belonging to Θ_n , e.g., its centre if Θ_{d-1} is the surface of a ball centred in X with random radius R) and its random “shape” $Z := \Theta_n - X$, then we may provide sufficient conditions on Θ_n such that our main result (21) holds. Note that in this case $\Lambda(d(x, s))$ represents the probability that the point X is in the infinitesimal region dx with mark in ds . For instance, if the “shape” does not depend on the position and X is uniformly distributed in a bounded region $W \subset \mathbb{R}^d$, then $\Lambda(d(x, s)) = dx Q(ds)/\mathcal{H}^d(W)$. Then, it emerges that the key assumption on the random closed set Θ_n which implies (21) is the geometric regularity assumption (A1) on its grains. As a matter of fact, (A1) can be seen as the stochastic version of the condition (5) which ensures the existence of the n -dimensional Minkowski content of each grain, whereas (A2) and (A3) are just technical assumptions; in particular (A3) allows us to prove the statement of Proposition 6 (in the Boolean case, it is already contained in (A1) and (A2)).

Under the above assumptions, it follows in particular that Θ_n admits the so-called *local mean n -dimensional Minkowski content*, which has been introduced in [1]; namely Θ_n is said to admit local mean n -dimensional Minkowski content if the following limit exists finite for any $A \subset \mathbb{R}^d$ such that $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap \partial A)] = 0$

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{n\oplus r} \cap A)]}{b_{d-n}r^{d-n}} = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)].$$

Proposition 11. *If Θ_n satisfies assumptions (A1) and (A2), then it admits local mean n -dimensional Minkowski content.*

Sketch of the proof. It is sufficient to prove that Θ_n satisfies the hypotheses of Theorem 4 in [1].

We already observed in Remark 4 that $\mathbb{E}[\mu_{\Theta_n}]$ is finite on bounded sets.

By proceeding along the same lines as in the proof of Theorem 3.9 in [26] (here, by defining $\tilde{\Theta}(\omega) := \bigcup_{(x_i, s_i) \in \Phi(\omega)} x_i + \Xi(s_i)$, where $\Xi(s_i) \supseteq Z(s_i)$ as in (A1), and $\eta(\cdot) := \mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 2} \cap \cdot) / \mathcal{H}^n(\tilde{\Theta}(\omega) \cap W_{\oplus 2})$), it is easy to see that the hypotheses of the above mentioned theorem are fulfilled with $Y := \mathcal{H}^n(\tilde{\Theta} \cap W_{\oplus 2}) / \gamma$. \square

Remark 12. By Theorem 7 and the above proposition, the following chain of equalities holds, for any $A \subset \mathbb{R}^d$ such that $\mathcal{H}^d(\partial A) = 0$ (which implies $\mathbb{E}[\mathcal{H}^n(\Theta_n \cap \partial A)] = 0$, being $\mathbb{E}[\mu_{\Theta_n}] \ll \mathcal{H}^d$):

$$\begin{aligned} \int_A \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n}r^{d-n}} dx &= \int_A \lambda_{\Theta_n}(x) dx = \mathbb{E}[\mathcal{H}^n(\Theta_n \cap A)] \\ &= \lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d(\Theta_{n\oplus r} \cap A)]}{b_{d-n}r^{d-n}} = \lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{n\oplus r})}{b_{d-n}r^{d-n}} dx \end{aligned}$$

as if we might exchange limit and integral, answering to the open problem raised in [1], Remark 8.

As mentioned in [1], several problems in real applications are related to the estimation of the mean density of lower dimensional inhomogeneous random sets (see also [10] and reference therein); in particular, as a computer graphics representation of lower dimensional sets in \mathbb{R}^2 is anyway provided in terms of pixels, which can offer only a 2-D box approximation of points in \mathbb{R}^2 , it might be useful to have statistical estimators of the mean density λ_{Θ_n} based on the volume measure \mathcal{H}^d of the Minkowski enlargement of Θ_n . To this end, a consistent and asymptotically unbiased estimator $\widehat{\lambda}_{\Theta_n}(x)$ of $\lambda_{\Theta_n}(x)$ has been introduced in [26], based on equation (21), for a class of Boolean models with typical grain Z_0 . Having now proved that (21) holds for more general random closed sets, that is not only in stationary settings or for Boolean models, but also for non-stationary germ-grains models whose grains are not assumed to be independent each other, the same simple proof of Proposition 6.1 in [26] still applies, so that we may state the following result.

Corollary 13. *Let Θ_n satisfy the assumptions, and $\{\Theta_n^i\}_{i \in \mathbb{N}}$ be a sequence of random closed sets i.i.d. as Θ_n ; then the estimator $\widehat{\lambda}_{\Theta_n}^N(x)$ of $\lambda_{\Theta_n}(x)$ so defined*

$$\widehat{\lambda}_{\Theta_n}^N(x) := \frac{\sum_{i=1}^N \mathbf{1}_{\Theta_n^i \cap B_{R_N}(x) \neq \emptyset}}{N b_{d-n} R_N^{d-n}}$$

is asymptotically unbiased and weakly consistent for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$, if R_N is such that

$$\lim_{N \rightarrow \infty} R_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} N R_N^{d-n} = \infty.$$

Remark 14. $\widehat{\lambda}_{\Theta_n}^N(x)$ can be written also in terms of the so-called *empirical capacity functional* of Θ_n , which we recall to be defined as [16] $\widehat{T}_{\Theta_n}^N(K) := \frac{1}{N} \sum_{i=1}^N \mathbf{1}_{\Theta_n^i \cap K \neq \emptyset}$ for any compact $K \subset \mathbb{R}^d$:

$$\widehat{\lambda}_{\Theta_n}^N(x) := \frac{\widehat{T}_{\Theta_n}^N(B_{R_N}(x))}{b_{d-n} R_N^{d-n}}.$$

For a more detailed discussion on $\widehat{\lambda}_{\Theta_n}^N(x)$ and related open problems, we refer to [26], Section 6.

4. Mean surface density and spherical contact distribution

Let us now consider a random closed set Θ in \mathbb{R}^d , with $\mathcal{H}^d(\Theta) > 0$. A problem of interest is then the existence (and which is its value) of the limit

$$\sigma_{\Theta}(x) := \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r}.$$

The quantity $\sigma_{\Theta}(x)$ is usually called the *specific area* of Θ at point x , and it has been introduced in [22], page 50. The name *specific area* comes from the fact that, under suitable regularity assumptions on the boundary of Θ (e.g., when Θ has Lipschitz boundary, or it is union of convex sets, etc.), $\sigma_{\Theta}(x)$ might coincide with the mean density $\lambda_{\partial\Theta}(x)$ of $\partial\Theta$, that is the density of the measure $\mathbb{E}[\mu_{\partial\Theta}]$ on \mathbb{R}^d . Moreover, it is clearly related to the existence of the right partial derivative at $r = 0$ of the so-called *local spherical contact distribution function* H_{Θ} of Θ , the function from $\mathbb{R}_+ \times \mathbb{R}^d$ to $[0, 1]$ so defined

$$H_{\Theta}(r, x) := \mathbb{P}(x \in \Theta_{\oplus r} | x \notin \Theta). \quad (24)$$

We refer to [26] and [27] (and reference therein) for a more detailed discussion on σ_{Θ} ; we point out that only results for Boolean models with position-independent grains has been given there, whereas in [19] general germ-grains models are considered assuming that the grains are convex, so that results and techniques from convex and integral geometry can be applied. In this last mentioned paper, some formulae for contact distributions and mean densities of inhomogeneous germ-grain models are to be taken in *weak form* (e.g., [19], Theorem 4.1), unless to add further

suitable integrability assumptions (e.g., in [19], Remark 4.4, the existence of a dominating integrable function is assumed). Nevertheless, the assumption of convexity of the grains in [19] seems to be too restrictive in possible real applications, and it hides the fact that σ_Θ may differ from the mean boundary density $\lambda_{\partial\Theta}$ of Θ , as discussed in [26]. Indeed, we remind that the value of σ_Θ is strictly related to the value of the so-called *mean outer Minkowski content* of Θ (and so of its grains), which depends on the \mathcal{H}^{d-1} -measure of the set of the boundary points of Θ where the d -dimensional density of Θ is 0 or 1 or 1/2 (e.g., see [25,26] for more details on this subject). In order to extend some results provided in [26] to general random closed sets, we briefly recall basics on the outer Minkowski content notion.

4.1. d -dimensional densities and outer Minkowski content

Let $A \in \mathcal{B}_{\mathbb{R}^d}$; the quantity $\mathcal{SM}(A)$ defined as (see [2])

$$\mathcal{SM}(A) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A_{\oplus r} \setminus A)}{r},$$

provided that the limit exists finite, is called *outer Minkowski content* of A . Note that if A is lower dimensional, then $\mathcal{SM}(A) = 2\mathcal{M}^{d-1}(A)$, whereas if A is a d -dimensional set, closure of its interior, then $A_{\oplus r} \setminus A$ coincides with the outer Minkowski enlargement of ∂A at distance r .

In [25] two general classes of subsets of \mathbb{R}^d which admit outer Minkowski content has been introduced; in particular we remind the definition of the so-called *class \mathcal{O}* and a related result.

Definition 15 (The class \mathcal{O}). Let \mathcal{O} be the class of Borel sets A of \mathbb{R}^d with countably \mathcal{H}^{d-1} -rectifiable and bounded topological boundary, such that

$$\eta(B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial A, \forall r \in (0, 1)$$

holds for some $\gamma > 0$ and some probability measure η in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^{d-1} .

The d -dimensional density (briefly, *density*) of A at a point $x \in \mathbb{R}^d$ is defined by [3]

$$\delta_d(A, x) := \lim_{r \downarrow 0} \frac{\mathcal{H}^d(A \cap B_r(x))}{\mathcal{H}^d(B_r(x))},$$

provided that the limit exists. It is clear that $\delta_d(A, x)$ equals 1 for all x in the interior of A , and 0 for all x into the interior of the complement set of A , whereas different values can be attained at its boundary points. It is well known (e.g., see [3], Theorem 3.61) that if $\mathcal{H}^{d-1}(\partial A) < \infty$, then A has density either 0 or 1 or 1/2 at \mathcal{H}^{d-1} -almost every point of its boundary. For every $t \in [0, 1]$ and every \mathcal{H}^d -measurable set $A \subset \mathbb{R}^d$ let

$$A^t := \{x \in \mathbb{R}^d : \delta_d(A, x) = t\}.$$

The set of points $\partial^*A := \mathbb{R}^d \setminus (A^0 \cup A^1)$ where the density of A is neither 0 nor 1 is called *essential boundary* of A . It is proved (e.g., see [3]) that all the sets A^t are Borel sets, and that $\mathcal{H}^{d-1}(\partial^*A \cap B) = \mathcal{H}^{d-1}(A^{1/2} \cap B)$ for all $B \in \mathcal{B}_{\mathbb{R}^d}$. It follows that for any A with $\mathcal{H}^{d-1}(A) < \infty$, it holds

$$\mathcal{H}^{d-1}(A) = \mathcal{H}^{d-1}(A^{1/2}) + \mathcal{H}^{d-1}(A^0 \cap \partial A) + \mathcal{H}^{d-1}(A^1 \cap \partial A). \quad (25)$$

As Theorem 1 gives general sufficient conditions on the existence of the Minkowski content of a lower dimensional set, as the following theorem gives similar general sufficient conditions for the existence of the outer Minkowski content.

Theorem 16 ([25]). *The class \mathcal{O} is stable under finite unions and any $A \in \mathcal{O}$ admits outer Minkowski content, given by*

$$\mathcal{SM}(A) = \mathcal{H}^{d-1}(A^{1/2}) + 2\mathcal{H}^{d-1}(\partial A \cap A^0) = \mathcal{H}^{d-1}(\partial^*A) + 2\mathcal{H}^{d-1}(\partial A \cap A^0).$$

Note that $\mathcal{SM}(A) = \mathcal{H}^{d-1}(A)$ if $\mathcal{H}^{d-1}(\partial\Theta \cap (\Theta^0 \cup \Theta^1)) = 0$. A local version of the outer Minkowski content is given in [25], Proposition 4.13.

We also remind that Theorem 2 is a generalization of Theorem 1; similarly, the next theorem might be seen as a generalization of Theorem 16.

Theorem 17 ([26]). *Let μ be a positive measure in \mathbb{R}^d absolutely continuous with respect to \mathcal{H}^d with locally bounded density f , and let A belong to \mathcal{O} . If $\mathcal{H}^{d-1}(\text{disc } f) = 0$, then*

$$\lim_{r \downarrow 0} \frac{\mu(A \oplus r \setminus A)}{r} = \int_{\partial^*A} f(x) \mathcal{H}^{d-1}(dx) + 2 \int_{\partial A \cap A^0} f(x) \mathcal{H}^{d-1}(dx). \quad (26)$$

4.2. Specific area and mean surface density

Let us consider a random closed set Θ in \mathbb{R}^d with $\mathcal{H}^d(\Theta) > 0$, such that it might be represented as an “one-grain” random set by giving its random shape Z and its random location y , that is by giving a marked point process $\Phi = (y, s)$ with $\mathbb{P}(\Phi(\mathbb{R}^d \times \mathbf{K}) > 1) = 0$, so that

$$\Theta = y + Z(s)$$

as discussed in Remark 10. For sake of simplicity, let Z be compact (the case in which Z is locally compact might be handled by introducing a suitable compact window containing the point x considered). Of course $\partial\Theta = x + \partial Z$, and so the regularity properties of $\partial\Theta$ coincide with the regularity properties of ∂Z . Let Φ have intensity measure $\Lambda(d(x, s)) = \lambda(x, s) dx Q(ds)$ such that

(A1') for any $(y, s) \in \mathbb{R}^d \times \mathbf{K}$, $y + \partial Z(s)$ is a countably \mathcal{H}^{d-1} -rectifiable and compact subset of \mathbb{R}^d , such that there exists a closed set $\Xi(s) \supseteq \partial Z(s)$ such that $\int_{\mathbf{K}} \mathcal{H}^{d-1}(\Xi(s)) Q(ds) < \infty$ and

$$\mathcal{H}^{d-1}(\Xi(s) \cap B_r(x)) \geq \gamma r^{d-1} \quad \forall x \in \partial Z(s), \quad \forall r \in (0, 1)$$

for some $\gamma > 0$ independent on y and s ;

(A2') for any $s \in \mathbf{K}$, $\mathcal{H}^{d-1}(\text{disc}(\lambda(\cdot, s))) = 0$ and $\lambda(\cdot, s)$ is locally bounded such that for any compact $K \subset \mathbb{R}^d$

$$\sup_{x \in K_{\oplus \text{diam}(Z(s))}} \lambda(x, s) \leq \tilde{\xi}_K(s)$$

for some $\tilde{\xi}_K(s)$ with $\int_{\mathbf{K}} \mathcal{H}^{d-1}(\Xi(s)) \tilde{\xi}_K(s) Q(ds) < \infty$.

Note that assumption (A1') guarantees that Z , and so Θ , belongs to the class \mathcal{O} ; in particular it is easy to see that Θ satisfies the hypotheses of Lemma 3.10 in [26], which implies that Θ admits *local mean outer Minkowski content*, that is:

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_{\oplus r} \setminus \Theta) \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial^* \Theta \cap A)] + 2\mathbb{E}[\mathcal{H}^{d-1}(\Theta^0 \cap \partial \Theta \cap A)] \quad (27)$$

for any Borel set A with $\mathbb{E}[\mathcal{H}^{d-1}(\partial \Theta \cap \partial A)] = 0$ (and so for any A with $\mathcal{H}^d(\partial A) = 0$, being $\mathbb{E}[\mu_{\partial \Theta}] \ll \mathcal{H}^d$).

The assumption (A2') allows us to apply Theorem 17 to prove that

$$\sigma_{\Theta}(x) = \lambda_{\partial^* \Theta}(x) + 2\lambda_{\Theta^0 \cap \partial \Theta}(x), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

having denoted by $\lambda_{\partial^* \Theta}$ and $\lambda_{\Theta^0 \cap \partial \Theta}$ the density of the measure $\mathbb{E}[\mathcal{H}^{d-1}(\partial^* \Theta \cap \cdot)]$ and $\mathbb{E}[\mathcal{H}^{d-1}(\Theta^0 \cap \partial \Theta \cap \cdot)]$, respectively; namely, we prove the following theorem.

Theorem 18. *Let $\Theta = y + Z$ be a random closed set as above, satisfying assumption (A1') and (A2'); then*

$$\sigma_{\Theta}(x) := \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} = \lambda_{\partial^* \Theta}(x) + 2\lambda_{\Theta^0 \cap \partial \Theta}(x), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (28)$$

In particular, if

$$\int_{\mathbf{K}} \mathcal{H}^{d-1}(\partial^* Z(s)) Q(ds) = \int_{\mathbf{K}} \mathcal{H}^{d-1}(\partial Z(s)) Q(ds), \quad (29)$$

then

$$\sigma_{\Theta}(x) = \lambda_{\partial \Theta}(x) = \int_{\mathbf{K}} \int_{x - \partial Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. By applying the same arguments used in the proof of Proposition 3.8 in [26], it follows that $\mathbb{E}[\mathcal{H}^{d-1}(\partial \Theta \cap \cdot)]$ is absolutely continuous with respect to \mathcal{H}^d (and so $\mathbb{E}[\mathcal{H}^{d-1}(\partial^* \Theta \cap \cdot)]$ and $\mathbb{E}[\mathcal{H}^{d-1}(\Theta^0 \cap \partial \Theta \cap \cdot)]$ as well, being $\partial^* \Theta$ and $\Theta^0 \cap \partial \Theta$ disjoint subsets of $\partial \Theta$); the equation (27) is equivalent to write

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} = \int_A (\lambda_{\partial^* \Theta}(x) + 2\lambda_{\Theta^0 \cap \partial \Theta}(x)) dx. \quad (30)$$

We want to apply the dominated convergence theorem in order to exchange limit and integral in the equation above.

Let us first prove that there exist the limit of $\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)/r$ for $r \downarrow 0$:

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} &= \lim_{r \downarrow 0} \frac{\mathbb{P}(\Phi\{(y, s): x \in (y + Z(s))_{\oplus r} \setminus (y + Z(s))\} > 0)}{r} \\ &= \lim_{r \downarrow 0} \frac{\Lambda(\{(y, s): y \in (x - Z(s))_{\oplus r} \setminus (x - Z(s))\})}{r} \\ &= \lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{K}} \int_{(x-Z(s))_{\oplus r} \setminus (x-Z(s))} \lambda(y, s) \, dy \, Q(ds). \end{aligned}$$

By applying now Theorem 17, we get

$$\begin{aligned} &\lim_{r \downarrow 0} \frac{1}{r} \int_{(x-Z(s))_{\oplus r} \setminus (x-Z(s))} \lambda(y, s) \, dy \\ &\stackrel{(26)}{=} \int_{x-\partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds) + 2 \int_{\mathbf{K}} \int_{(x-\partial Z(s)) \cap (x-Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy), \end{aligned}$$

besides we observe that

$$\begin{aligned} \frac{1}{r} \int_{(x-Z(s))_{\oplus r} \setminus (x-Z(s))} \lambda(y, s) \, dy &\leq \frac{1}{r} \int_{(x-\partial Z(s))_{\oplus r}} \lambda(y, s) \, dy \\ &\leq \frac{\mathcal{H}^{d-1}(Z(s))}{r} \sup_{y \in (x-\partial Z(s))_{\oplus r}} \lambda(y, s) \\ &\stackrel{(17), (A2')}{\leq} \mathcal{H}^{d-1}(\Xi(s)) \frac{2^{3d-1} b_d \tilde{\xi}_{B_2(x)}(s)}{\gamma} \quad \forall r < 2. \end{aligned}$$

Therefore, assumption (A2') and the dominated convergence theorem imply

$$\begin{aligned} \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} &= \lim_{r \downarrow 0} \frac{1}{r} \int_{\mathbf{K}} \int_{(x-Z(s))_{\oplus r} \setminus (x-Z(s))} \lambda(y, s) \, dy \, Q(ds) \\ &= \int_{\mathbf{K}} \left(\int_{x-\partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds) \right. \\ &\quad \left. + 2 \int_{\mathbf{K}} \int_{(x-\partial Z(s)) \cap (x-Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy) \right) Q(ds). \end{aligned} \tag{31}$$

Analogously, for any fixed bounded Borel set A and for any $r < 2$,

$$\frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} \leq \int_{\mathbf{K}} \frac{\mathcal{H}^{d-1}(\Xi(s))}{\gamma} 2^{3d-1} b_d \tilde{\xi}_K(s) Q(ds) \stackrel{(A2')}{=} c \in \mathbb{R},$$

where K is a compact subset of \mathbb{R}^d containing $A_{\oplus 2}$.

Thus we may change limit and integral in (30), and we get

$$\lim_{r \downarrow 0} \int_A \frac{\mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)}{r} = \int_A \sigma_{\Theta}(x) \, dx = \int_A (\lambda_{\partial^* \Theta}(x) + 2\lambda_{\Theta^0 \cap \partial \Theta}(x)) \, dx \quad (32)$$

for any A with $\mathcal{H}^d(\partial A) = 0$, and so equation (28) holds.

Assumption (29) ensures that the \mathcal{H}^{d-1} -measure of the boundary of Z equals the \mathcal{H}^{d-1} -measure of its essential boundary, and so $\mathbb{E}[\mathcal{H}^{d-1}(\partial^* \Theta \cap \cdot)] = \mathbb{E}[\mathcal{H}^{d-1}(\partial \Theta \cap \cdot)]$; in particular it follows that $\lambda_{\partial^* \Xi}(x) = \lambda_{\partial \Xi}(x)$ and $\lambda_{\partial \Xi \cap \Xi^0}(x) = 0$ for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$, and that

$$\begin{aligned} & \int_{x - \partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) + 2 \int_{(x - \partial Z(s)) \cap (x - Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy) \\ &= \int_{x - \partial Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy). \end{aligned}$$

Thus, by (31) and (32) we get

$$\sigma_{\Theta}(x) = \lambda_{\partial \Theta}(x) = \int_{\mathbf{K}} \int_{x - \partial Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad \square$$

Remark 19. The above theorem answers also to the open problem posed by Matheron in [22], page 50, about the equality between the specific area σ_{Θ} and the mean boundary density λ_{Θ} for a general random set Θ . Again, such an equality strongly depends on the geometric regularities of $\partial \Theta$; of course the cases in which $\sigma_{\Theta} \neq \lambda_{\partial \Theta}$ are, in a certain sense, “pathological,” because condition (29) is usually fulfilled in applications.

Of course the specific area σ_{Θ} may be evaluated for germ-grain processes whose grains have integer dimension $n < d$ ($n = 0$ is trivial), but it is clear that $\sigma_{\Theta}(x) \equiv 0$ if $n < d - 1$.

In the case $d - 1$, that is $Z(s) = \partial Z(s)$ for any $s \in \mathbf{K}$, assumptions (A1) and (A2) given in the previous section coincide with (A1') and (A2') above; by noticing that $\partial Z(s) = Z^0(s) \cap \partial Z(s)$, and that $\mathbb{P}(x \in \Theta) = 0$ a.s., the results (21) and (18) proved in Theorem 7 and Proposition 5, respectively, are in accordance with Theorem 18:

$$\begin{aligned} \sigma_{\Theta}(x) &= \lim_{r \downarrow 0} \frac{\mathbb{P}(x \in \Theta_{\oplus r})}{r} = 2\lambda_{\Theta}(x) \\ &= 2 \int_{\mathbf{K}} \int_{x - Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds). \end{aligned}$$

We point out that it seems to be hard to find out explicit expressions for σ_{Θ} when Θ is a general germ-grain model (i.e., non-Boolean) with $\mathcal{H}^d(\Theta) > 0$, in terms of its grains as we did for λ_{Θ_n} in Proposition 5 in the n -dimensional case. Indeed, due to the fact that the interior of the grains is in general not empty, we cannot follow the same lines of the proof of the mentioned proposition, because $\mathbb{E}[\mathcal{H}^{d-1}(\partial \Theta \cap \cdot)] \neq \mathbb{E}[\sum_{(y_i, s_i) \in \Phi} \mathcal{H}^{d-1}((y_i + \partial Z(s_i)) \cap \cdot)]$.

Instead, when Θ is a Boolean model, and so thanks to the independence property of its grains and to the knowledge of the associated capacity functional, it is possible to prove an explicit expression for its specific area, as proved in [26], Proposition 3.7, in the case of position-independent grains. By similar arguments of the previous sections, it is easy to extend it to the case of a general Boolean model Θ whose grains satisfy the above assumption (A1') and (A2'), obtaining that

$$\begin{aligned} \sigma_{\Theta}(x) = \mathbb{P}(x \notin \Theta) & \left[\int_{\mathbf{K}} \int_{x - \partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds) \right. \\ & \left. + 2 \int_{\mathbf{K}} \int_{(x - \partial Z(s)) \cap (x - Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds) \right]. \end{aligned} \quad (33)$$

We may notice that the above expression for σ_{Θ} applies only to Boolean models, thanks to independence properties of the underlying point process Φ , and that it cannot be true for different germ-grain models: it is sufficient to consider the case when Θ is an ‘‘one-grain’’ random set as in Theorem 18, and observe that its specific area given in (31) differs from (33), being $\mathbb{P}(x \notin \Theta) \neq 1$, in general.

4.3. The spherical contact distribution function

We are now able to give a general expression for the derivative in $r = 0$ of the spherical contact distribution function H_{Θ} , defined in (24), under the same general assumptions on the random set Θ given in the previous section.

By noticing that $\mathbb{P}(x \notin \Theta)H_{\Theta}(r, x) = \mathbb{P}(x \in \Theta_{\oplus r} \setminus \Theta)$ and $H_{\Theta}(0, x) \equiv 0$, the following corollary of Theorem 18 is easily proved.

Corollary 20. *Let Θ be a random closed set as in Theorem 18; then*

$$\begin{aligned} \frac{\partial}{\partial r} H_{\Theta}(r, x)|_{r=0} &= \frac{\sigma_{\Theta}(x)}{\mathbb{P}(x \notin \Theta)} \\ &= \frac{\lambda_{\partial^* \Theta}(x) + 2\lambda_{\Theta^0 \cap \partial \Theta}(x)}{\mathbb{P}(x \notin \Theta)}, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d, \end{aligned}$$

where the above derivative has to be intended the right derivative in 0.

If in particular (29) is satisfied, then

$$\frac{\partial}{\partial r} H_{\Theta}(r, x)|_{r=0} = \frac{\lambda_{\partial \Theta}(x)}{\mathbb{P}(x \notin \Theta)}, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Remark 21 (Boolean model and ‘‘one-grain’’ random set). By the corollary above and by (33) and (31), we get the following explicit formulas in the case Θ is a Boolean model (reobtain-

ing [26], equation (4.1), as particular case), or Θ is an “one-grain” random set:

$$\frac{\partial}{\partial r} H_{\Theta}(r, x)|_{r=0} = \begin{cases} \text{Boolean model} \\ \int_{\mathbf{K}} \int_{x-\partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds) \\ + 2 \int_{\mathbf{K}} \int_{(x-\partial Z(s)) \cap (x-Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds), \\ \text{“one-grain” random set} \\ \frac{\int_{\mathbf{K}} [\int_{x-\partial^* Z(s)} \lambda(y, s) \mathcal{H}^{d-1}(dy) + 2 \int_{(x-\partial Z(s)) \cap (x-Z^0(s))} \lambda(y, s) \mathcal{H}^{d-1}(dy)] Q(ds)}{\int_{\mathbb{R}^d \times \mathbf{K}} (1 - \mathbf{1}_{x-Z(s)(y)}) \lambda(y, s) dy Q(ds)}. \end{cases}$$

In [26], Theorem 4.1, has been proved a result about the differentiability of H_{Θ} with respect to r for a quite general class of Boolean models with typical grain having *positive reach*. Such a result can be easily extended for Boolean models with position dependent grains by considering an intensity measure $\Lambda(d(y, s))$ of the type $\lambda(y, s) dy Q(ds)$, instead of the type $f(y) dy Q(ds)$, and by modifying the assumption of the cited theorem accordingly. Here we reformulate such a result also for “one-grain” random sets. In order to do this, we briefly recall some basic definitions from geometric measure theory.

For any closed subset A of \mathbb{R}^d , let $\text{Unp}(A) := \{x \in \mathbb{R}^d : \exists! a \in A \text{ such that } \text{dist}(x, A) = |a - x|\}$. The definition of $\text{Unp}(A)$ implies the existence of a projection mapping $\xi_A: \text{Unp}(A) \rightarrow A$ which assigns to $x \in \text{Unp}(A)$ the unique point $\xi_A(x) \in A$ such that $\text{dist}(x, A) = |x - \xi_A(x)|$; then for all $x \in \text{Unp}(A)$ with $\text{dist}(x, A) > 0$ we may define $u_A(x) := (x - \xi_A(x))/\text{dist}(x, A)$. The set of all $x \in \mathbb{R}^d \setminus A$ for which $\xi_A(x)$ is not defined it is called *exoskeleton* of A , and it is denoted by $\text{exo}(A)$. The *normal bundle* of A is the measurable subset of $\partial A \times \mathbf{S}^{d-1}$ defined by $N(A) := \{(\xi_A(x), u_A(x)) : x \notin A \cup \text{exo}(A)\}$. For any $x \in \partial^+ A := \{x \in \partial A : (x, u) \in N(A) \text{ for some } u \in \mathbf{S}^{d-1}\}$, we define

$$N(A, x) := \{u \in \mathbf{S}^{d-1} : (x, u) \in N(A)\}$$

and

$$\partial^1 A := \{x \in \partial^+ A : \text{card } N(A, x) = 1\}.$$

Note that for any $x \in \partial^1 A$, the unique element of $N(A, x)$ is the *outer normal* of A at x , denoted here by n_x . The reach of a compact set A is defined by (see [14])

$$\text{reach}(A) := \inf_{a \in A} \sup\{r > 0 : B_r(a) \subset \text{Unp}(A)\};$$

for any set $A \subset \mathbb{R}^d$ with positive reach, the *curvature measures* $\Phi_i(A; \cdot)$ on \mathbb{R}^d , for $i = 1, \dots, d - 1$, introduced in [14], are well defined.

Then, by following the same lines of Section 4 in [26], it is not difficult to prove the following proposition for an “one-grain” random set.

Proposition 22. Let Θ be a random closed set as in Theorem 18, with $\text{reach}(Z(s)) > R$ for some $R > 0$ and such that $\mathcal{H}^0(N(Z(s), x)) = 1$ for \mathcal{H}^{d-1} -a.e. $x \in \partial Z(s)$, for all $s \in \mathbf{K}$. Moreover, we assume that

$$\int_{\mathbf{K}} |\Phi_i|(Z(s)) Q(ds) < \infty \quad \forall i = 1, \dots, d-1,$$

where $|\Phi_i|(Z(s))$ is the total variation of the measure $\Phi_i(Z(s); \cdot)$, and that the intensity $\lambda(\cdot, s)$ is bounded, Lipschitz with Lipschitz constant $\text{Lip } f(\cdot, s)$ such that $\int_{\mathbf{K}} \text{Lip } f(\cdot, s) Q(ds) < \infty$, and the set where $\lambda(\cdot, s)$ is not differentiable is \mathcal{H}^{d-1} -negligible. Then, for all $x \in \mathbb{R}^d$,

$$\begin{aligned} \frac{\partial}{\partial r} H_{\Theta}(r, x) &= \frac{\int_{\mathbf{K}} \int_{x - \partial Z(s) \oplus r} \lambda(y, s) \mathcal{H}^{d-1}(dy) Q(ds)}{\int_{\mathbb{R}^d \times \mathbf{K}} (1 - \mathbf{1}_{x-Z(s)}(y)) \lambda(y, s) dy Q(ds)} \quad \forall r \in [0, R), \\ \frac{\partial^2}{\partial r^2} H(r, x)|_{r=0} &= \frac{\int_{\mathbf{K}} [2\pi \int_{\mathbb{R}^d} \lambda(y, s) \Phi_{d-2}(x - Z(s); dy) + \int_{x - \partial^1 Z(s)} D_{n_y} \lambda(y, s) \mathcal{H}^{d-1}(dy)] Q(ds)}{\int_{\mathbb{R}^d \times \mathbf{K}} (1 - \mathbf{1}_{x-Z(s)}(y)) \lambda(y, s) dy Q(ds)}, \end{aligned}$$

where $D_{n_y} \lambda(\cdot, s)$ is the directional derivative of $\lambda(\cdot, s)$ along $n_y \in \mathbf{S}^{d-1}$.

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References

- [1] Ambrosio, L., Capasso, V. and Villa, E. (2009). On the approximation of mean densities of random closed sets. *Bernoulli* **15** 1222–1242. [MR2597590](#)
- [2] Ambrosio, L., Colesanti, A. and Villa, E. (2008). Outer Minkowski content for some classes of closed sets. *Math. Ann.* **342** 727–748. [MR2443761](#)
- [3] Ambrosio, L., Fusco, N. and Pallara, D. (2000). *Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs*. New York: The Clarendon Press Oxford Univ. Press. [MR1857292](#)
- [4] Anderson, A.R.A. (2003). Effects of cell adhesion on solid tumour geometry. In *Morphogenesis and Pattern Formation in Biological Systems* (T. Sekimura et al., eds.). Tokyo: Springer.
- [5] Armendáriz, I., Cuevas, A. and Fraiman, R. (2009). Nonparametric estimation of boundary measures and related functionals: Asymptotic results. *Adv. in Appl. Probab.* **41** 311–322. [MR2541178](#)
- [6] Baddeley, A., Bárány, I., Schneider, R. and Weil, W. (2007). *Stochastic Geometry. Lecture Notes in Math.* **1892**. Berlin: Springer. [MR2327289](#)
- [7] Baddeley, A.J. and Molchanov, I.S. (1997). On the expected measure of a random set. In *Proceedings of the International Symposium on Advances in Theory and Applications of Random Sets (Fontainebleau, 1996)* 3–20. River Edge, NJ: World Sci. Publ. [MR1654394](#)

- [8] Beneš, V. and Rataj, J. (2004). *Stochastic Geometry*. Dordrecht: Kluwer.
- [9] Capasso, V. and Morale, D. (2009). Stochastic modelling of tumour-induced angiogenesis. *J. Math. Biol.* **58** 219–233. [MR2448429](#)
- [10] Capasso, V. and Villa, E. (2007). On mean densities of inhomogeneous geometric processes arising in material science and medicine. *Image Anal. Stereol.* **26** 23–36. [MR2337324](#)
- [11] Cuevas, A. and Fraiman, R. (2010). Set estimation. In *New Perspectives in Stochastic Geometry* 374–397. Oxford: Oxford Univ. Press. [MR2654684](#)
- [12] Daley, D.J. and Vere-Jones, D. (1988). *An Introduction to the Theory of Point Processes. Springer Series in Statistics*. New York: Springer. [MR0950166](#)
- [13] Falconer, K.J. (1986). *The Geometry of Fractal Sets. Cambridge Tracts in Mathematics* **85**. Cambridge: Cambridge Univ. Press. [MR0867284](#)
- [14] Federer, H. (1959). Curvature measures. *Trans. Amer. Math. Soc.* **93** 418–491. [MR0110078](#)
- [15] Federer, H. (1969). *Geometric Measure Theory. Die Grundlehren der Mathematischen Wissenschaften* **153**. New York: Springer. [MR0257325](#)
- [16] Feng, D.J. and Feng, D. (2004). On a statistical framework for estimation from random set observations. *J. Theoret. Probab.* **17** 85–110. [MR2054580](#)
- [17] Galerne, B. (2011). Computation of the perimeter of measurable sets via their covariogram. Applications to random sets. *Image Anal. Stereol.* **30** 39–51. [MR2816305](#)
- [18] Hoffmann, L.M. (2007). Intersection densities of nonstationary Poisson processes of hypersurfaces. *Adv. in Appl. Probab.* **39** 307–317. [MR2341574](#)
- [19] Hug, D. and Last, G. (2000). On support measures in Minkowski spaces and contact distributions in stochastic geometry. *Ann. Probab.* **28** 796–850. [MR1782274](#)
- [20] Hug, D., Last, G. and Weil, W. (2002). A survey on contact distributions. In *Morphology of Condensed Matter* (K. Mecke and D. Stoyan, eds.). *Lecture Notes in Physics* **600** 317–357. Berlin: Springer.
- [21] Last, G. and Brandt, A. (1995). *Marked Point Processes on the Real Line: The Dynamic Approach. Probability and Its Applications (New York)*. New York: Springer. [MR1353912](#)
- [22] Matheron, G. (1975). *Random Sets and Integral Geometry*. New York-London-Sydney: Wiley. [MR0385969](#)
- [23] Prokešová, M. (2010). Inhomogeneity in spatial Cox point processes – location dependent thinning is not the only option. *Image Anal. Stereol.* **29** 133–141. [MR2780337](#)
- [24] Stoyan, D., Kendall, W.S. and Mecke, J. (1987). *Stochastic Geometry and Its Applications. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics*. Chichester: Wiley. [MR0895588](#)
- [25] Villa, E. (2009). On the outer Minkowski content of sets. *Ann. Mat. Pura Appl.* (4) **188** 619–630. [MR2533959](#)
- [26] Villa, E. (2010). Mean densities and spherical contact distribution function of inhomogeneous Boolean models. *Stoch. Anal. Appl.* **28** 480–504. [MR2739571](#)
- [27] Villa, E. (2010). On the specific area of inhomogeneous Boolean models. Existence results and applications. *Image Anal. Stereol.* **29** 111–119. [MR2724819](#)
- [28] Zähle, M. (1982). Random processes of Hausdorff rectifiable closed sets. *Math. Nachr.* **108** 49–72. [MR0695116](#)

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