Bernoulli **19**(5A), 2013, 2010–2032
DOI: 10.3150/12-BEJ441

Generalization of the Blumenthal–Getoor index to the class of homogeneous diffusions with jumps and some applications

ALEXANDER SCHNURR

TU Dortmund, Faculty of Mathematics, Vogelpothsweg 87, 44227 Dortmund, Germany.
E-mail: alexander.schnurr@math.tu-dortmund.de

We introduce the probabilistic symbol for the class of homogeneous diffusions with jumps (in the sense of Jacod/Shiryaev). This concept generalizes the well-known characteristic exponent of a Lévy process. Using the symbol, we introduce eight indices which generalize the Blumenthal–Getoor index β and the Pruitt index δ. These indices are used afterwards to obtain growth and Hölder conditions of the process. In the future, the technical main results will be used to derive further fine properties. Since virtually all examples of homogeneous diffusions in the literature are Markovian, we construct a process which does not have this property.

Keywords: COGARCH process; Feller process; fine continuity; fine properties; generalized indices; Itô process; semimartingale; symbol

1. Introduction

Two of the main tools in order to analyze and describe Lévy processes are the characteristic exponent and the Blumenthal–Getoor index. In the present paper, we show that there exist analogous of these concepts for a much wider class of processes, namely homogeneous diffusions with jumps (h.d.w.j.) in the sense of Jacod and Shiryaev ([15], Definition III.2.18). These indices are used to derive growth and Hölder conditions for the paths of the process.

A Lévy process X is a stochastic process with stationary and independent increments which has a.s. càdlàg paths (cf. [20]). It is a well-known fact that the characteristic function of X_t can be written as

$$\varphi_{X_t}(\xi) = \mathbb{E}^0 e^{iX_t'\xi} = e^{-t\psi(\xi)}, \tag{1}$$

where the characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is a continuous negative definite function (c.n.d.f.) in the sense of Schoenberg (cf. [2], Chapter 2). In fact, one obtains by the relation (1) a one-to-one correspondence between the class of c.n.d.f.'s and Lévy processes. The Blumenthal–Getoor index was first introduced in [3] in order to analyze Hölder conditions, the γ-variation and the Hausdorff-dimension of the paths of Lévy processes.

The idea of the present paper is to use the state-space dependent right derivative at $t = 0$ of the characteristic function to obtain the symbol p of the process which generalizes the characteristic exponent of a Lévy process. The formula reads as follows (for details, see Definition 3.5 below):

for $x, \xi \in \mathbb{R}^d$

$$p(x,\xi) := -\lim_{t\downarrow 0} \frac{\mathbb{E}^x \mathrm{e}^{\mathrm{i}(X_t^\sigma - x)'\xi} - 1}{t},$$

where σ is the first-exit time of a compact neighborhood of x. Since for every fixed $t > 0$, the function $\xi \mapsto \mathbb{E}^x \mathrm{e}^{\mathrm{i}(X_t^\sigma - x)'\xi}$ is the characteristic function of the random variable $X_t^\sigma - x$ it is continuous and positive definite. By Corollary 3.6.10 of [12], we conclude that $\xi \longmapsto -(\mathbb{E}^x \mathrm{e}^{\mathrm{i}(X_t^\sigma - x)'\xi} - 1)$ is a continuous negative definite function. Dividing by t preserves this property since the c.n.d.f.'s form a convex cone. By Lemma 3.6.7 of [12], the above limit is a negative definite function which is continuous if the convergence is locally uniform. The idea to analyze objects of this type was proposed first in [11] in the context of universal Markov processes.

We have thus shown that the symbol is a state-space dependent c.n.d.f. Therefore, we can define and analyze eight indices along the same lines as in Schilling's article [23] where the case of rich Feller processes was analyzed. These are Feller processes with the property that the test functions $C_c^\infty(\mathbb{R}^d)$ are contained in the domain of their generator. The multiplier in the Fourier representation of the generator of such a process is also a state-space dependent c.n.d.f. (cf. Example 4.1 below and for details the monograph by Jacob [12–14]). For these c.n.d.f.'s, we write $q(x,\xi)$ to distinguish them from the $p(x,\xi)$ above. In order to introduce and use the indices, Schilling needed the following two conditions (G) and (S) which we state here since they play a role in our considerations, too. The growth condition is fulfilled, if there exists a $c > 0$ such that

$$\big\|q(\cdot,\xi)\big\|_\infty \le c\big(1 + \|\xi\|^2\big) \tag{G}$$

for every $\xi \in \mathbb{R}^d$. The sector condition, which is needed only for some of the results, is fulfilled, if there exists a $c_0 > 0$ such that for every $x, \xi \in \mathbb{R}^d$

$$\big|\Im\big(q(x,\xi)\big)\big| \le c_0 \Re\big(p(x,\xi)\big). \tag{S}$$

In [27], we have shown that every rich Feller process is an Itô process in the sense of Cinlar, Jacod, Protter and Sharpe (cf. [7], Section 7), that is, a Hunt semimartingale with characteristics of the form

$$\begin{aligned}
B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}\big(X_s(\omega)\big)\,\mathrm{d}s, && j = 1,\dots,d, \\
C_t^{jk}(\omega) &= \int_0^t Q^{jk}\big(X_s(\omega)\big)\,\mathrm{d}s, && j,k = 1,\dots,d, \\
\nu(\omega;\mathrm{d}s,\mathrm{d}y) &= N\big(X_s(\omega),\mathrm{d}y\big)\,\mathrm{d}s,
\end{aligned} \tag{2}$$

where for every $x \in \mathbb{R}^d$ $\ell(x)$ is a vector in $\mathbb{R}^d$, $Q(x)$ is a positive semi-definite matrix and N is a Borel transition kernel such that $N(x,\{0\}) = 0$. The triplet $(\ell(x), Q(x), N(x,\mathrm{d}y))$ appears in the symbol again (cf. Theorem 3.6). Since the characteristics describe the local dynamics of the process, it is not surprising that the symbol, as well as the associated indices, contain a lot of information about the global and the path properties of the process, like conservativeness (cf. [21], Theorem 5.5), strong γ-variation (cf. [24], Corollary 5.10) or Hausdorff-dimension (cf. [22], Theorem 4). By now, all results of this type were restricted to rich Feller processes. The above considerations show that Itô processes would be a natural candidate to generalize the results on symbols, indices and fine properties. In the present paper we go even one step further: semimartingales having characteristics of the form (2) are called h.d.w.j. It is this class we are dealing with. In Section 2, we have included an example of this kind, which is not a Markov process. Philosophically speaking we show that the symbol, as well as the derived indices, are a concept related to the underlying semimartingale structure rather than the property of being memoryless. To this end, new techniques of proof had to be developed.

Here and in the following, we mean by a stochastic process a family of processes $(X, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ which is normal, that is, $\mathbb{P}^x(X_0 = x) = 1$. Such a process is called a martingale, continuous, ... iff it is w.r.t. every $\mathbb{P}^x$ $(x \in \mathbb{R}^d)$ a martingale, continuous, ... A stochastic basis $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ is always meant to be in the background. We assume that the usual hypotheses are satisfied.

Before closing this section, we give an overview on what was known before the present paper. We consider the following classes of processes:

$$\begin{matrix}\text{symmetric}\\ \alpha\text{-stable}\end{matrix} \subseteq \text{Lévy} \subseteq \begin{matrix}\text{rich}\\ \text{Feller}\end{matrix} \subseteq \text{Itô} \subseteq \text{h.d.w.j.} \tag{3}$$

The symbol was generalized to Itô processes in [27]. The indices were known for rich Feller processes satisfying (G) and (S). Fine properties were obtained for the same class, sometimes under additional assumptions (cf. [22]). Let us mention that even in the known case of rich Feller processes we generalize Schilling's results: instead of (G) we only need a local version of this property which is automatically fulfilled by every rich Feller process.

Let us give a brief outline on how the paper is organized: in the subsequent section we show that there exists a h.d.w.j. which is not Markovian. In particular the last inclusion in (3) is strict. In Section 3 we present the definitions and main results. Complementary results and several examples, including the COGARCH process which is used to model financial data, are contained in Section 4. The proofs are postponed to Section 5, since they are rather technical. Our main results are Theorems 3.6, 3.11 and 3.12.

The notation we are using is more or less standard. Vectors are column vectors. Transposed vectors or matrices are denoted by $'$. Vector entries are written as follows: $v = (v^{(1)}, \dots, v^{(d)})'$. In the context of semimartingales we follow mainly [15]. Multivariate stochastic integrals are always meant componentwise. This is true for integrals w.r.t. processes as well as for those w.r.t. random measures. A function $\chi : \mathbb{R}^d \to \mathbb{R}$ is called cut-off function if it is Borel measurable, with compact support and equal to one in a neighborhood of zero. In this case $h(y) := \chi(y) \cdot y$ is a truncation function in the sense of [15]. Finally, let $\mathbb{N} := \{0, 1, \dots\}$.

2. A non-Markovian homogeneous diffusion

Virtually all examples of homogeneous diffusions (with or without jumps) in the literature are Markov processes. Here we construct an example which is not Markovian.

Example 2.1. We use the construction principle for deterministic processes which we introduced in [28] and generalized in [26]. Let $\mathbb{T}$ denote the unit sphere in $\mathbb{R}^2$.

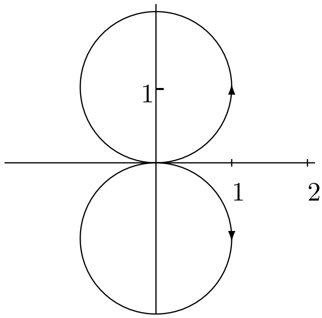
Within the set $((0,1)' + \mathbb{T}) \cup ((0,-1)' + \mathbb{T})$, we consider the following ODE on $[0,\infty[$:

$$\begin{aligned} y_1' &= 1 - y_2, \qquad y_2' = y_1 \qquad \text{for } y_2 \geq 0, \\ y_1' &= y_2 + 1, \qquad y_2' = -y_1 \qquad \text{for } y_2 < 0 \end{aligned}$$

with the initial value $y(0) = (y_1(0), y_2(0))' = (0,0)'$ having the (non-unique) solution

$$y(t) = \sum_{n \in 2\mathbb{N}} \begin{pmatrix} \sin(t) \\ 1 - \cos(t) \end{pmatrix} \cdot 1_{[2n\pi, 2(n+1)\pi[}(t) + \begin{pmatrix} \sin(t) \\ \cos(t) - 1 \end{pmatrix} \cdot 1_{[2(n+1)\pi, 2(n+2)\pi[}(t).$$

For the readers convenience, we include the following picture:



We denote by $\widetilde{y}$ the restriction of y to $[0, 4\pi[$. On this interval the function is bijective. The process X is defined as follows: under the law $\mathbb{P}^x$ we have

$$X_t := \begin{cases} y\big(\widetilde{y}^{-1}(x) + t\big), & \text{for } x \in \big((0,1)' + \mathbb{T}\big) \cup \big((0,-1)' + \mathbb{T}\big), \\ x, & \text{else.} \end{cases}$$

This process is not Markovian, since

$$\mathbb{P}^{(0,2)'}\left(X_{2\pi} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \middle| X_\pi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 1 \neq 0 = \mathbb{P}^{(0,-2)'}\left(X_{2\pi} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \middle| X_\pi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right).$$

On the other hand, X is a homogeneous diffusion with ℓ given by

$$\ell(x) = \begin{cases} \begin{pmatrix} 1 - x^{(2)} \\ x^{(1)} \end{pmatrix}, & \text{if } x \in \big((0,1)' + \mathbb{T}\big), \\ \begin{pmatrix} x^{(2)} + 1 \\ x^{(1)} \end{pmatrix}, & \text{if } x \in \big((0,-1)' + \mathbb{T}\big) \setminus \big\{(0,0)'\big\}, \\ 0, & \text{else.} \end{cases}$$

Anticipating an important concept of the next section, let us mention that ℓ is not continuous on $\mathbb{R}^2$, but it is X-finely continuous (cf. Definition 3.3).

3. Definitions and main results

We have decided to postpone the proofs to Section 5.

Definition 3.1. *A* homogeneous diffusion with jumps (h.d.w.j., *for short*) $(X, \mathbb{P}^x)_{x\in\mathbb{R}^d}$ *is a semimartingale with characteristics of the form*

$$\begin{aligned} B_t^{(j)}(\omega) &= \int_0^t \ell^{(j)}\big(X_s(\omega)\big)\,\mathrm{d}s, \qquad j = 1, \dots, d, \\ C_t^{jk}(\omega) &= \int_0^t Q^{jk}\big(X_s(\omega)\big)\,\mathrm{d}s, \qquad j, k = 1, \dots, d, \\ \nu(\omega; \mathrm{d}s, \mathrm{d}y) &= N\big(X_s(\omega), \mathrm{d}y\big)\,\mathrm{d}s \end{aligned} \tag{4}$$

for every $x \in \mathbb{R}^d$ with respect to a fixed cut-off function χ. Here $\ell(x) = (\ell^{(1)}(x), \dots, \ell^{(d)}(x))'$ is a vector in $\mathbb{R}^d$, $Q(x)$ is a positive semi-definite matrix and N is a Borel transition kernel such that $N(x, \{0\}) = 0$. We call ℓ, Q and $n := \int_{y\neq 0}(1 \wedge \|y\|^2)N(\cdot, \mathrm{d}y)$ the differential characteristics *of the process.*

Remark 3.2. In the monograph [15], this class of processes is called homogeneous diffusion with jumps, but even there this name was qualified as ‘misleading’, since the term ‘diffusion’ is often used for continuous Markov processes: a diffusion with jumps is not continuous and in Section 2 we have seen that it does not have to be Markovian. However, we decided to stick to the classical name, since it has become canonical.

In our considerations, it turned out that the most general assumption on the differential characteristics, under which we are able to prove our main results, read as follows.

Definition 3.3. *Let X be a h.d.w.j. and $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel-measurable function. f is called* X-finely continuous (*or* finely continuous, *for short*) *if the function*

$$t \mapsto f(X_t) = f \circ X_t \tag{5}$$

is right continuous at zero $\mathbb{P}^x$-a.s. for every $x \in \mathbb{R}^d$.

Remark 3.4.

(a) In the context of Markov processes, fine continuity is introduced differently (see [4], Section II.4, and [9]). By Theorem 4.8 of [4], this is equivalent to (5).

(b) If the differential characteristics are continuous, the condition stated in Definition 3.3 is obviously fulfilled, since the paths of X are càdlàg.

The other important assumption on the differential characteristics is that they are locally bounded. By Lemma 3.3 of [25], this is equivalent to the local version of the growth condition:

for every compact set $K \subseteq \mathbb{R}^d$ there exists a constant $c_K > 0$ such that

$$\left|p(x,\xi)\right| \le c_K\left(1+\|\xi\|^2\right) \tag{LG}$$

for every $x \in K$. This condition is fulfilled by every rich Feller process (Lemma 3.3 of [25]).

Definition 3.5. *Let X be a h.d.w.j., which is conservative and normal, that is, $\mathbb{P}^x(X_0 = x) = 1$. Fix a starting point x and define $\sigma = \sigma_k^x$ to be the first exit time from a compact neighborhood $K := K_x$ of x:*

$$\sigma := \inf\left\{t \ge 0\colon\ X_t^x \notin K\right\}.$$

For $\xi \in \mathbb{R}^d$, we call $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ given by

$$p(x,\xi) := -\lim_{t\downarrow 0} \mathbb{E}^x \frac{\mathrm{e}^{\mathrm{i}(X_t^\sigma - x)'\xi} - 1}{t} \tag{6}$$

the symbol *of the process, if the limit exists and coincides for every choice of K.*

In Example 4.1, we show that this symbol coincides with the classical functional analytic symbol in the case of rich Feller process. This motivates the name.

Theorem 3.6. *Let X be a h.d.w.j. such that the differential characteristics ℓ, Q and n are locally bounded and finely continuous. In this case, the limit (6) exists and the symbol of X is*

$$p(x,\xi) = -\mathrm{i}\ell(x)'\xi + \frac{1}{2}\xi' Q(x)\xi - \int_{y\neq 0}\left(\mathrm{e}^{\mathrm{i}y'\xi} - 1 - \mathrm{i}y'\xi\cdot\chi(y)\right)N(x,\mathrm{d}y). \tag{7}$$

Remark 3.7.

(a) If the differential characteristics are continuous, the conditions of the theorem are fulfilled.

(b) If the differential characteristics are globally bounded, that is, if (G) is satisfied, the limit (6) without stopping time exists and coincides with the above limit (the proof is similar).

(c) Let us mention that the symbol of a Lévy process is just its characteristic exponent, that is, $p(x,\cdot) = \psi(\cdot)$ for every $x \in \mathbb{R}^d$. Further examples can be found in the next section.

Now, we define the following helpful quantities for $x \in \mathbb{R}^d$ and $R > 0$:

$$H(x,R) := \sup_{\|y-x\|\le 2R}\ \sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{R}\right)\right|, \tag{8}$$

$$H(R) := \sup_{y\in\mathbb{R}^d}\ \sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{R}\right)\right|, \tag{9}$$

$$h(x,R) := \inf_{\|y-x\|\le 2R} \sup_{\|\varepsilon\|\le 1} \Re p\left(y, \frac{\varepsilon}{4\kappa R}\right), \tag{10}$$

$$h(R) := \inf_{y\in\mathbb{R}^d} \sup_{\|\varepsilon\|\le 1} \Re p\left(y, \frac{\varepsilon}{4\kappa R}\right). \tag{11}$$

In (10) and (11) $\kappa = (4\arctan(1/2c_0))^{-1}$ where c_0 comes from the sector condition (S) as defined in the introduction. In particular, $h(x,R)$ and $h(R)$ are only defined if (S) is satisfied and only in this case they will be used below.

Definition 3.8. *The quantities* (*cf.* [23], *Definitions* 4.2 *and* 4.5)

$$\beta_0 := \sup\left\{\lambda \ge 0\colon \limsup_{R\to\infty} R^\lambda H(R) = 0\right\},$$

$$\underline{\beta_0} := \sup\left\{\lambda \ge 0\colon \liminf_{R\to\infty} R^\lambda H(R) = 0\right\},$$

$$\overline{\delta_0} := \sup\left\{\lambda \ge 0\colon \limsup_{R\to\infty} R^\lambda h(R) = 0\right\},$$

$$\delta_0 := \sup\left\{\lambda \ge 0\colon \liminf_{R\to\infty} R^\lambda h(R) = 0\right\}$$

are called indices of X at the origin, *while*

$$\beta^x_\infty := \inf\left\{\lambda > 0\colon \limsup_{R\to 0} R^\lambda H(x,R) = 0\right\},$$

$$\underline{\beta^x_\infty} := \inf\left\{\lambda > 0\colon \liminf_{R\to 0} R^\lambda H(x,R) = 0\right\},$$

$$\overline{\delta^x_\infty} := \inf\left\{\lambda > 0\colon \limsup_{R\to 0} R^\lambda h(x,R) = 0\right\},$$

$$\delta^x_\infty := \inf\left\{\lambda > 0\colon \liminf_{R\to 0} R^\lambda h(x,R) = 0\right\}$$

are the indices of X at infinity.

Example 3.9. In the case of symmetric α-stable processes, all indices coincide and they are equal to α. For so called stable-like Feller processes (cf. [1,18]) with uniformly bounded exponential function, that is, $0 < \alpha_0 \le \alpha(x) \le \alpha_\infty < 1$ one obtains $\beta_0 = \underline{\beta_0} = \alpha_0$ and $\delta_0 = \overline{\delta_0} = \alpha_\infty$ (see [23], Example 5.5). For more examples, consult the next section.

The following proposition is the key ingredient for using the symbol to analyze fine properties of a stochastic process. Similar results were proved for Lévy processes by Pruitt in [19] and for rich Feller processes satisfying (G) and (S) by Schilling in [23]. We write

$$(X_\cdot - x)^*_t := \sup_{s\le t} \|X_s - x\|$$

for the maximum process.

Proposition 3.10. *Let X be a h.d.w.j. such that the differential characteristics of X are locally bounded and finely continuous. In this case, we have*

$$\mathbb{P}^x\big((X_\cdot - x)_t^* \geq R\big) \leq c_d \cdot t \cdot H(x,R) \tag{12}$$

for $t \geq 0$, $R > 0$ and a constant $c_d > 0$ which can be written down explicitly and only depends on the dimension d.

If (S) *holds in addition, we have*

$$\mathbb{P}^x\big((X_\cdot - x)_t^* < R\big) \leq c_\kappa \cdot \frac{1}{t} \cdot \frac{1}{h(x,R)} \tag{13}$$

for a constant c_κ only depending on the c_0 of the sector condition.

Using this result and standard Borel–Cantelli techniques, we obtain the following two theorems which describe the behavior of the process at infinity respective zero.

Theorem 3.11. *Let X be a h.d.w.j. such that the differential characteristics of X are locally bounded and finely continuous. Then we have*

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \qquad \text{for all } \lambda < \beta_0, \tag{14}$$

$$\liminf_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \qquad \text{for all } \beta_0 \leq \lambda < \underline{\beta_0}. \tag{15}$$

If the symbol p of the process X satisfies (S), *then we have in addition*

$$\limsup_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \qquad \text{for all } \overline{\delta_0} < \lambda \leq \delta_0, \tag{16}$$

$$\lim_{t\to\infty} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \qquad \text{for all } \delta_0 < \lambda. \tag{17}$$

All these limits are meant $\mathbb{P}^x$-a.s. with respect to every $x \in \mathbb{R}^d$.

Theorem 3.12. *Let X be a h.d.w.j. such that the differential characteristics of X are locally bounded and finely continuous. Then we have*

$$\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \qquad \text{for all } \lambda > \beta_\infty^x, \tag{18}$$

$$\liminf_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \qquad \text{for all } \beta_\infty^x \geq \lambda > \underline{\beta_\infty^x}. \tag{19}$$

If the symbol p of the process X satisfies (S), *then we have in addition*

$$\limsup_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \qquad \text{for all } \overline{\delta_\infty^x} > \lambda \geq \delta_\infty^x, \tag{20}$$

$$\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = \infty \qquad \text{for all } \delta_\infty^x > \lambda. \tag{21}$$

All these limits are meant $\mathbb{P}^x$-a.s with respect to every $x \in \mathbb{R}^d$.

The relation between indices of this type associated with Lévy processes and the classical Blumenthal–Getoor respective Pruitt indices were analyzed in Section 5 of [23].

4. Examples, applications, complementary results

In the present section, we show how the above results can be used for some classes of processes. The first example explains the connection with the classical Markovian theory. The second one deals with Lévy driven SDEs having unbounded coefficients and the third one with the COGARCH process.

Example 4.1. Let X be a Feller processes, that is, a strong Markov process such that

(F1) $T_t : C_\infty(\mathbb{R}^d) \to C_\infty(\mathbb{R}^d)$ for every $t \geq 0$,
(F2) $\lim_{t\downarrow 0} \|T_t u - u\|_\infty = 0$ for every $u \in C_\infty(\mathbb{R}^d)$,

where

$$T_t u(x) := \mathbb{E}^x u(X_t), \qquad t \geq 0, x \in \mathbb{R}^d$$

and $C_\infty(\mathbb{R}^d)$ denotes the real-valued continuous functions vanishing at infinity. The generator $(A, D(A))$ of the process is the closed operator given by

$$Au := \lim_{t\downarrow 0} \frac{T_t u - u}{t} \qquad \text{for } u \in D(A), \tag{22}$$

where the domain $D(A)$ consists of all $u \in C_\infty(\mathbb{R}^d)$ for which the limit (22) exists uniformly. Using a classical result due to Courrège [8], Jacob (cf. [12], Section 4.5) showed that the generator A of a process of this kind can be written in the following way:

$$Au(x) = -\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x'\xi} q(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi \qquad \text{for } u \in C_c^\infty\big(\mathbb{R}^d\big),$$

where $\widehat{u}(\xi) = (2\pi)^{-d} \int \mathrm{e}^{-\mathrm{i}y'\xi} u(y)\,\mathrm{d}y$ denotes the Fourier transform. The functional analytic symbol $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ has the following properties: it is locally bounded, $q(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^d$ and $q(x,\cdot)$ is a c.n.d.f. for every $x \in \mathbb{R}^d$. The last point means that the symbol admits a 'state-space dependent' Lévy–Khinchine formula like (7). In Lemma 3.3 of [25], we have shown that the symbol q always satisfies (LG).

By Theorem 3.10 of [27], every rich Feller process is an Itô process and the differential characteristics are equal to the Lévy triplet of the symbol. From Corollary 4.5 of the same thesis, we deduce that for a rich Feller process with finely continuous differential characteristics the functional analytic symbol and the probabilistic symbol do coincide, that is, $p(x,\xi) = q(x,\xi)$ for every $x, \xi \in \mathbb{R}^d$. Furthermore, this shows that the case treated in Schilling [23] is encompassed by our considerations. Having a look at his Theorem 3.5, this does not seem to be the case, because the characteristics look differently, but this is due to a different choice of the cut-off function.

Example 4.2. Let $(Z_t)_{t\geq 0}$ be an $\mathbb{R}^n$-valued Lévy process. The solution of the stochastic differential equation

$$\begin{aligned} \mathrm{d}X_t^x &= \Phi\big(X_{t-}^x\big)\,\mathrm{d}Z_t, \\ X_0^x &= x, \qquad x \in \mathbb{R}^d, \end{aligned} \tag{23}$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is locally Lipschitz continuous and satisfies the standard linear growth condition, admits the symbol

$$p(x,\xi) = \psi\big(\Phi(x)'\xi\big),$$

where $\psi : \mathbb{R}^n \to \mathbb{C}$ denotes the characteristic exponent of the Lévy process. This was shown in [24]. Fine properties could only be obtained for the case of bounded Φ, because in general the solution of the above SDE is not rich Feller. Using the classical characterization of Itô processes due to Cinlar and Jacod ([6], Theorem 3.33), it is straightforward to show that X belongs to this class. Since Φ and ψ are continuous, the symbol is finely continuous. Along the same lines as in [24], we obtain the following two results.

Theorem 4.3. *Let $p(x,\xi)$ be a state-space dependent c.n.d.f. which can be written as $p(x,\xi) = \psi(\Phi(x)'\xi)$ where $\psi : \mathbb{R}^n \to \mathbb{C}$ is a c.n.d.f. and $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is locally Lipschitz continuous and satisfies the linear growth condition. In this case there exists a corresponding Itô process, that is, a process X with symbol $p(x,\xi)$.*

Theorem 4.4. *Let Z be a driving Lévy process with non-constant symbol. Let X be the solution of* (23) *such that $d = n$ and the rank of Φ is equal to d in every point. Then*

$$\lim_{t\to 0} t^{-1/\lambda}(X_\cdot - x)_t^* = 0 \qquad \text{if } \lambda > \beta_\infty,$$

where β_∞ is the index of the driving Lévy process Z.

Example 4.5. Let us recall how the COGARCH process is defined (cf. [16]):

Let $Z = (Z_t)_t$ be a Lévy process with triplet (ℓ, Q, N) and fix $0 < \delta < 1$, $\beta > 0$, $\lambda \geq 0$. The volatility process $(S_t)_{t\geq 0}$ is the solution of the SDE

$$\begin{aligned} \mathrm{d}S_t^2 &= \beta\,\mathrm{d}t + S_t^2\left(\log\delta\,\mathrm{d}t + \frac{\lambda}{\delta}\,\mathrm{d}\left(\sum_{0<s\leq t}(\Delta Z_s)^2\right)\right), \\ S_0 &= S\,(>0). \end{aligned}$$

The process

$$G_t := g + \int_0^t S_{s-}\,\mathrm{d}Z_t, \qquad g \in \mathbb{R}$$

is called *COGARCH process*. The pair (G_t, S_t) is a (normal) Markov process which is is homogeneous in space in the first component. It is not a Feller process, at least not a C_∞-

Feller process. Furthermore (G_t, S_t^2) is an Itô process, which follows by combining Theorem 3.33 of [6] with Proposition IX.5.2 of [15]. To avoid problems which might arise for processes defined on $\mathbb{R} \times \mathbb{R}_+$, we consider the logarithmic squared volatility, that is, the process $(G_t, V_t) = (G_t, \log(S_t^2))$. This process admits the symbol $p : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}$ given by

$$\begin{aligned}
&p\left(\begin{pmatrix} g \\ v \end{pmatrix}, \xi\right) \\
&\quad = -\mathrm{i}\xi_1\left(\ell \mathrm{e}^{v/2} + \mathrm{e}^{v/2}\int_{\mathbb{R}\setminus\{0\}} y \cdot (1_{\{|\mathrm{e}^{v/2}y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)y^2)|<1\}} - 1_{\{|y|<1\}}) N(\mathrm{d}y)\right) \\
&\qquad - \mathrm{i}\xi_2\left(\frac{\beta}{\mathrm{e}^v} + \log\delta + \int_{\mathbb{R}\setminus\{0\}} \log\left(1 + \frac{\lambda}{\delta}y^2\right) \cdot (1_{\{|\mathrm{e}^{v/2}y|<1\}} \cdot 1_{\{|\log(1+(\lambda/\delta)y^2)|<1\}}) N(\mathrm{d}y)\right) \\
&\qquad + \frac{1}{2}\xi_1^2 \mathrm{e}^v Q \\
&\qquad - \int_{\mathbb{R}^2\setminus\{0\}} \big(\mathrm{e}^{\mathrm{i}(z_1,z_2)\xi} - 1 - \mathrm{i}z'\xi \cdot (1_{\{|z_1|<1\}} \cdot 1_{\{|z_2|<1\}})\big) \tilde{N}\left(\begin{pmatrix} g \\ v \end{pmatrix}, \mathrm{d}z\right),
\end{aligned}$$

where $\tilde{N}$ is the image measure

$$\tilde{N}\left(\begin{pmatrix} g \\ v \end{pmatrix}, \mathrm{d}z\right) = N(f_v \in \mathrm{d}z)$$

under $f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f_v(w) = \begin{pmatrix} \mathrm{e}^{v/2} w \\ \log\big(1 + (\lambda/\delta)w^2\big) \end{pmatrix}.$$

This was shown in [29]. A typical driving term in mathematical finance is the variance gamma process (cf. [5] and [17]). This is a pure jump Lévy process with

$$N(\mathrm{d}y) = \frac{C}{|y|}\exp\big(-(2C)^{-1/2}|y|\big)\,\mathrm{d}y$$

for a constant $C > 0$. In order to have a concrete example, let $\lambda = 2$, $\delta = 1/2$, $\beta = 10$ and $C = 2$. Using standard calculus we obtain that $\beta_0 = 1$. The calculations are elementary but tedious. By Theorem 3.11, we obtain for $g \in \mathbb{R}$

$$\lim_{t\to\infty} t^{-1/\lambda}(G_\cdot - g)_t^* = 0 \qquad \text{for all } \lambda < 1.$$

In the future, the indices will be used in order to obtain other fine properties of non-Feller processes.

Now we consider the special case of a process which consists of independent components.

Proposition 4.6. *Let X be a d-dimensional vector of independent h.d.w.j.'s $X^{(j)}$ with symbols $p^{(j)}$, $j=1,\dots,d$. The process X admits the symbol*

$$p(x,\xi)=p^{(1)}\bigl(x^{(1)},\xi^{(1)}\bigr)+\cdots+p^{(d)}\bigl(x^{(d)},\xi^{(d)}\bigr).$$

Proof. We give the proof for two components. The general case follows inductively. Let X and Y be independent h.d.w.j.'s with symbols $p(x,\xi_1)$, respectively, $q(y,\xi_2)$, where the sum of the dimensions of x and y is d, and consider:

$$\begin{aligned}
&\mathbb{E}^{(x,y)}\frac{e^{i(X_t-x)'\xi_1+i(Y_t-y)'\xi_2}-1}{t}\\
&\quad=\frac{\mathbb{E}^{(x,y)}(e^{i(X_t-x)'\xi_1+i(Y_t-y)'\xi_2})-1}{t}\\
&\quad=\frac{\mathbb{E}^{x}(e^{i(X_t-x)'\xi_1})\cdot\mathbb{E}^{y}(e^{i(Y_t-y)'\xi_2})-1}{t}\\
&\quad=\frac{\mathbb{E}^{x}(e^{i(X_t-x)'\xi_1})\cdot\mathbb{E}^{y}(e^{i(Y_t-y)'\xi_2})-\mathbb{E}^{y}(e^{i(Y_t-y)'\xi_2})+\mathbb{E}^{y}(e^{i(Y_t-y)'\xi_2})-1}{t}\\
&\quad=\frac{\mathbb{E}^{x}(e^{i(X_t-x)'\xi_1})-1}{t}\cdot\mathbb{E}^{y}\bigl(e^{i(Y_t-y)'\xi_2}\bigr)+\frac{\mathbb{E}^{y}(e^{i(Y_t-y)'\xi_2})-1}{t}.
\end{aligned}$$

The three terms on the right-hand side tend to $-p(x,\xi_1)$, 1 and $-q(y,\xi_2)$, respectively. Hence, the result. □

5. Proofs of the main results

In this section, we present the proofs of the main results.

Proof of Theorem 3.6. Let $x\in\mathbb{R}^d$ and let the stopping time defined as in Definition 3.5 where K is an arbitrary compact neighborhood of x. We give the one dimensional proof, since the multidimensional version works alike; only the notion becomes more involved. First, we use Itô's formula under the expectation and obtain

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\bigl(e^{i(X_t^\sigma-x)\xi}-1\bigr)\\
&\quad=\frac{1}{t}\mathbb{E}^x\left(\int_{0+}^t i\xi e^{i(X_{s-}^\sigma-x)\xi}\,dX_s^\sigma\right) &&\text{(I)}\\
&\qquad+\frac{1}{t}\mathbb{E}^x\left(\frac{1}{2}\int_{0+}^t -\xi^2 e^{i(X_{s-}^\sigma-x)\xi}\,d\bigl[X^\sigma,X^\sigma\bigr]_s^c\right) &&\text{(II)}\\
&\qquad+\frac{1}{t}\mathbb{E}^x\left(e^{-ix\xi}\sum_{0<s\le t}\bigl(e^{i\xi X_s^\sigma}-e^{i\xi X_{s-}^\sigma}-i\xi e^{i\xi X_{s-}^\sigma}\Delta X_s^\sigma\bigr)\right). &&\text{(III)}
\end{aligned}$$

The left-continuous process X^σ_{t-} is bounded on $[[0,\sigma]]$. Furthermore, we have $(\Delta X)^\sigma = (\Delta X^\sigma)$ and X^σ admits the stopped characteristics

$$\begin{aligned}
B^\sigma_t(\omega) &= \int_0^{t\wedge\sigma(\omega)} \ell\big(X_s(\omega)\big)\,\mathrm{d}s = \int_0^t \ell\big(X_s(\omega)\big)1_{[[0,\sigma]]}(\omega,s)\,\mathrm{d}s,\\
C^\sigma_t(\omega) &= \int_0^t Q\big(X_s(\omega)\big)1_{[[0,\sigma]]}(\omega,s)\,\mathrm{d}s, \\
\nu^\sigma(\omega;\mathrm{d}s,\mathrm{d}y) &:= 1_{[[0,\sigma]]}(\omega,s)N\big(X_s(\omega),\mathrm{d}y\big)\,\mathrm{d}s
\end{aligned} \tag{24}$$

with respect to the fixed cut-off function χ. One can now set the integrand at the right endpoint of the stochastic support to zero, as we are integrating with respect to Lebesgue measure:

$$\begin{aligned}
B^\sigma_t(\omega) &= \int_0^t \ell\big(X_s(\omega)\big)1_{[[0,\sigma[[}(\omega,s)\,\mathrm{d}s,\\
C^\sigma_t(\omega) &= \int_0^t Q\big(X_s(\omega)\big)1_{[[0,\sigma[[}(\omega,s)\,\mathrm{d}s,\\
\nu^\sigma(\omega;\mathrm{d}s,\mathrm{d}y) &= 1_{[[0,\sigma[[}(\omega,s)N\big(X_s(\omega),\mathrm{d}y\big)\,\mathrm{d}s.
\end{aligned}$$

In the first two lines, the integrand is now bounded, because ℓ and Q are locally bounded and $\|X^\sigma_s(\omega)\| < k$ on $[0,\sigma(\omega)[$ for every $\omega\in\Omega$. In what follows, we will deal with the terms one-by-one. To calculate the first term, we use the canonical decomposition of the semimartingale (see [15], Theorem II.2.34) which we write as follows

$$\begin{aligned}
X^\sigma_t &= X_0 + X^{\sigma,c}_t + \int_0^{t\wedge\sigma} \chi(y)y\big(\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y) - \nu^\sigma(\cdot;\mathrm{d}s,\mathrm{d}y)\big)\\
&\quad + \check{X}^\sigma(\chi) + B^\sigma_t(\chi),
\end{aligned} \tag{25}$$

where $\check{X}_t = \sum_{s\le t}(\Delta X_s(1-\chi(\Delta X_s)))$. Therefore, term (I) can be written as

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\bigg(\int_{0+}^t \mathrm{i}\xi\mathrm{e}^{\mathrm{i}(X^\sigma_{s-}-x)\xi}\,\mathrm{d}\Big(\underbrace{X^{\sigma,c}_t}_{\text{(IV)}} + \underbrace{\int_0^{t\wedge\sigma} \chi(y)y\big(\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y) - \nu^\sigma(\cdot;\mathrm{d}s,\mathrm{d}y)\big)}_{\text{(V)}}\\
&\qquad + \underbrace{\check{X}^\sigma(\chi)}_{\text{(VI)}} + \underbrace{B^\sigma_t(\chi)}_{\text{(VII)}}\Big)\bigg).
\end{aligned}$$

We use the linearity of the stochastic integral. Our first step is to prove for term (IV)

$$\mathbb{E}^x \int_{0+}^t \mathrm{i}\xi\mathrm{e}^{\mathrm{i}(X^\sigma_{s-}-x)\xi}\,\mathrm{d}X^{\sigma,c}_s = 0.$$

The integral $\mathrm{e}^{\mathrm{i}(X_{t-}^\sigma - x)\xi} \bullet X_t^{\sigma,c}$ is a local martingale, since $X_t^{\sigma,c}$ is a local martingale. To see that it is indeed a martingale, we calculate the following:

$$\begin{aligned}\big[\mathrm{e}^{\mathrm{i}(X^\sigma - x)\xi} \bullet X^{\sigma,c}, \mathrm{e}^{\mathrm{i}(X^\sigma - x)\xi} \bullet X^{\sigma,c}\big]_t &= \big[\mathrm{e}^{\mathrm{i}(X^\sigma - x)\xi} \bullet X^{c}, \mathrm{e}^{\mathrm{i}(X^\sigma - x)\xi} \bullet X^{c}\big]_t^\sigma \\ &= \int_0^t \big(\mathrm{e}^{\mathrm{i}(X_s^\sigma - x)\xi}\big)^2 1_{[[0,\sigma]]}(s)\,\mathrm{d}\big[X^c, X^c\big]_s \\ &= \int_0^t \big(\big(\mathrm{e}^{\mathrm{i}(X_s^\sigma - x)\xi}\big)^2 1_{[[0,\sigma[[}(s) Q(X_s)\big)\,\mathrm{d}s,\end{aligned}$$

where we have used several well known facts about the square bracket. The last term is uniformly bounded in ω and therefore, finite for every $t \geq 0$. This means that $\mathrm{e}^{\mathrm{i}(X_t^\sigma - x)\xi} \bullet X_t^{\sigma,c}$ is an L^2-martingale which is zero at zero and therefore, its expected value is constantly zero.

The same is true for the integrand (V). We show that the function $H_{x,\xi}(\omega, s, y) := \mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi} \cdot y\chi(y)$ is in the class F_p^2 of Ikeda and Watanabe (see [10], Section II.3), that is,

$$\mathbb{E}^x \int_0^t \int_{y\neq 0} \big|\mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi} \cdot y\chi(y)\big|^2 \nu^\sigma(\cdot; \mathrm{d}s, \mathrm{d}y) < \infty.$$

To prove this, we observe

$$\begin{aligned}&\mathbb{E}^x \int_0^t \int_{y\neq 0} \big|\mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi}\big|^2 \cdot \big|y\chi(y)\big|^2 \nu^\sigma(\cdot; \mathrm{d}s, \mathrm{d}y) \\ &\quad = \mathbb{E}^x \int_0^t \int_{y\neq 0} \big|y\chi(y)\big|^2 1_{[[0,\sigma[[}(\omega, s) N(X_s, \mathrm{d}y)\,\mathrm{d}s.\end{aligned}$$

Since we have by hypothesis $\| \int_{y\neq 0} (1 \wedge y^2) 1_{[[0,\sigma[[} N(\cdot, \mathrm{d}y)\|_\infty < \infty$, this expected value is finite. Therefore, the function $H_{x,\xi}$ is in F_p^2 and we conclude that

$$\begin{aligned}&\int_0^t \mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi}\,\mathrm{d}\bigg(\int_0^{s\wedge\sigma} \int_{y\neq 0} \chi(y) y \big(\mu^{X^\sigma}(\cdot; \mathrm{d}r, \mathrm{d}y) - \nu^\sigma(\cdot; \mathrm{d}r, \mathrm{d}y)\big)\bigg) \\ &\quad = \int_0^t \int_{y\neq 0} \big(\mathrm{e}^{\mathrm{i}(X_{s-} - x)\xi} \chi(y) y\big)\big(\mu^{X^\sigma}(\cdot; \mathrm{d}s, \mathrm{d}y) - \nu^\sigma(\cdot; \mathrm{d}s, \mathrm{d}y)\big)\end{aligned}$$

is a martingale. The last equality follows from [15], Theorem I.1.30.

Now we deal with the second term (II). Here we have

$$\big[X^\sigma, X^\sigma\big]_t^c = \big[X^c, X^c\big]_t^\sigma = C_t^\sigma = \big(Q(X_t) \bullet t\big)^\sigma = \big(Q(X_t) \cdot 1_{[[0,\sigma[[}(t)\big) \bullet t$$

and therefore,

$$\frac{1}{2} \int_{0+}^t -\xi^2 \mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi}\,\mathrm{d}\big[X^\sigma, X^\sigma\big]_s^c = -\frac{1}{2}\xi^2 \int_0^t \mathrm{e}^{\mathrm{i}(X_{s-}^\sigma - x)\xi} Q(X_s) \cdot 1_{[[0,\sigma[[}(t)\,\mathrm{d}s.$$

Since Q is finely continuous and locally bounded, we obtain by dominated convergence

$$-\lim_{t\downarrow 0}\frac{1}{2}\xi^2\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{\mathrm{i}(X_s-x)\xi}Q(X_s)1_{[[0,\sigma[[}(s)\,\mathrm{d}s=-\frac{1}{2}\xi^2 Q(x).$$

For the finite variation part of the first term, that is, (VII), we obtain analogously

$$\lim_{t\downarrow 0}\mathrm{i}\xi\frac{1}{t}\mathbb{E}^x\int_0^t \mathrm{e}^{\mathrm{i}(X_s-x)\xi}\ell(X_s)1_{[[0,\sigma[[}(s)\,\mathrm{d}s=\mathrm{i}\xi\ell(x).$$

Now we have to deal with the various jump parts. At first, we write the sum in (III) as an integral with respect to the jump measure μ^{X^σ} of the process:

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}x\xi}\sum_{0<s\le t}\big(\mathrm{e}^{\mathrm{i}X_s\xi}-\mathrm{e}^{\mathrm{i}X_{s-}\xi}-\mathrm{i}\xi\mathrm{e}^{\mathrm{i}\xi X_{s-}}\Delta X_s\big)\\
&\quad=\mathrm{e}^{-\mathrm{i}x\xi}\sum_{0<s\le t}\big(\mathrm{e}^{\mathrm{i}X_{s-}\xi}\big(\mathrm{e}^{\mathrm{i}\xi\Delta X_s}-1-\mathrm{i}\xi\Delta X_s\big)\big)\\
&\quad=\int_{]0,t]\times\mathbb{R}^d}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\big)1_{\{y\neq 0\}}\big)\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad=\int_{]0,t]\times\{y\neq 0\}}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)\big)\big)\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\qquad+\int_{]0,t]\times\{y\neq 0\}}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(-\mathrm{i}\xi y\cdot\big(1-\chi(y)\big)\big)\big)\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y).
\end{aligned}$$

The last term cancels with the one we left behind from (I), given by (VI). For the remainder-term, we get:

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)\big)\big)1_{[[0,\sigma[[}(\cdot,s)\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)\big)\big)1_{[[0,\sigma[[}(\cdot,s)\nu^{\sigma}(\cdot;\mathrm{d}s,\mathrm{d}y)\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\underbrace{\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)\big)\big)1_{[[0,\sigma[[}(\cdot,s)N(X_s,\mathrm{d}y)}_{:=g(s-,\cdot)}\mathrm{d}s\\
&\quad=\frac{1}{t}\mathbb{E}^x\int_{]0,t]\times\{y\neq 0\}}\big(\mathrm{e}^{\mathrm{i}(X_{s-}-x)\xi}\big(\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)\big)\big)1_{[[0,\sigma[[}(\cdot,s)N(X_s,\mathrm{d}y)\,\mathrm{d}s.
\end{aligned}$$

Here we have used the fact that it is possible to integrate with respect to the compensator of a random measure instead of the measure itself, if the integrand is in F_p^1 (see [10], Section II.3). The function $g(s,\omega)$ is measurable and bounded by our assumption, since $|\mathrm{e}^{\mathrm{i}\xi y}-1-\mathrm{i}\xi y\chi(y)|\le$

$const \cdot (1 \wedge \|y\|^2)$. Hence, $g \in F_p^1$. Again by bounded convergence, we obtain

$$\begin{aligned}\lim_{t\downarrow 0} \frac{1}{t}\mathbb{E}^x \int_0^t \mathrm{e}^{\mathrm{i}(X_s - x)\xi} \int_{y\neq 0} \bigl(\mathrm{e}^{\mathrm{i}y\xi} - 1 - \mathrm{i}y\xi\chi(y)\bigr) N(X_s, \mathrm{d}y)\,\mathrm{d}s \\ = \int_{y\neq 0} \bigl(\mathrm{e}^{\mathrm{i}y\xi} - 1 - \mathrm{i}y\xi\chi(y)\bigr) N(x, \mathrm{d}y).\end{aligned}$$

This is the last part of the symbol. Here, we have used the continuity assumption on $N(x, \mathrm{d}y)$. □

Now we prepare the proof of Proposition 3.10, our technical main result. It will turn out to be useful to have a closer look at the symbol (7). The real part of p is $\Re(p(x,\xi)) = (1/2)\xi' Q(x)\xi - \int_{y\neq 0}(\cos(y'\xi) - 1)N(x, \mathrm{d}y)$ and therefore, we obtain

$$\int_{y\neq 0} \bigl(1 - \cos(y'\eta)\bigr) N(x, \mathrm{d}y) \le \Re\bigl(p(x,\xi)\bigr). \tag{26}$$

We assume for the remainder of this section: $R > 0$ and $S > 2R$. χ is a fixed cut-off function such that

$$\chi \in C_c^\infty(\mathbb{R}^d); \qquad 1_{B_R(0)} \le \chi \le 1_{B_{2R}(0)}; \qquad \chi(y) = \chi(-y) \qquad \text{for every } y \in \mathbb{R}^d.$$

The stopping time $\sigma = \sigma_R$ is defined as follows

$$\sigma := \inf\bigl\{t \ge 0\colon \|X_t - x\| > S\bigr\}.$$

We need the following two lemmas.

Lemma 5.1. *For every $z \in \mathbb{R}^d$, we have*

$$\bigl(\|z\|^2 \wedge 1\bigr) \le c\bigl(1 - \mathrm{e}^{-\|z\|^2/2}\bigr) \le c\bigl(\|z\|^2 \wedge 1\bigr),$$

where $c = 1/(1 - \exp(-1/2))$ and

$$\bigl(1 - \mathrm{e}^{-\|z\|^2/2}\bigr) = \int_{\mathbb{R}^d} \bigl(1 - \cos(z'\eta)\bigr) h_d \,\mathrm{d}\eta$$

with

$$h_d(\eta) = \frac{1}{(\sqrt{2\pi})^d}\mathrm{e}^{-\|\eta\|^2/2}.$$

The proof is elementary and hence omitted.

Lemma 5.2. *Let $p(x,\xi)$ be the symbol (7) and $R > 0$. Then we have*

$$\int_{z\neq 0} \left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N(y, \mathrm{d}z) \le \widetilde{c}_d \sup_{\|\varepsilon\|\le 1} \left| p\left(y, \frac{\varepsilon}{2R}\right)\right|,$$

where $\widetilde{c}_d = 2c(d+1)$ *with the* c *of Lemma* 5.1.

Proof. By the above lemma, we obtain

$$\begin{aligned}
LHS &\le c\int_{z\neq 0}\left(1-\exp\left(-\left\|\frac{z}{2R}\right\|^2\Big/2\right)\right)N(y,\mathrm{d}z)\\
&= c\int_{z\neq 0}\int_{\mathbb{R}^d}\left(1-\cos\left(\frac{1}{2R}(z'\eta)\right)\right)\frac{1}{(\sqrt{2\pi})^d}\mathrm{e}^{-\|\eta\|^2/2}\,\mathrm{d}\eta\, N(y,\mathrm{d}z)\\
&\le c\int_{\mathbb{R}^d}\Re p\left(y,\frac{\eta}{2R}\right)h_d(\eta)\,\mathrm{d}\eta\\
&\le 2c\int_{\mathbb{R}^d}\sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{2R}\right)\right|\left(1+\|\eta\|^2\right)h_d(\eta)\,\mathrm{d}\eta\\
&= \sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{2R}\right)\right|\int_{\mathbb{R}^d}2c\left(1+\|\eta\|^2\right)h_d(\eta)\,\mathrm{d}\eta,
\end{aligned}$$

where we have used the Tonelli–Fubini theorem, the inequality (26) and a standard estimate of the c.n.d.f. $\eta\mapsto p(y,\eta/(2R))$ as it can be found in the proof of Lemma 3.2 in [27]. □

Proof of Proposition 3.10. Let X be a h.d.w.j. such that the differential characteristics (ℓ, Q, n) of X are locally bounded and finely continuous. At first, we show that for S, R and σ as above we have

$$\mathbb{P}^x\left(\left(X_\cdot^\sigma - x\right)_t^* \ge 2R\right) \le c_d\cdot t\cdot \sup_{\|y-x\|\le S}\sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{2R}\right)\right|, \tag{27}$$

where $c_d = 4d + 16\widetilde{c}_d$. Having proved this the result follows easily.

The semimartingale characteristics of the stopped process X^σ are given in (24) above. Now, we use a double stopping technique introducing

$$\tau_R := \inf\left\{t\ge 0\colon \left\|\Delta X_t^\sigma\right\| > R\right\}.$$

We start with

$$\mathbb{P}^x\left(\left(X_\cdot^\sigma - x\right)_t^* \ge 2R\right) \le \mathbb{P}^x\left(\left(X_\cdot^\sigma - x\right)_t^* \ge 2R, \tau_R > t\right) + \mathbb{P}^x(\tau_R\le t) \tag{28}$$

and deal with the terms on the right-hand side one after another, starting with the first one.

We show how to separate the first term of (28) again in order to get control over the big jumps. Let $\check{X}$ be as defined in equation (25). The semimartingale $\check{X}^\sigma$ admits the following third characteristic: $\chi(y)1_{[[0,\sigma]]}(s)N(X_s,\mathrm{d}y)\,\mathrm{d}s$. Now let $u=(u_1,\dots,u_d)'\colon\mathbb{R}^d\to\mathbb{R}^d$ be such that $u_j\in C_b^2(\mathbb{R}^d)$ is 1-Lipschitz continuous, u_j depends only on $x^{(j)}$ and is zero in zero for $j=$

$1, \dots, d$. We define the auxiliary process

$$\check{M}_t := u\big(\check{X}_t^\sigma - x\big) - \int_0^{t\wedge\sigma} F_s \, \mathrm{d}s,$$

where

$$\begin{aligned} F_s^{(j)} = {} & \partial_j u(\check{X}_{s-} - x)\ell^{(j)}(X_{s-}) \\ & - \frac{1}{2}\partial_j \partial_j u(\check{X}_{s-} - x) Q^{jj}(X_{s-}) \\ & - \int_{z\neq 0} \big(u(\check{X}_{s-} - x + z) - u(\check{X}_{s-} - x) \\ & \qquad - \chi(z) z^{(j)} \partial_j u(\check{X}_{s-} - x)\big)\chi(z) N(X_{s-}, \mathrm{d}z). \end{aligned} \tag{29}$$

$\check{M}$ is a local martingale by [15], Theorem II.2.42 and by Lemma 3.7 of [27] we have under (LG):

$$\big|F_s^{(j)}\big| \le const \cdot \sum_{0\le|\alpha|\le 2} \big\|\partial^\alpha u\big\|_\infty$$

since $u_j \in C_b^2(\mathbb{R}^d)$. In particular, for every fixed $t > 0$ $\check{M}$ is an L^2-martingale on $[0, t]$. Now we define

$$D := \left\{\omega \in \Omega\colon \int_0^{t\wedge\sigma(\omega)} \big\|F_s(\omega)\big\| \, \mathrm{d}s \le R\right\}$$

and obtain

$$\mathbb{P}^x\big(\big(X_\cdot^\sigma - x\big)_t^* \ge 2R, \tau_R > t\big) \le \mathbb{P}^x\big(\big(X_\cdot^\sigma - x\big)_t^* \ge 2R, \tau_R > t, D\big) + \mathbb{P}^x\big(D^c\big). \tag{30}$$

Using Doob's inequality and the Lipschitz property of u, we obtain at first

$$\begin{aligned} \mathbb{P}^x\big(u\big(X_\cdot^\sigma - x\big)_t^* \ge 2R, \tau_R > t, D\big) & \le \mathbb{P}^x\left(u\big(X_\cdot^\sigma - x\big)_t^* - \int_0^{\cdot\wedge\sigma} F_s \, \mathrm{d}s \ge R, \tau_R > t, D\right) \\ & \le \mathbb{P}^x\big(\check{M}_{t\wedge\sigma}^* \ge R\big) \\ & \le \frac{1}{R^2}\mathbb{E}^x\big(\big\|\check{M}_t^\sigma\big\|^2\big) \\ & \le \frac{1}{R^2}\sum_{j=1}^d \mathbb{E}^x\big(\big[\check{X}_\cdot^{(j)}, \check{X}_\cdot^{(j)}\big]_t^\sigma\big). \end{aligned}$$

Since

$$\mathbb{E}^x\big(\big[\check{X}_\cdot^{(j)}, \check{X}_\cdot^{(j)}\big]_t^\sigma\big) = \mathbb{E}^x\big(\big\langle\check{X}_\cdot^{(j),c}, \check{X}_\cdot^{(j),c}\big\rangle_t^\sigma\big) + \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\int_{z\neq 0} \big(z^{(j)}\big)^2 \chi(z)^2 N(X_s, \mathrm{d}z) \, \mathrm{d}s\right)$$

we obtain

$$\begin{aligned}
&\mathbb{P}^x\big(u\big(X^\sigma_\cdot - x\big)^*_t \geq 2R,\, \tau_R > t,\, D\big)\\
&\leq \frac{1}{R^2}\sum_{j=1}^d \mathbb{E}^x \int_0^{t\wedge\sigma} Q^{jj}(X_s)\,\mathrm{d}s + \mathbb{E}^x \int_0^{t\wedge\sigma}\int_{z\neq 0} \frac{\|z\|^2}{R^2}\chi(z)^2 N(X_s, z)\,\mathrm{d}s\\
&\leq 4\sum_{j=1}^d \mathbb{E}^x \int_0^{t\wedge\sigma} \left(\frac{e_j'}{2R} Q(X_s)\frac{e_j}{2R}\right)\mathrm{d}s + 4^2\mathbb{E}^x \int_0^{t\wedge\sigma}\int_{z\neq 0}\left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N(X_s, \mathrm{d}z)\,\mathrm{d}s\\
&\leq 4t\sum_{j=1}^d \sup_{s<t\wedge\sigma} \Re p\left(X_s, \frac{e_j}{2R}\right) + 4^2 \sup_{\|y-x\|\leq S}\int_0^{t\wedge\sigma}\int_{z\neq 0}\left(\left\|\frac{z}{2R}\right\|^2 \wedge 1\right) N(y, \mathrm{d}z)\,\mathrm{d}s\\
&\leq 4td \sup_{\|y-x\|\leq S}\sup_{\|\varepsilon\|\leq 1}\left|p\left(y, \frac{\varepsilon}{2R}\right)\right| + 4^2 t \sup_{\|y-x\|\leq S}\widetilde{c}_d \sup_{\|\varepsilon\|\leq 1}\left|p\left(y, \frac{\varepsilon}{2R}\right)\right|,
\end{aligned}$$

where we have used Lemma 5.2 on the second term. By choosing a sequence $(u_n)_{n\in\mathbb{N}}$ of functions of the type described above which tends to the identity in a monotonous way, we obtain

$$\mathbb{P}^x\big(\big(X^\sigma_\cdot - x\big)^*_t \geq 2R,\, \tau_R > t,\, D\big) \leq \big(4d + 4^2\widetilde{c}_d\big)t \sup_{\|y-x\|\leq S}\sup_{\|\varepsilon\|\leq 1}\left|p\left(y, \frac{\varepsilon}{2R}\right)\right|. \tag{31}$$

Now we deal with the second term of (30). By the Markov inequality, we get

$$\mathbb{P}^x\big(D^c\big) = \mathbb{P}^x\left(\int_0^{t\wedge\sigma}\|F_s\|\,\mathrm{d}s > R\right) \leq \frac{1}{R}\sum_{j=1}^d \mathbb{E}^x\left(\int_0^{t\wedge\sigma}\big|F_s^{(j)}\big|\,\mathrm{d}s\right) =: (*).$$

Again, we chose a sequence $(u_n)_{n\in\mathbb{N}}$ of functions as we described in (29), but this time it is important that the first and second derivatives are uniformly bounded. Since the u_n converge to the identity, the first partial derivatives tend to 1 and the second partial derivatives to 0. In the limit ($n\to\infty$), we obtain

$$\begin{aligned}
(*) &\leq \frac{1}{R}\sum_{j=1}^d \mathbb{E}^x\int_0^{t\wedge\sigma}\left|\ell^{(j)}(X_s) + \int_{z\neq 0}\big(-z^{(j)}\chi(z) + \big(\chi(z)\big)^2 z^{(j)}\big)N(X_s, \mathrm{d}z)\right|\mathrm{d}s\\
&\leq 2\sum_{j=1}^d \mathbb{E}^x\int_0^{t\wedge\sigma}\left|\frac{\ell^{(j)}(X_s)}{2R} + \int_{z\neq 0}\sin\left(\frac{z'e_j}{2R}\right) - \frac{z^{(j)}\chi(z)}{2R}N(X_s, \mathrm{d}z)\right|\mathrm{d}s \qquad (32)\\
&\quad + 2\sum_{j=1}^d \mathbb{E}^x\int_0^{t\wedge\sigma}\left|\int_{z\neq 0}\frac{(\chi(z))^2 z^{(j)}}{2R} - \sin\left(\frac{z'e_j}{2R}\right)N(X_s, \mathrm{d}z)\right|\mathrm{d}s. \qquad (33)
\end{aligned}$$

For term (32), we get

$$
\begin{aligned}
&2\sum_{j=1}^{d}\mathbb{E}^x\int_0^{t\wedge\sigma}\left|\frac{\ell(X_s)'e_j}{2R}+\int_{z\neq 0}\sin\left(\frac{z'e_j}{2R}\right)-\frac{z'e_j\chi(z)}{2R}N(X_s,\mathrm{d}z)\right|\mathrm{d}s\\
&\quad\le 2td\sup_{s\le t\wedge\sigma}\mathbb{E}^x\left|\frac{\ell(X_s)'e_j}{2R}+\int_{z\neq 0}\sin\left(\frac{z'e_j}{2R}\right)-\frac{z'e_j\chi(z)}{2R}N(X_s,\mathrm{d}z)\right| \qquad (34)\\
&\quad\le 2td\sup_{\|y-x\|\le S}\sup_{\|\varepsilon\|\le 1}\left|\Im p\left(y,\frac{\varepsilon}{2R}\right)\right|
\end{aligned}
$$

and for term (33)

$$
\begin{aligned}
&2\sum_{j=1}^{d}\mathbb{E}^x\int_0^{t\wedge\sigma}\left|\int_{z\neq 0}\frac{(\chi(z))^2z'e_j}{2R}-\sin\left(\frac{z'e_j}{2R}\right)N(X_s,\mathrm{d}z)\right|\mathrm{d}s\\
&\quad\le 2\sum_{j=1}^{d}\mathbb{E}^x\int_0^{t\wedge\sigma}\left|\int_{B_{2R}(0)\setminus\{0\}}1-\cos\left(\frac{z'e_j}{2R}\right)N(X_s,\mathrm{d}z)\right|\\
&\qquad+\left|\int_{B_{2R}(0)^c}1N(X_s,\mathrm{d}z)\right|\mathrm{d}s \qquad (35)\\
&\quad\le 2td\sup_{\|y-x\|\le S}\sup_{\|\varepsilon\|\le 1}\Re p\left(y,\frac{\varepsilon}{2R}\right)+2^2td\sup_{\|y-x\|\le S}\widetilde{c}_d\sup_{\|\varepsilon\|\le 1}\left|p\left(y,\frac{\varepsilon}{2R}\right)\right|,
\end{aligned}
$$

where we have used again Lemma 5.2 on the second term.

It remains to deal with the second term of (28). Let $\delta>0$ be fixed (at first) and $m:\mathbb{R}\to\,]1,1+\delta[$ a strictly monotone increasing auxiliary function. Since $m\ge 1$ and since we have at least one jump of size $>R$ on $\{\tau_R\le t\}$, we obtain

$$
\begin{aligned}
\mathbb{P}^x(\tau_R\le t)&\le\mathbb{P}^x\left(\int_0^t\int_{\|z\|\ge R}m\big(\|z\|\big)\mu^{X^\sigma}(\cdot;\mathrm{d}s,\mathrm{d}z)\ge m(R)\right)\\
&\le\frac{1}{m(R)}\mathbb{E}^x\left(\int_0^t\int_{\|z\|\ge R}m\big(\|z\|\big)1_{[[0,\sigma]]}(s)\mu^{X}(\cdot;\mathrm{d}s,\mathrm{d}z)\right)\\
&=\frac{1}{m(R)}\mathbb{E}^x\left(\int_0^t\int_{z\neq 0}m\big(\|z\|\big)1_{[[0,\sigma[[}(s)1_{B_R(0)^c}(z)N(X_s,\mathrm{d}z)\,\mathrm{d}s\right)\\
&\le(1+\delta)t\sup_{s\le t\wedge\sigma}N\big(X_s,B_R(0)^c\big)\\
&\le(1+\delta)t\sup_{\|y-x\|\le S}N\big(y,B_R(0)^c\big)\\
&\le(1+\delta)4t\sup_{\|y-x\|\le S}\int_{z\neq 0}\left(\left\|\frac{z}{2R}\right\|^2\wedge 1\right)N(y,\mathrm{d}z)
\end{aligned}
$$

because $m(\|z\|)1_{[[0,\sigma[[}(s)1_{B_R(0)^c}(z)$ is in class F_p^1 of Ikeda and Watanabe (see [10], Section II.3). Since δ can be chosen arbitrarily small, we obtain by Lemma 5.2

$$\mathbb{P}^x(\tau_R \le t) \le 4t \sup_{\|y-x\|\le S} \widetilde{c}_d \sup_{\|\varepsilon\|\le 1} \left| p\left(y, \frac{\varepsilon}{2R}\right)\right|. \tag{36}$$

Plugging together (31), (35), (35) and (36), we obtain (27).

For the particular case $\sigma = \sigma^x_{3\widetilde{R}}$, we have

$$\left\{\left(X^\sigma_\cdot - x\right)^*_t \ge 2\widetilde{R}\right\} = \left\{(X_\cdot - x)^*_t \ge 2\widetilde{R}\right\}$$

and therefore, for every $\widetilde{R} > 0$

$$\mathbb{P}^x\left((X_\cdot - x)^*_t \ge 2\widetilde{R}\right) \le c_d \cdot t \cdot \sup_{\|y-x\|\le 3\widetilde{R}} \sup_{\|\varepsilon\|\le 1} \left| p\left(y, \frac{\varepsilon}{2\widetilde{R}}\right)\right|. \tag{37}$$

Setting $R := (1/2)\widetilde{R}$, we obtain (12). The proof of (13) works literally as in the case of rich Feller processes satisfying (G) and (S). Compare in this context [23], Lemma 6.3 and Lemma 4.1. The condition (G) is not used in the proofs of these lemmas. □

Proof of Theorems 3.11 and 3.12. Since the proofs of the analogue statements for rich Feller processes can be adapted and since all eight proofs are very similar, we decided to give only on exemplary proof, namely of (18): Fix $x \in \mathbb{R}^d$. Let $\lambda > \beta^x_\infty$ and choose $\lambda > \alpha_1 > \alpha_2 > \beta^x_\infty$. We have

$$\mathbb{P}^x\left((X_\cdot - x)^*_t \ge t^{1/\alpha_1}\right) \le c_d \cdot t \cdot H\left(x, t^{1/\alpha_1}\right) \le c'_d \cdot t\left(t^{1/\alpha_1}\right)^{-\alpha_2} = c'_d t^{1-(\alpha_2/\alpha_1)}$$

for t small enough, say $t < T_0$, since the lim sup is considered. Now let $t_k := (1/2)^k$ for $k \in \mathbb{N}$. We obtain

$$\sum_{k=k_0}^\infty \mathbb{P}^x\left((X_\cdot - x)^*_{t_k} \ge t_k^{1/\alpha_1}\right) \le c'_d \sum_{k=k_0}^\infty 2^{-k(1-(\alpha_2/\alpha_1))} < \infty,$$

where k_0 depends on T_0. By the Borel–Cantelli lemma, we obtain

$$\mathbb{P}^x\left(\limsup_{k\to\infty} (X_\cdot - x)^*_{t_k} \ge (t_k)^{1/\alpha_1}\right) = 0$$

and hence $(X_\cdot - x)^*_{t_k} < (t_k)^{1/\alpha_1}$ for all $k \ge k_1(\omega)$ on a set of probability one. For fixed ω in this set and $t_{k+1} \le t \le t_k$ and $k \ge k_1(\omega) \ge k_0$, we have

$$\left(X_\cdot(\omega) - x\right)^*_t \le \left(X_\cdot(\omega) - x\right)^*_{t_k} \le t_k^{1/\alpha_1} \le 2^{1/\alpha_1} t^{1/\alpha_1}$$

and since $\lambda > \alpha_1$

$$t^{-1/\lambda}(X_\cdot - x)^*_t \le 2^{1/\alpha_1} t^{(1/\alpha_1)-(1/\lambda)}$$

which converges $\mathbb{P}^x$-a.s to zero for $t \downarrow 0$. □

Acknowledgements

The author wishes to thank René Schilling (TU Dresden) for suggesting the problem and Peter Furlan (TU Dortmund) for interesting discussions on ODEs. Furthermore, he wishes to thank an anonymous referee for his/her work. Financial support by the German Science Foundation (DFG) for the project SCHN1231/1-1 is gratefully acknowledged.

References

[1] Bass, R.F. (1988). Uniqueness in law for pure jump Markov processes. *Probab. Theory Related Fields* **79** 271–287. MR0958291

[2] Berg, C. and Forst, G. (1975). *Potential Theory on Locally Compact Abelian Groups*. *Ergebnisse der Mathematik und ihrer Grenzgebiete* **87**. New York: Springer. MR0481057

[3] Blumenthal, R.M. and Getoor, R.K. (1961). Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.* **10** 493–516. MR0123362

[4] Blumenthal, R.M. and Getoor, R.K. (1968). *Markov Processes and Potential Theory*. *Pure and Applied Mathematics* **29**. New York: Academic Press. MR0264757

[5] Carr, P., Chang, E.C. and Madan, D.B. (1998). The variance gamma process and option pricing. *European Finance Review* **2** 79–105.

[6] Çinlar, E. and Jacod, J. (1981). Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures. In *Seminar on Stochastic Processes*, 1981 (*Evanston*, *IL*, 1981). *Progr. Prob. Statist.* **1** 159–242. Boston, MA: Birkhäuser. MR0647786

[7] Çinlar, E., Jacod, J., Protter, P. and Sharpe, M.J. (1980). Semimartingales and Markov processes. *Z. Wahrsch. Verw. Gebiete* **54** 161–219. MR0597337

[8] Courrège, P. (1965/66). Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. In *Sém. Théorie du Potentiel* **10** 1–38. Paris: Secrétariat mathematique.

[9] Fuglede, B. (1972). *Finely Harmonic Functions*. *Lecture Notes in Mathematics* **289**. Berlin: Springer. MR0450590

[10] Ikeda, N. and Watanabe, S. (1981). *Stochastic Differential Equations and Diffusion Processes*. Tokio: North-Holland.

[11] Jacob, N. (1998). Characteristic functions and symbols in the theory of Feller processes. *Potential Anal.* **8** 61–68. MR1608650

[12] Jacob, N. (2001). *Pseudo Differential Operators and Markov Processes*. *Vol. I. Fourier Analysis and Semigroups*. London: Imperial College Press. MR1873235

[13] Jacob, N. (2002). *Pseudo Differential Operators and Markov Processes*. *Vol. II. Generators and Their Potential Theory*. London: Imperial College Press. MR1917230

[14] Jacob, N. (2005). *Pseudo Differential Operators and Markov Processes*. *Vol. III. Markov Processes and Applications*. London: Imperial College Press. MR2158336

[15] Jacod, J. and Shiryaev, A.N. (1987). *Limit Theorems for Stochastic Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR0959133

[16] Klüppelberg, C., Lindner, A. and Maller, R. (2004). A continuous-time GARCH process driven by a Lévy process: Stationarity and second-order behaviour. *J. Appl. Probab.* **41** 601–622. MR2074811

[17] Klüppelberg, C., Maller, R. and Szimayer, A. (2011). The COGARCH: A review, with news on option pricing and statistical inference. In *Surveys in Stochastic Processes*. *EMS Ser. Congr. Rep.* 29–58. Zürich: Eur. Math. Soc. MR2883852

[18] Negoro, A. (1994). Stable-like processes: Construction of the transition density and the behavior of sample paths near $t = 0$. *Osaka J. Math.* **31** 189–214. MR1262797

[19] Pruitt, W.E. (1981). The growth of random walks and Lévy processes. *Ann. Probab.* **9** 948–956. MR0632968

[20] Sato, K.i. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original, revised by the author. MR1739520

[21] Schilling, R.L. (1998). Conservativeness and extensions of Feller semigroups. *Positivity* **2** 239–256. MR1653474

[22] Schilling, R.L. (1998). Feller processes generated by pseudo-differential operators: On the Hausdorff dimension of their sample paths. *J. Theoret. Probab.* **11** 303–330. MR1622573

[23] Schilling, R.L. (1998). Growth and Hölder conditions for the sample paths of Feller processes. *Probab. Theory Related Fields* **112** 565–611. MR1664705

[24] Schilling, R.L. and Schnurr, A. (2010). The symbol associated with the solution of a stochastic differential equation. *Electron. J. Probab.* **15** 1369–1393. MR2721050

[25] Schnurr, A. (2012). On the semimartingale nature of Feller processes with killing. *Stochastic Process. Appl.* **122** 2758–2780. MR2926174

[26] Schnurr, A. On deterministic Markov processes: Expandability and related topics. Preprint.

[27] Schnurr, A. (2009). The symbol of a Markov semimartingale. Ph.D. thesis, TU Dresden.

[28] Schnurr, A. (2011). A classification of deterministic Hunt processes with some applications. *Markov Process. Related Fields* **17** 259–276. MR2856243

[29] Schnurr, A. (2011). COGARCH: Symbol, generator and characteristics. In *Proceedings of the* 8*th Congress of the ISAAC*. Singapore: World Scientific.

Received February 2012.