

# On hitting times, Bessel bridges and Schrödinger's equation

GERARDO HERNANDEZ-DEL-VALLE

*Statistics Dept., Columbia University, 1255 Amsterdam Ave. Room 1005, New York, N.Y. 10027, USA.  
E-mail: gerardo@stat.columbia.edu*

In this paper, we establish relationships between four important concepts: (a) hitting time problems of Brownian motion, (b) 3-dimensional Bessel bridges, (c) Schrödinger's equation with linear potential, and (d) heat equation problems with moving boundary. We relate (a) and (b) by means of Girsanov's theorem, which suggests a strategy to extend our ideas to problems in  $\mathbb{R}^n$  and general diffusions. This approach also leads to (c) because we may relate, through a Feynman–Kac representation, functionals of a Bessel bridge with two Schrödinger-type problems. In particular, we also find a fundamental solution to a class of parabolic partial differential equations with linear potential. Finally, the relationship between (c) and (d) suggests a possible link between Generalized Airy processes and their hitting times.

*Keywords:* Bessel bridge; fundamental solution; heat equation; hitting times; Schrödinger's equation

## 1. Introduction

Finding the density for the first time, a Brownian motion reaches a moving boundary is an old and classic problem with a wide range of applications in both mathematics and applied sciences. The problem may be traced back to Louis Bachelier's doctoral thesis Bachelier, L. [1], *Théorie de la Spéculation*, and to a paper by the Austrian physicist Erwin Schrödinger [13].

On the other hand, [10] and later [6,12] and [9] are among the first to explicitly establish the relationship between hitting time problems and parabolic partial differential equations.

The aim of this work is not to provide a new set of solutions to the problem of boundary crossing. Our goal is instead conceptual: we establish relationships between (a) hitting time problems of Brownian motion, (b) 3-dimensional Bessel bridge, (c) Schrödinger's equation with linear potential, and (d) heat equation problems with moving boundary. We determine conditions under which these problems are equivalent. As a corollary, we derive a Green function to a class of parabolic partial differential equations with linear potential.

The paper is organized as follows: The hitting time problem is introduced in Section 2, and by use of Girsanov's theorem, we relate it to a functional of a 3-dimensional Bessel bridge. This relationship is established in Section 3.

In Section 4, we derive the Feynman–Kac representation that will allow us to determine the correspondence between Bessel bridges and Schrödinger's equation with linear potential. Next, in Section 5 we derive a pair of equivalent problems which in Section 6 are used to link boundary crossing probabilities with the heat equation. In Section 7, we find a fundamental solution to the equation introduced in Section 4, and discuss its properties. We conclude in Section 8 with some final remarks.

## 2. The problem

Throughout this paper,  $B$  stands for a one-dimensional standard Brownian motion, and  $f(\cdot)$  is a real-valued function denoting a “moving boundary.” Moreover,

$$T := \inf\{t \geq 0 \mid B_t = f(t)\} \tag{2.1}$$

is the first time that  $B$  reaches the moving boundary  $f$ . The main motivation of this work is to study the density of  $T$ . To be more precise, we consider the following problem.

**Problem 2.1.** Determine the density of  $T$ , given that, for  $a > 0$

$$f(t) := a + \int_0^t f'(u) \, du, \quad f''(t) \geq 0 \quad \text{and} \quad \int_0^t (f'(u))^2 \, du < \infty \tag{2.2}$$

for  $t > 0$ .

To analyze Problem 2.1, it is convenient to note that using (2.1) and setting

$$\tilde{B}_t := B_t - \int_0^t f'(u) \, du \tag{2.3}$$

we obtain the following alternative representation of  $T$ :

$$\begin{aligned} T &= \inf\left\{t \geq 0 \mid B_t = a + \int_0^t f'(u) \, du\right\} \\ &= \inf\left\{t \geq 0 \mid B_t - \int_0^t f'(u) \, du = a\right\} \\ &= \inf\{t \geq 0 \mid \tilde{B}_t = a\}. \end{aligned} \tag{2.4}$$

Since our study will make use of Girsanov’s theorem (see, for instance, Section 3.5, Chapter 3 in [8]), let us first introduce the heuristics that link these ideas.

### 2.1. Girsanov’s theorem and hitting time problems

Notice that if there exists a measure  $\tilde{\mathbb{P}}$  under which the process  $\tilde{B}$  is a one-dimensional standard Brownian motion, then for  $T$  as in (2.1) [or (2.4)] and  $\tilde{B}$  as in (2.3), then

$$\tilde{\mathbb{P}}(T < t) = \int_0^t h(s, a) \, ds, \quad t > 0,$$

where

$$h(s, a) := \frac{|a|}{\sqrt{2\pi s^3}} \exp\left\{-\frac{a^2}{2s}\right\}, \quad s > 0, a \neq 0. \tag{2.5}$$

This function  $h$  is the hitting time density of Brownian motion reaching a fixed level  $a$ . Alternatively,  $h$  is also referred to as the *derived* source solution of the heat equation (see, for instance, [11]).

This last statement follows from observing, in (2.4), that the original problem is reduced to that of hitting a 'fixed' boundary  $a$ .

Assuming such measure  $\tilde{\mathbb{P}}$  exists, how is it related to the original measure  $\mathbb{P}$ ? More explicitly, what is the connection (if any) between  $\mathbb{P}(T < t)$  and  $\tilde{\mathbb{P}}(T < t)$ ?

**Remark (Exponential martingale and Novikov's condition).** Girsanov's theorem states that the process  $\tilde{B}$  in (2.3) will be a one-dimensional standard Brownian motion under the measure  $\tilde{\mathbb{P}}$ , as long as the following exponential process

$$Z_t := \exp\left\{ \int_0^t f'(u) dB_u - \frac{1}{2} \int_0^t (f'(u))^2 du \right\}, \quad 0 \leq t < \infty, \tag{2.6}$$

is a  $\mathbb{P}$ -martingale. In turn, for this last statement to hold, a sufficient condition is *Novikov's condition* (see, for instance, Proposition 3.5.12 in [8]). Now, given the  $\mathbb{P}$ -martingale  $Z$  in (2.6), together with the  $\tilde{\mathbb{P}}$ -martingale

$$\tilde{Z}_t := \exp\left\{ - \int_0^t f'(u) d\tilde{B}_u - \frac{1}{2} \int_0^t (f'(u))^2 du \right\}, \quad 0 \leq t < \infty, \tag{2.7}$$

where  $\tilde{B}$  is a Brownian motion under  $\tilde{\mathbb{P}}$ , it follows from Girsanov's theorem that the measures  $\mathbb{P}$  and  $\tilde{\mathbb{P}}$  are equivalent.

This sequence of ideas leads to the following representation.

**Proposition 2.1.** *Given that  $T$  is as in (2.1) and condition (2.2) holds, the exponential process  $\tilde{Z}$  in (2.7) is a  $\tilde{\mathbb{P}}$ -martingale. Furthermore, given the hitting time density  $h$  in (2.5), it follows*

$$\mathbb{P}(T < t) = \int_0^t e^{-f'(s)a - (1/2) \int_0^s (f'(u))^2 du} \tilde{\mathbb{E}}\left[ e^{\int_0^s f''(u)\tilde{B}_u du} | T = s \right] h(s, a) ds \tag{2.8}$$

for  $t \geq 0$ .

**Proof.** The first claim is justified from (2.2) and Novikov's condition. Next, from Girsanov's theorem, we have the following equality

$$\mathbb{P}(T < t) = \tilde{\mathbb{E}}[\tilde{Z}_t \mathbb{I}_{(T < t)}],$$

where  $\tilde{Z}$  is as in (2.7).

From the fact that  $\tilde{Z}$  is a  $\tilde{\mathbb{P}}$ -martingale and  $\mathbb{I}_{(T < t)}$  is  $\mathcal{F}_{T \wedge t}$ -measurable, it follows from the Optional Sampling theorem that

$$\mathbb{P}(T < t) = \tilde{\mathbb{E}}[\tilde{\mathbb{E}}(\tilde{Z}_t | \mathcal{F}_{T \wedge t}) \mathbb{I}_{(T < t)}] = \tilde{\mathbb{E}}[\tilde{Z}_{T \wedge t} \mathbb{I}_{(T < t)}] = \tilde{\mathbb{E}}[\tilde{Z}_T \mathbb{I}_{(T < t)}]. \tag{2.9}$$

Finally, from (2.9) together with the integration by parts formula:

$$e^{-\int_0^t f'(u) d\tilde{B}_u - (1/2) \int_0^t (f'(u))^2 du} = e^{-\tilde{B}_t f'(t) + \int_0^t f''(u) \tilde{B}_u du - (1/2) \int_0^t (f'(u))^2 du}.$$

Then the terminal condition  $\tilde{B}_T = a$ , and conditioning with respect to  $T$  under  $\tilde{\mathbb{P}}$  completes the proof of the proposition.  $\square$

In view of (2.8), our study now focuses on the following expectation:

$$\tilde{\mathbb{E}}[e^{\int_0^s f''(u) \tilde{B}_u du} | T = s].$$

Thus, our next task is to understand the dynamics of the process  $\tilde{B}_u$  (for  $0 \leq u \leq s$ ), which is conditioned to hit the fixed level  $a$  for the *first time* at time  $s$ .

### 3. Three dimensional Bessel bridge

The first idea that comes to mind is that the conditioned process  $\tilde{B}_u$  (for  $0 \leq u \leq s$ ) is a Brownian bridge, since at  $t = 0$ ,  $\tilde{B}_0 = 0$  and at time  $s$ ,  $\tilde{B}_s = a$ . However, this argument is flawed because, in contrast to a Brownian bridge, the process  $\tilde{B}$  can *never* be above level  $a$  before time  $s$ .

Elaborating on the heuristics, let us for the moment write the conditioned process as:  $(\hat{B}_u)_{0 \leq u \leq s}$ . Alternatively, setting

$$\tilde{X}_u := a - \hat{B}_u, \quad 0 \leq u \leq s,$$

we shall analyze the dynamics of the process  $\tilde{X}$ :

$$\tilde{X}_0 = a; \quad \tilde{X}_u > 0 \text{ for } 0 < u < s; \quad \tilde{X}_s = 0. \tag{3.1}$$

That is,  $\tilde{X}$  is *strictly positive* (Bessel part), except at time  $s$ , when it is *pinned* down at zero (bridge part). It may be formally proved that the process just described is in fact the following.

**Definition 3.1.** *A 3-dimensional Bessel bridge is a stochastic process which has the following dynamics:*

$$d\tilde{X}_t := \left[ \frac{1}{\tilde{X}_t} - \frac{\tilde{X}_t}{(s-t)} \right] dt + dW_t, \quad \tilde{X}_0 = a \geq 0, \tag{3.2}$$

$0 \leq t \leq s$ , and  $W$  is a one-dimensional standard Brownian motion. Furthermore, letting

$$k(\sigma, \kappa) := \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{\kappa^2}{2\sigma}\right\}, \quad \sigma > 0, \kappa \in \mathbb{R}, \tag{3.3}$$

also known as the source solution of the heat equation (see, for instance, [11]), the transition probability of the process  $\tilde{X}$  is:

$$\begin{aligned}
 G(t, x; \tau, y) &:= \tilde{\mathbb{P}}^{t,x}(\tilde{X}_\tau \in dy) \\
 &= \frac{y}{x} \frac{s-t}{s-\tau} \frac{k(s-\tau, y)}{k(s-t, x)} [k(\tau-t, y-x) - k(\tau-t, x+y)]
 \end{aligned}
 \tag{3.4}$$

for  $0 \leq t < \tau \leq s$ , and  $x, y \geq 0$ .

### 3.1. Hitting problems and 3-dimensional Bessel bridges

From Proposition 2.1 and Definition 3.1, we are now able to represent a class of first-hitting time problems in terms of a 3-dimensional Bessel process, namely the following.

**Theorem 3.1.** *Given that  $h$  is as in (2.5), the process  $\tilde{X}$  is a 3 dimensional Bessel bridge, Definition 3.1. Then, the distribution of  $T$  equals*

$$\begin{aligned}
 \mathbb{P}(T < t) &= \int_0^t \tilde{\mathbb{E}} \left[ \exp \left\{ - \int_0^s f''(u) \tilde{X}_u du \right\} \right] \\
 &\quad \times \exp \left\{ - \frac{1}{2} \int_0^s (f'(u))^2 du - f'(0)a \right\} h(s, a) ds
 \end{aligned}
 \tag{3.5}$$

for  $t \geq 0$ .

**Proof.** Directly verified from Proposition 2.1 and Definition 3.1. □

## 4. Feynman–Kac and 3 dimensional Bessel bridge

In this section, we will provide a Feynman–Kac representation (see for instance: Theorem 4.4.2 in [8]) of

$$v(t, a) = \tilde{\mathbb{E}} \left[ \exp \left\{ - \int_t^s f''(u) \tilde{X}_u du \right\} \middle| \tilde{X}_t = a \right],$$

where  $\tilde{X}$  is the 3-dimensional Bessel bridge, equation (3.2), introduced in Definition 3.1.

The idea is to apply Ito's rule to the process

$$v(y, \tilde{X}_y) \exp \left\{ - \int_t^y f''(u) \tilde{X}_u du \right\}, \quad y \in [t, s],$$

given the following growth and/or boundary condition

$$0 \leq v(t, a) \leq 1, \quad 0 \leq t < s, a \geq 0.$$

We will also make use of the fact that  $f''(u)\tilde{X}_u \geq 0$  for  $0 \leq u < s$ ; that the process  $\tilde{X}$  is continuous and strictly positive on  $u \in [0, s)$ , and that the moments of the running maximum of the process  $\tilde{X}$  are known and finite.

**Theorem 4.1.** *Suppose that  $v(t, a) : [0, s] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous, of class  $C^{1,2}([0, s] \times \mathbb{R}^+)$ , and satisfies the Cauchy problem*

$$\begin{aligned}
 -\frac{\partial v}{\partial t} + f''(t)av &= \frac{1}{2} \frac{\partial^2 v}{\partial a^2} + \left( \frac{1}{a} - \frac{a}{s-t} \right) \frac{\partial v}{\partial a}, & [0, s) \times \mathbb{R}^+, \\
 v(s, a) &= 1, & a \in \mathbb{R}^+,
 \end{aligned}
 \tag{4.1}$$

as well as

$$0 \leq v(t, a) \leq 1 \quad \forall t, a \in \mathbb{R}^+.
 \tag{4.2}$$

Then  $v(t, a)$  admits the stochastic representation:

$$v(t, a) = \mathbb{E}^{t,a} \left[ \exp \left\{ - \int_t^s f''(u) \tilde{X}_u \, du \right\} \right].$$

**Proof.** We proceed as in the proof of Theorem 5.7.6, pages 366–367, [8]. □

Our next step is to relate Theorem 4.1 to two alternative, better known problems. Namely, that of Schrödinger’s equation with linear potential. To this end, we first have the following proposition.

**Proposition 4.1.** *Let  $h, v$ , and  $w$  be of class  $C^{1,2}$ . Suppose that  $v, w$  and  $h$  satisfy the following identity*

$$v(t, a) = \frac{w(t, a)}{h(s-t, a)}
 \tag{4.3}$$

and  $h \neq 0$  for some strip in  $\mathbb{R}^+ \times \mathbb{R}$ . If  $h$  satisfies (2.5) and

$$-w_t(t, a) + f''(t)aw(t, a) = \frac{1}{2}w_{aa}(t, a), \quad [0, s) \times \mathbb{R}^+,
 \tag{4.4}$$

then (4.1) holds.

**Proof.** Setting

$$u(t, a) = 1/h(s-t, a)$$

and

$$v(t, a) := u(t, a)w(t, a),
 \tag{4.5}$$

we have that

$$\begin{aligned}
 u_t(t, a) &= \left[ \frac{a^2}{2(s-t)^2} - \frac{3}{2(s-t)} \right] u(t, a), \\
 u_a(t, a) &= - \left[ \frac{1}{a} - \frac{a}{s-t} \right] u(t, a), \\
 u_{aa}(t, a) &= \left[ \frac{2}{a^2} + \frac{a^2}{(s-t)^2} - \frac{1}{s-t} \right] u(t, a).
 \end{aligned}$$

Alternatively, from (4.5) it follows

$$\begin{aligned}
 v_t &= u_t w + u w_t, & v_a &= u_a w + u w_a, \\
 v_{aa} &= u_{aa} w + 2u_x w_w + u w_{aa}.
 \end{aligned}$$

Hence from (4.1) and (4.4), we conclude that

$$\left[ -u_t - \frac{1}{2}u_{aa} - \left( \frac{1}{a} - \frac{a}{s-t} \right) u_a \right] w = \left[ u_a + \left( \frac{1}{a} - \frac{a}{s-t} \right) u \right] w_a,$$

as claimed. □

The previous proposition may be generalized by first noting that

$$\left( \frac{1}{x} - \frac{x}{s-t} \right) = \frac{h_x(s-t, x)}{h(s-t, x)}$$

for  $h$  as in (2.5). Second, recall that  $h$  is a solution to the heat equation

$$h_t = \frac{1}{2} h_{xx}.$$

This leads to the next proposition.

**Proposition 4.2.** *If  $h, v,$  and  $w$  satisfy the following system of partial differential equations*

$$\begin{cases}
 -\frac{1}{\sigma_t} \cdot h_t = \frac{1}{2} h_{xx}, & \text{(a)} \\
 -v_\tau + k(\tau, x)v = \frac{1}{2} v_{xx} + \frac{h_x}{h} v_x, & \text{(b)} \\
 -w_\tau + k(\tau, x)w = \frac{1}{2} w_{xx}, & \text{(c)}
 \end{cases} \tag{4.6}$$

where  $\sigma_t$  is only a function of  $t$ , then they satisfy the identity:

$$v(\tau, x) = \frac{w(\tau, x)}{h(\tau, x)}, \tag{4.7}$$

where

$$\tau := \int_s^t \sigma_u \, du. \tag{4.8}$$

**Proof.** Given  $\tau$  is as in (4.8), it follows from the chain rule that

$$\frac{\partial h}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial h}{\partial \tau}.$$

Next, given that  $v$  is as in (4.7) we will compute the following partial derivatives

$$\begin{aligned} v_\tau &= \frac{w_\tau h - wh_\tau}{h^2}, & v_x &= \frac{w_x h - wh_x}{h^2}, \\ v_{xx} &= \frac{(w_{xx}h - wh_{xx})h^2 - 2hh_x(w_x h - wh_x)}{h^4}. \end{aligned}$$

Substituting the previous identities in equation in (4.6.b), we have:

$$\begin{aligned} -\frac{w_\tau h - wh_\tau}{h^2} + k \frac{w}{h} &= \frac{1}{2} \frac{w_{xx}}{h} - \frac{1}{2} \frac{h_{xx}w}{h^2} \\ &\quad - \frac{h_x(w_x h - wh_x)}{h^3} + \frac{h_x}{h} \left[ \frac{w_x h - wh_x}{h^2} \right]. \end{aligned}$$

Observe that the last two terms on the right-hand side of the previous equality cancel out, hence

$$-\frac{w_\tau h - wh_\tau}{h^2} + k \frac{w}{h} = \frac{1}{2} \frac{w_{xx}}{h} - \frac{1}{2} \frac{h_{xx}w}{h^2}.$$

After rearranging terms

$$\frac{1}{h} \left( -w_\tau + k \cdot w - \frac{1}{2} w_{xx} \right) = \frac{w}{h^2} \left( -h_\tau - \frac{1}{2} h_{xx} \right).$$

However, from (4.6.a) and (4.6.c) the previous identity equals zero as claimed. □

## 5. Schrödinger’s boundary value problems with linear potential

The main objective of Theorem 4.1 is to characterize the problem of finding the first hitting time densities of Brownian motion in terms of the *Cauhy* problem (4.1). However, in this section, the same problem is represented in terms of two different, but equivalent, *Boundary value* problems.

To this end, we will make use of the subsequent facts.

**Remark 5.1 (On notation).** Throughout this section, let functions  $v$ ,  $w$ , and  $h$  satisfy the conditions in Proposition 4.1. Furthermore, let  $v$  satisfy the growth condition

$$0 \leq v(t, a) \leq 1, \quad 0 \leq t \leq s, a \in \mathbb{R}^+, \tag{5.1}$$

introduced in Theorem 4.1.

**Remark 5.2.** Given the assumptions in Remark 5.1, it follows that

(a) Function  $w$  is nonnegative and bounded by  $h$

$$0 \leq w(t, a) \leq h(s - t, a) \quad \forall 0 \leq t \leq s, a \in \mathbb{R}^+. \tag{5.2}$$

(b) Function  $h$ , defined in (2.5), (which can also be interpreted as the first hitting time density of Brownian motion to a fixed level  $a$ ) satisfies the following two boundary conditions

$$\lim_{t \rightarrow s} h(s - t, a) = 0, \tag{5.3}$$

$$\lim_{a \rightarrow 0} h(s - t, a) = \delta(s). \tag{5.4}$$

(c) From the terminal condition  $v(s, a) = 1$ , in Theorem 4.1, and equality (5.3) it follows that

$$\lim_{t \rightarrow s} w(t, a) = 0$$

as well as

$$\lim_{a \rightarrow 0s} w(t, a) = \delta(s).$$

**Remark 5.3.** Part (b) of Remark 5.2 should be interpreted in the following sense: Suppose  $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a  $C^1$  function, then

$$\lim_{x \rightarrow 0} \int_0^s h(s - t, x) \lambda(t) dt = \lambda(s), \quad 0 \leq t < s < \infty. \tag{5.5}$$

*First Schrödinger boundary value problem.* From Remark 5.2, it follows that the Cauchy problem (4.1), stated in Theorem 4.1, is equivalent to the following Schrödinger boundary value problem.

**Problem 5.1.** Let function  $w$  be the unique solution to

$$\begin{aligned} & -\frac{\partial w}{\partial t}(t, a) + f''(t)aw(t, a) \\ & = \frac{1}{2} \frac{\partial^2 w}{\partial a^2}(t, a), \quad t \in [0, s], a \in \mathbb{R}^+, \end{aligned} \tag{5.6}$$

with

$$\begin{aligned} & \lim_{t \rightarrow s} w(t, a) = 0, \\ & \lim_{a \rightarrow 0} w(t, a) = \delta(s), \end{aligned}$$

and the compatibility condition

$$\lim_{t \rightarrow s} \frac{w(t, a)}{h(s - t, a)} = 1.$$

For our second problem, we will make use of the following proposition.

**Proposition 5.1.** *Let  $h, k, v$  and  $w$  be of class  $C^{1,2}$ . Suppose that  $v, w$  and  $h$  satisfy the following identity*

$$v(\tau, x) = \frac{w(\tau, x)}{h(\tau, x)},$$

where

$$\tau := \int_s^t \sigma_u \, du \tag{5.7}$$

Then they also satisfy the following system of partial differential equations

$$\begin{cases} -\frac{1}{\sigma_t} \cdot h_t = \frac{1}{2} h_{xx}, \\ -k_\tau = \frac{1}{2} k_{xx}, \\ -v_\tau + \sigma(\tau, x)v = \frac{1}{2} v_{xx} + \frac{k_x}{k} v_x, \\ -w_\tau + \left[ \sigma(\tau, x) + \left( \frac{k_x}{k} - \frac{h_x}{h} \right) \frac{h_x}{h} \right] w = \frac{1}{2} w_{xx} + w_x \left( \frac{k_x}{k} - \frac{h_x}{h} \right). \end{cases}$$

**Proof.** We proceed as in the proof of Proposition 4.2. □

*Second Schrödinger problem.* Given the fundamental or source solution of the heat equation  $k$ , equation (3.3), we may relate  $v$  to another Schrödinger-type problem, by setting

$$v(t, a) = \frac{u(t, a)}{k(s - t, a)}$$

and making use of Proposition 5.1:

**Problem 5.2.** *Let  $v$  be as in Theorem 4.1 and  $k$  as in equation (3.3), then the following relationship holds:*

$$v(t, a) = \frac{u(t, a)}{k(s - t, a)}, \quad t \in [0, s] \times \mathbb{R}^+.$$

Furthermore, it follows from Proposition 5.1, that  $u$  is a solution to the Schrödinger equation with linear potential:

$$\begin{aligned}
 & -\frac{\partial u}{\partial t}(t, a) + \left[ -\frac{1}{(s-t)} + f''(t)a \right] u(t, a) \\
 & = \frac{1}{2} \frac{\partial^2 u}{\partial a^2}(t, a) + \frac{1}{a} \frac{\partial u}{\partial a}(t, a), \quad t \in [0, s], a \in \mathbb{R}^+,
 \end{aligned}
 \tag{5.8}$$

satisfying the following boundary conditions:

$$\begin{aligned}
 \lim_{t \rightarrow s} u(t, a) &= \delta(a), \\
 \lim_{a \rightarrow 0} u(t, a) &= 0,
 \end{aligned}$$

as well as the compatibility condition

$$\lim_{t \rightarrow s} \frac{u(t, a)}{k(s-t, a)} = 1.$$

The boundary conditions of this second problem follow, once more, by using the growth condition in Theorem 4.1, that is,

$$0 \leq u(t, a) \leq k(s-t, a) \quad \forall t \in [0, s], a \in \mathbb{R}.$$

We conclude this section by making a few remarks. First, note that the problem described in Remark 5.2 is a *Cauchy* problem, the one appearing in Remark 5.1 is not.

We have introduced the consistency conditions in Remarks 5.1 and 5.2 in order to make them compatible with Theorem 4.1. Boundary conditions for the density of  $T$  will be derived in the following section.

The next observation is that for the equations (5.6) and (5.8) one may construct *particular* solutions. There is in fact *one* solution of (5.6) which may be used to find a new set of solutions of (5.8). To do so, one may use a technique introduced in [3]. However, solutions to the problems described in Remarks 5.1 and 5.2, for arbitrary bounds  $f$  – which do not involve *integral equations* – have yet to be found.

This last remark will be clarified in the following section. The idea stems from the following fact (yet to be shown): There is a particular set of solutions of either (5.6) and/or (5.8) which transform our original problem – that is, finding the density of hitting times – into that of finding particular solutions to the heat equation killed at a moving boundary! Unfortunately, in practice this is *not* very useful to us, since the problem leads back to Volterra integral equations as described in [4].

A final curious remark is that equation (5.6) may be transformed into a non-homogeneous Burgers' equation. The homogeneity is only time dependent and it corresponds precisely to  $f''$ :

$$\kappa_t(t, a) + \kappa(t, a) \cdot \kappa_a(t, a) - \frac{1}{2} \kappa_{aa}(t, a) = f''(t).
 \tag{5.9}$$

Alternatively,  $\kappa$  and  $w$  – equation (5.6) – introduced in Remark 5.1, are related through:

$$\kappa(t, a) = -\frac{\partial}{\partial a} \log w(t, a).$$

From Proposition 4.2, it follows the following.

**Proposition 5.2.** *If  $h, v, w$  and  $u$  satisfy the following system of partial differential equations*

$$\begin{cases} -\frac{1}{\sigma_t} \cdot h_t = \frac{1}{2} h_{xx}, \\ -v_\tau + k(\tau, x)v = \frac{1}{2} v_{xx} + \frac{h_x}{h} v_x, \\ -w_\tau + k(\tau, x)w = \frac{1}{2} w_{xx}, \\ -u_\tau + u \cdot u_x = \frac{1}{2} u_{xx} + k_x, \end{cases}$$

where  $\sigma_t$  is only a function of  $t$ , then they satisfy the identities:

$$v(\tau, x) = \frac{w(\tau, x)}{h(\tau, x)}, \quad u = -\frac{w_x}{w}, \quad w = e^{-\int u dx},$$

where

$$\tau := \int_s^t \sigma_u du.$$

Of course, at first sight, this is not so surprising since this relationship is well known. The possibly “interesting” part lies in the fact that so many probability-related papers have been written regarding Burgers’ equation (5.9), see for instance [2] and references therein.

## 6. Particular solutions to Schrödinger’s equation with linear potential (heat equation)

Particular solutions to Schrödinger’s equation with linear potential have been obtained, for instance, by [5] an references therein.

The strategy employed in [5] is not only interesting but it is also useful. This follows since it ultimately leads to heat equation problems. In this section, we propose yet another *equivalent* approach using Fourier transforms:

**Theorem 6.1.** *Solutions to (4.4) are given by*

$$\begin{aligned} w(t, a) &= e^{(1/2) \int_t^s (f'(u))^2 du + af'(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-(1/2)y^2(s-t) + iy(a + \int_t^s f'(u) du)} dy \\ &= e^{(1/2) \int_t^s (f'(u))^2 du + af'(t)} \omega\left(s - t, a + \int_t^s f'(u) du\right), \end{aligned} \tag{6.1}$$

where  $\omega$  is a solution to the heat equation and  $\Pi$  is an arbitrary function, as long as the integral is well defined.

**Proof.** Let

$$\hat{w}(t, \lambda) := \int_{-\infty}^{\infty} e^{-i\lambda a} w(t, a) da. \tag{6.2}$$

Applying the Fourier transform to (4.4), we have

$$-\hat{w}_t(t, \lambda) + i f''(t) \hat{w}_\lambda(t, \lambda) + \frac{1}{2} \lambda^2 \hat{w}(t, \lambda) = 0, \quad i := \sqrt{-1}.$$

Next, set  $y = \lambda + i f'(t)$  and  $\hat{w}(t, \lambda) = \tilde{w}(t, y)$ , that is:

$$\hat{w}_t = \tilde{w}_t + i f''(t) \tilde{w}_y, \quad \hat{w}_\lambda = \tilde{w}_y,$$

which after substitution in (6.2) leads to

$$-\tilde{w}_t(t, y) + \frac{1}{2} (y - i f'(t))^2 \tilde{w}(t, y) = 0. \tag{6.3}$$

Consequently, setting  $\Pi(y) = \tilde{w}(0, y)$  it follows that

$$\begin{aligned} \tilde{w}(t, y) &= \Pi(y) \exp \left\{ -\frac{1}{2} \int_t^s (y - i f'(u))^2 du \right\} \\ &= \Pi(y) \exp \left\{ -\frac{1}{2} y^2 (s - t) + iy \int_t^s f'(u) du + \frac{1}{2} \int_t^s (f'(u))^2 du \right\}. \end{aligned} \tag{6.4}$$

[Note that equation (6.4) will be a solution of (6.3) for arbitrary  $\Pi$ .] Alternatively, this implies that

$$\begin{aligned} w(t, a) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-(1/2)y^2(s-t) + iy \int_t^s f'(u) du + (1/2) \int_t^s (f'(u))^2 du} e^{i\lambda a} d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-(1/2)y^2(s-t) + iy \int_t^s f'(u) du + (1/2) \int_t^s (f'(u))^2 du} e^{iy a + a f'(t)} dy \\ &= e^{(1/2) \int_t^s (f'(u))^2 du + a f'(t)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(y) e^{-(1/2)y^2(s-t) + iy(a + \int_t^s f'(u) du)} dy \end{aligned}$$

as claimed. □

This leads to our already anticipated claim, that boundary crossing probabilities, are equivalent to a set of heat equation problems.

**Theorem 6.2.** *The density  $\varphi$  of the first passage time  $T$  defined in (2.1) is related to a solution of the heat equation  $\omega$  as follows*

$$\varphi(s, a) = \omega \left( s, a + \int_0^s f'(u) du \right),$$

and it is bounded by

$$\begin{aligned} h(s, a) & e^{-af'(0)-(1/2)\int_0^s (f'(u))^2 du - \int_0^s f''(u)\tilde{\mathbb{E}}^a(\tilde{X}_u) du} \\ & \leq \omega\left(s, a + \int_0^s f'(u) du\right) \\ & \leq h(s, a)e^{-af'(0)-(1/2)\int_0^s (f'(u))^2 du} \end{aligned}$$

where  $h$  is as in (2.5).

**Proof.** It follows from Jensen’s inequality,

$$\exp\left\{-\int_0^s f''(u)\tilde{\mathbb{E}}^{t,a}(\tilde{X}_u) du\right\} \leq \tilde{\mathbb{E}}^{t,a}\left[\exp\left\{-\int_0^s f''(u)\tilde{X}_u du\right\}\right] \leq 1,$$

equation (4.3), and (6.1). □

As a corollary, we obtain the *necessary* boundary conditions for density  $\varphi$ .

**Corollary 6.1.** *From the previous theorem, the following boundary conditions hold:*

$$\begin{aligned} \lim_{a \rightarrow 0} \omega\left(s, a + \int_0^s f'(u) du\right) & = \delta(0), \\ \lim_{s \rightarrow 0} \omega\left(s, a + \int_0^s f'(u) du\right) & = 0 \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} & \frac{1}{\sqrt{2\pi s^3}} e^{-(1/2)\int_0^s (f'(u))^2 du - 2\sqrt{2/\pi}\int_0^s f''(u)\sqrt{s-u}\sqrt{u/s} du} \\ & \leq \lim_{a \rightarrow 0} \omega_a\left(s, a + \int_0^s f'(u) du\right) \leq \frac{1}{\sqrt{2\pi s^3}} e^{-(1/2)\int_0^s (f'(u))^2 du} \end{aligned}$$

or given the definition of Fractional Integral

$$J^\alpha g(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x - y)^{\alpha-1} g(y) dy, \quad x > 0, \alpha \in \mathbb{R}^+$$

we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi s^3}} e^{-(1/2)\int_0^s (f'(u))^2 du - s^{-1/2} J^{3/2} f''(s)\sqrt{2s}} \\ & \leq \lim_{a \rightarrow 0} \omega_a\left(s, a + \int_0^s f'(u) du\right) \\ & \leq \frac{1}{\sqrt{2\pi s^3}} e^{-(1/2)\int_0^s (f'(u))^2 du}. \end{aligned}$$

That is, Theorem 6.2 together with Corollary 6.1 uniquely characterize the density of  $T$ . Alternatively, from Theorem 6.2, we have obtained – for free – lower bounds for the problems described in Remarks 5.1 and 5.2.

### 7. A Green's function

In this section, we derive a fundamental solution of equation (4.1). To this end, let us first recall the following.

**Definition 7.1** (See, for instance, Definition 5.7.9 in [8]). *A fundamental solution of the second-order partial differential equation*

$$-\frac{\partial u}{\partial t} + ku = \mathcal{A}_t u \tag{7.1}$$

is a nonnegative function  $G(t, a; \tau, b)$  defined for  $0 \leq t \leq \tau \leq s$ ,  $a \in \mathbb{R}^+$ ,  $b \in \mathbb{R}^+$  with the property that for every  $\tilde{f} \in C_0(\mathbb{R}^+)$ ,  $\tau \in (0, s]$ , the function

$$u(t, a) := \int_0^\infty G(t, a; \tau, b) \tilde{f}(b) db; \quad 0 \leq t < \tau, a \in \mathbb{R}^+,$$

is bounded, of class  $C^{1,2}$ , satisfies (7.1), and

$$\lim_{t \uparrow \tau} u(t, a) = \tilde{f}(a), \quad a \in \mathbb{R}^+.$$

**Theorem 7.1.** *For  $h$  as in (2.5), the following function, is a fundamental solution of (4.1):*

$$G(t, a; \tau, b) = \frac{h(s - \tau, b)}{h(s - t, a)} H(t, a; \tau, b), \quad 0 \leq t < \tau < s, \tag{7.2}$$

where

$$\begin{aligned} H(t, a; \tau, b) = & \frac{1}{\sqrt{2\pi(\tau - t)}} \exp \left\{ \frac{1}{2} \int_t^\tau (f'(u))^2 du - f'(\tau)b + f'(t)a \right\} \\ & \times \left[ \exp \left\{ -\frac{(b - a - \int_t^\tau f'(u) du)^2}{2(\tau - t)} \right\} \right. \\ & \left. - \exp \left\{ -\frac{(b + a - \int_t^\tau f'(u) du)^2}{2(\tau - t)} \right\} \right]. \end{aligned}$$

Furthermore, it is also a solution to the forward Kolmogorov equation

$$\frac{\partial \psi}{\partial \tau}(\tau, b) + f''(\tau)b\psi(\tau, b) = \frac{1}{2} \frac{\partial^2 \psi}{\partial b^2}(\tau, b) - \frac{\partial}{\partial b} \left[ \left( \frac{1}{b} - \frac{b}{s - \tau} \right) \psi(\tau, b) \right]$$

in the forward variables  $(\tau, b) \in (t, s) \times \mathbb{R}^+$ .

**Proof.** It follows by applying the same techniques as in the previous section. □

**Remark 7.1.** (a) When  $f' = 0$ , then  $G$ , in (7.2) is the transition density of a 3-dimensional Bessel bridge. (b) Although  $G$  is a Green function of (4.1), and hence

$$v(t, a) = \int_0^\infty G(t, a; \tau, b) db, \quad (t, a) \in \times [0, \tau) \times \mathbb{R}^+,$$

is also a solution to (4.1), it is unbounded, and hence it can be the solution to our problem. This last statement follows from taking the limit as  $a \rightarrow 0$ .

## 8. Concluding remarks

As was already mentioned in the [Introduction](#), the aim of this paper is not to provide a new set of solutions to the problem of boundary crossing. Our goal is instead conceptual, in that we first make use of Girsanov’s theorem to connect hitting time problems to Bessel bridges. We do so by use of the Optional Sampling theorem. Next, we relate exponential functionals of 3 dimensional Bessel bridges to Schrödinger’s equation with linear potential. This is done by *uniquely* relating both problems through a Feynman–Kac representation. We conclude by characterizing the density of  $T$  in terms of a heat-equation problem and/or two alternate Schrödinger-equation problems.

Girsanov’s link is useful since it suggests how to extend this idea to higher dimensions or more general diffusions. Alternatively, the construction of Bessel bridges points to the fact that the concept can be extended. From the derivation of our Feynman–Kac representation we have learned that in some particular cases, sufficient conditions can be relaxed. Finally, Schrödinger’s equation with linear potential may be the way in which boundary crossing probabilities of some “higher order” processes can be described. For example, instead of having a parabolic equation – which describes the transition probability of Brownian motion – as in (5.6), it could be that a process with transition probability

$$\frac{\partial \theta}{\partial t}(t, x) = \alpha \frac{\partial^n \theta}{\partial x^n}(t, x), \quad [0, \infty) \times \mathbb{R}, n \geq 2,$$

has a first passage time probability which might be described by a Schrödinger type-problem:

$$\frac{\partial w}{\partial t}(t, x) + x f''(t)w(t, x) = \alpha \frac{\partial^n w}{\partial x^n}(t, x), \quad [0, \infty) \times \mathbb{R}.$$

In particular note that the techniques used in the proof of Theorem 6.2 still hold.

For further details see [7].

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