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Gambler's ruin estimates for random walks with symmetric spatially inhomogeneous increments

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Generalizing to higher dimensions the classical gambler's ruin estimates, we give pointwise estimates for the transition kernel corresponding to a spatially inhomogeneous random walk on the half-space. Our results hold under some strong but natural assumptions of symmetry, boundedness of the increments, and ellipticity. Among the most important steps in our proof are: discrete variants of the boundary Harnack estimate, as proven by Bauman, Bass and Burdzy, and Fabes *et al.*, based on comparison arguments and potential-theoretical tools; the existence of a positive $\tilde{L}$-harmonic function globally defined in the half-space; and some Gaussian inequalities obtained by a treatment inspired by Varopoulos.

Keywords: discrete potential theory; Gaussian estimates; Markov chains; transition kernels

1. Introduction and statement of results

An interesting modification of the gambler's ruin problem from classical random walk theory can be formulated in the following way. Imagine that the gambler is not forced to make the same wager with the same probabilities at each step but rather with transition probabilities which depend on his current fortune. How does this affect his probability of survival? In addition, physical applications of the classical ruin problem suggest the more flexible interpretation in terms of the motion of a particle on a half-line. This interpretation leads in a natural way to various generalizations, a typical example being the replacing of the half-line by the half-space.

Let us denote by

$$\mathbb{Z}^d_+ = \{x = (x_1, \ldots, x_{d-1}; x_d) = (x'; x_d) \in \mathbb{Z}^d = \mathbb{Z}^{d-1} \times \mathbb{Z},\, x_d > 0\},$$

the upper half-space and let $(S_j)_{j\in\mathbb{N}}$ denote a spatially inhomogeneous random walk with bounded symmetric increments in $\mathbb{Z}^d$. More precisely, let $\Gamma \subset \mathbb{Z}^d$ be a finite set such that $e \in \Gamma$ implies $-e \in \Gamma$ and let $\pi : \mathbb{Z}^d \times \Gamma \to [0, 1]$ such that

$$\sum_{e\in\Gamma} \pi(x, e) = 1, \quad \pi(x, e) = \pi(x, -e), \qquad e \in \Gamma,\, x \in \mathbb{Z}^d.$$

Then we let $(S_j)_{j\in\mathbb{N}}$ be the Markov chain defined by

$$\mathbb{P}\big[S_{j+1} = x + e \,|\, S_j = x\big] = \pi(x, e), \qquad e \in \Gamma, x \in \mathbb{Z}^d, j = 0, 1, \ldots.$$

To avoid unnecessary and trivial complications we shall also assume that Γ contains 0 and all unit vectors in $\mathbb{Z}^d$, that is, all e with $|e| = 1$ where $|\cdot|$ denotes the Euclidean norm. This eliminates periodic chains, that is, chains that live in a sublattice. Furthermore, it is vital for the considerations that follow that we should impose the following ellipticity condition:

$$\pi(x, e) \geqslant \alpha, \qquad x \in \mathbb{Z}^d, e \in \Gamma, \tag{1.1}$$

for some $\alpha > 0$.

The random walks $(S_j)_{j\in\mathbb{N}}$ that satisfy condition (1.1) are the discrete analogues of elliptic second-order operators in non-divergence form (cf. Kozlov 1985: 78, Table 1). They were studied in Lawler (1991), Kuo and Trudinger (1998) and more recently in Varopoulos (2000, 2001, 2003a, 2003b, 2005) and Mustapha (2006).

We shall denote by τ the first exit time from $\mathbb{Z}^d_+$, that is,

$$\tau = \inf\{j = 0, 1, \ldots, S_j \notin \mathbb{Z}^d_+\},$$

and we shall consider the transition kernel corresponding to the random walk $(S_j)_{j\in\mathbb{N}}$ with killing outside of $\mathbb{Z}^d_+$:

$$p_n(x, y) = \mathbb{P}_x[S_n = y, \tau > n], \qquad n = 1, 2, \ldots; x, y \in \mathbb{Z}^d_+.$$

The object of our study is to gain as much information as possible about the kernel $p_n(x, y)$. We shall see that precise estimates for $p_n(x, y)$ can be given in terms of two important 'functions': a global adjoint solution m and a positive normalized adjoint solution u vanishing on the boundary $\partial\mathbb{Z}^d_+$ (see Section 2 for a precise definition of the boundary $\partial\mathbb{Z}^d_+$). In one dimension, these estimates generalize the classical Gambler's ruin estimates.

To define the two functions m and u we need to introduce some notation. We shall denote by L the generator corresponding to the chain $(S_j)_{j\in\mathbb{N}}$, that is, the difference operator defined by

$$Lf(x) = \sum_{e\in\Gamma} \pi(x, e)(f(x + e) - f(x)), \qquad f : \mathbb{Z}^d \to \mathbb{R}.$$

It was shown in Mustapha (2006: Section 3.2) that there exists a positive measure $(m(x))_{x\in\mathbb{Z}^d}$ which is invariant for the chain $(S_j)_{j\in\mathbb{N}}$. This invariant measure is unique (up to a multiplicative constant) and satisfies the adjoint equation

$$\sum_{e\in\Gamma} \pi(x - e, e)m(x - e) = m(x), \qquad x \in \mathbb{Z}^d. \tag{1.2}$$

It was also shown in Mustapha (2006: Section 3.2) that the measure $(m(x))_{x\in\mathbb{Z}^d}$ satisfies the doubling property of the volume

$$v(x, r) = \sum_{y\in B_r(x)} m(y),$$

for the balls $B_r(x) = \{y \in \mathbb{Z}^d, |y - x| < r\}$ $(x \in \mathbb{Z}^d,\ r > 0)$. More precisely, we have

$$v(x, 2r) \leqslant Cv(x, r), \qquad x \in \mathbb{Z}^d,\ r > 0. \tag{1.3}$$

The fact that (1.2) holds opens up an important possibility. It enables us to define a normalized adjoint process $(S_j^*)_{j\in\mathbb{N}}$ by setting

$$\mathbb{P}\Big[S_{j+1}^* = x + e \,|\, S_j^* = x\Big] = \pi^*(x, e), \qquad j = 0, 1, \ldots;\ x \in \mathbb{Z}^d,\ e \in \mathbb{Z}^d,$$

where

$$\pi^*(x, e) = \begin{cases} \pi(x+e, e)\dfrac{m(x+e)}{m(x)}, & e \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that the generator of this process is the adjoint of L with respect to the invariant measure $(m(x))_{x\in\mathbb{Z}^d}$:

$$\tilde{L}f(x) = \sum_{e\in\Gamma} \pi^*(x, e)(f(x+e) - f(x)), \qquad f : \mathbb{Z}^d \to \mathbb{R}.$$

We shall prove that there exists one and only one (up to a multiplicative constant) positive solution $u(\cdot)$ of the equation

$$\tilde{L}u(x) = \sum_{e\in\Gamma} \pi^*(x, e)(u(x+e) - u(x)) = 0, \qquad x \in \mathbb{Z}_+^d,$$

vanishing on the boundary $\partial\mathbb{Z}_+^d$.

We can now state our main estimates.

Theorem 1. *Let $(S_j)_{j\in\mathbb{N}}$ be a spatially inhomogeneous random walk with bounded symmetric increments as above. Then the transition kernel $p_n(x, y)$ satisfies:*

$$p_n(x, y) \leqslant \frac{Cx_d m(y)}{\sqrt{n}v(x, \sqrt{n})}\exp\left(-\frac{|x-y|^2}{Cn}\right),$$
$$n \geqslant 1,\ x = (x'; x_d),\ y = (y'; y_d) \in \mathbb{Z}_+^d,\ y_d \geqslant C\sqrt{n}, \tag{1.4}$$

$$p_n(x, y) \leqslant \frac{Cx_d u(y)m(y)}{\sqrt{n}v(x, \sqrt{n})u(y'; C[\sqrt{n}])}\exp\left(-\frac{|x-y|^2}{Cn}\right),$$
$$n \geqslant 1,\ x = (x'; x_d),\ y = (y'; y_d) \in \mathbb{Z}_+^d,\ y_d \leqslant C\sqrt{n}, \tag{1.5}$$

where $C > 0$ is an integer constant depending only on d, Γ, α. Here $[\sqrt{n}]$ denotes the greatest integer $\leqslant \sqrt{n}$.

The following remarks will help to clarify this theorem. Firstly, the doubling property (1.3) implies that the volume factor $v(x, \sqrt{n})$ in (1.4) and (1.5) can be symmetrized and replaced by the factor $(v(x, \sqrt{n})v(y, \sqrt{n}))^{1/2}$. Secondly, the upper estimate (1.5) points to the important role played by the positive normalized adjoint solution u vanishing on the boundary $\partial\mathbb{Z}_+^d$. This function u is the multidimensional analogue of the positive solution of the Wiener–Hopf equation which plays a crucial role in fluctuation theory (cf. Spitzer (1976: Chapters 18, 19 and 27)).

Thirdly, in the homogeneous case, that is, the case where $\pi(x, e) = \pi(y, e)$, $x, y \in \mathbb{Z}^d$, $e \in \Gamma$, it is easy to see that $m(y) \equiv 1$, $v(x, \sqrt{n}) \approx n^{d/2}$ and $u(y) = y_d$. The upper estimates of Theorem 1 can be rewritten in this case as follows:

$$p_n(x, y) \leqslant \frac{Cx_d y_d}{n^{1+d/2}} \exp\left(-\frac{|x-y|^2}{Cn}\right), \qquad x, y \in \mathbb{Z}_+^d, \; n \geqslant 1.$$

This estimate follows also from the general results of Varopoulos (2003a). An approach which allows similar estimates to be obtained for inhomogeneous random walks on Lipschitz domains is formulated in Varopoulos (2003a). Our proof uses some ideas from Varopoulos (2003a).

Fourthly, let us consider, in the one-dimensional case, the spatially inhomogeneous discrete heat operator defined by the difference operator

$$[\mathcal{D}f](k) = \alpha_k(f(k+1) + f(k-1)) - 2\alpha_k f(k), \qquad k \in \mathbb{Z}, \tag{1.6}$$

where we assume that the coefficients α_k satisfy

$$0 < \alpha \leqslant \alpha_k < 1/2, \qquad k \in \mathbb{Z}.$$

The function m can be explicitely computed in this case and we obtain

$$m(k) = \frac{\alpha_0}{\alpha_k}, \qquad k \in \mathbb{Z}.$$

It then follows that the volume function $v(k, \sqrt{n})$ satisfies $v(k, \sqrt{n}) \approx \sqrt{n}$. On the other hand, one can easily show that the function u is given by

$$u(k) = k, \qquad k \in \mathbb{Z}^+ = \{1, 2, \ldots\}.$$

The discrete half-line heat kernel $p_n(k, l)$ generated by (1.6) therefore admits the bound

$$p_n(k, l) \leqslant \frac{Ckl}{n^{3/2}} \exp\left(-\frac{(l-k)^2}{Cn}\right), \qquad l, k \in \mathbb{Z}^+, \; n = 1, 2, \ldots.$$

This shows that the classical gambler's ruin estimate (cf. Feller 1966; Spitzer 1976) is still valid when we deal with spatially inhomogeneous walks.

Finally, the upper estimates (1.4) and (1.5) can easily be extended to more general half-spaces of the type

$$\Pi_{\mathcal{Z}} = \{x \in \mathbb{Z}^d, \langle \mathcal{Z}, x\rangle > 0\},$$

where $\mathcal{Z} \in \mathbb{Z}^d \backslash \{0\}$ is fixed and where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product on $\mathbb{R}^d$. It is routine to generalize the considerations of Sections 2 and 3 to prove the existence of a unique (up to a multiplicative constant) positive solution of the equation $\tilde{L}u = 0$ vanishing on $\partial\Pi_{\mathcal{Z}}$. If we denote by $u_{\mathcal{Z}}$ this function and by $p_n^{\mathcal{Z}}(x, y)$ the iterated transition kernel with killing at the boundary in $\Pi_{\mathcal{Z}}$ then our estimates take the following form:

$$p_n^{\mathcal{Z}}(x, y) \leqslant \frac{C\langle \mathcal{Z}, x\rangle m(y)}{\sqrt{n}v(x, \sqrt{n})} \exp\left(-\frac{|x-y|^2}{Cn}\right),$$
$$n \geqslant 1;\ x,\ y \in \Pi_{\mathcal{Z}},\ \langle \mathcal{Z}, y\rangle \geqslant C\sqrt{n},$$
$$p_n^{\mathcal{Z}}(x, y) \leqslant \frac{C\langle \mathcal{Z}, x\rangle u_{\mathcal{Z}}(y) m(y)}{\sqrt{n}v(x, \sqrt{n}) u_{\mathcal{Z}}([\pi(y)] + C[\sqrt{n}]\mathcal{Z})} \exp\left(-\frac{|x-y|^2}{Cn}\right),$$
$$n \geqslant 1;\ x,\ y \in \Pi_{\mathcal{Z}},\ \langle \mathcal{Z}, y\rangle \leqslant C\sqrt{n},$$

where the constant $C > 0$ depends only on d, Γ, α and $\mathcal{Z}$. Here π denotes the orthogonal projection $\pi : \Pi_{\mathcal{Z}} \to \{x \in \mathbb{R}^d, \langle \mathcal{Z}, x\rangle = 0\}$ and $[\pi(y)]$ is a point of $\partial\Pi_{\mathcal{Z}}$ which minimizes the distance between $\pi(y)$ and $\partial\Pi_{\mathcal{Z}}$.

2. Notation and known results

Two points in $\mathbb{Z}^d$ will be said to be *adjacent* if the distance between them is unity. A subset A of cardinality $|A| \geqslant 2$ will be called *connected* if for any two points of A there is a path consisting of segments of unit length connecting them in such a manner that the end-points of these segments are all in A. A set of points is a *domain* if it is connected. The symbol A will be used in the following to denote a domain of $\mathbb{Z}^d$. We define the boundary ∂A of A by

$$\partial A = \{x \in A^c,\ x = z + e, \text{ for some } z \in A \text{ and } e \in \Gamma\},$$

Γ being as in the previous section. The closure of A will be denoted by $\bar{A}$ and defined by

$$\bar{A} = A \cup \partial A.$$

For a subset $B = A \times \{a \leqslant k \leqslant b\} \subset \mathbb{Z}^d \times \mathbb{Z}$ where $a < b \in \mathbb{Z}$, we define the lateral boundary and the parabolic boundary of B by

$$\partial_l B = \bigcup_{a<k<b} \partial A \times \{k\}, \qquad \partial_p B = \partial_l B \cup \left(\bar{A} \times \{a\}\right),$$

and we let

$$\bar{B} = B \cup \partial_p B.$$

We say that u: $\bar{A} \to \mathbb{R}$ is *L-harmonic* in $A \subset \mathbb{Z}^d$ if

$$Lu(x) = \sum_{e\in\Gamma} \pi(x, e)(u(x+e) - u(x)) = 0, \qquad x \in A.$$

We say that $u : \bar{B} \to \mathbb{R}$, where $B = A \times \{a \leqslant k \leqslant b\} \subset \mathbb{Z}^d \times \mathbb{Z}$ is *$\mathcal{L}$-caloric* in B if

$$\mathcal{L}u(x, k) = \sum_{e\in\Gamma} \pi(x, e)u(x+e, k) - u(x, k+1) = 0, \qquad (x, k) \in A \times \{a \leqslant k < b\}.$$

The following theorem (cf. Kuo and Trudinger 1998) is a random walk version of the well-known and fundamental result in the potential theory of second-order equations in non-divergence form (for an elliptic version, see Lawler 1991).

Theorem 2 (Parabolic Harnack principle). *Let u be a non-negative $\mathcal{L}$-caloric function in $B_{2r}(y) \times \{s - 4r^2 \leqslant k \leqslant s\}$, $(y, s) \in \mathbb{Z}^d \times \mathbb{Z}$, $r \geqslant 1$. Then*

$$\begin{aligned} &\sup\{u(x, k);\, x \in B_r(y),\, s - 3r^2 < k < s - 2r^2\} \\ &\leqslant C \inf\{u(x, k);\, x \in B_r(y),\, s - r^2 < k < s\}, \end{aligned} \tag{2.1}$$

where $C = C(d, \alpha, \Gamma) > 0$.

In the proof of Theorem 1, together with Theorem 2, we need the following boundary Harnack principle. Let $Q = \mathbb{Z}_+^d \times \mathbb{Z}$, let $Y = (y, s) \in \partial\mathbb{Z}_+^d \times \mathbb{Z}$, and let $r \geqslant 2\mathrm{diam}(\Gamma)$. We shall write

$$C_r(Y) = B_r(y) \times \{s - r^2 \leqslant k \leqslant s\}, \qquad Q_r(Y) = Q \cap C_r(Y),$$
$$\overline{Y}_r = (y_r, s + 2[r]^2), \qquad \underline{Y}_r = (y_r, s - 2[r]^2),$$

where $y_r = y + (0; [r]) \in \mathbb{Z}_+^d$.

Theorem 3 (Boundary Harnack principle). *Let $Y = (y, s) \in \partial\mathbb{Z}_+^d \times \mathbb{Z}$. Let $r \geqslant 2\mathrm{diam}(\Gamma)$. Let $K > 0$ large enough. Assume that u and v are two non-negative $\mathcal{L}$-caloric functions in $Q \cap (B_{3Kr}(y) \times \{s - 9K^2r^2 \leqslant k \leqslant s + 9K^2r^2\})$ and $u = 0$ on $(\partial\mathbb{Z}_+^d \times \mathbb{Z}) \cap (B_{2Kr}(y) \times \{s - 4K^2r^2 \leqslant k \leqslant s + 4K^2r^2\})$. Then*

$$\sup_{Q_r(Y)} \frac{u}{v} \leqslant C \frac{u(\overline{Y}_{Kr})}{v(\underline{Y}_{Kr})}, \tag{2.2}$$

where $C = C(d, \alpha, \Gamma) > 0$.

The estimate (2.2) is a discrete variant of the boundary Harnack estimate for non-negative solutions of second-order equations in non-divergence form (Bauman 1984; Bass and Burdzy 1994; Fabes *et al.* 1999). The proof of this estimate is a straightforward adaptation of the proof of the corresponding estimate for the boundary behaviour of non-negative $\mathcal{L}$-caloric functions in cylindrical domains given in Mustapha (2006).

Let us finally denote by

$$h_n(x, y) = \mathbb{P}_x[S_n = y], \qquad n = 1, 2, \ldots;\, x, y \in \mathbb{Z}^d, \tag{2.3}$$

the global transition kernel corresponding to the chain $(S_j)_{j \in \mathbb{N}}$. We have (cf. Mustapha 2006):

Theorem 4. *There exists $C > 0$, depending only on d, Γ and α, such that*

$$h_n(x, y) \leqslant \frac{Cm(y)}{v(x, \sqrt{n})} \exp\Big(-\frac{|x - y|^2}{Cn}\Big), \qquad n \geqslant 1;\, x, y \in \mathbb{Z}^d. \tag{2.4}$$

We shall use throughout the usual convention $f \approx g$ to indicate that $C^{-1} \leqslant f/g \leqslant C$ for an appropriate constant $C > 0$; C and c will denote different positive constants which depend only on d, α, $\mathrm{diam}(\Gamma)$.

3. Potential theory

Let $D \subset \mathbb{Z}^d$ denote a bounded domain and $a < b \in \mathbb{Z}$. We say that $v = v(y, t) : \overline{D \times \{a \leqslant t \leqslant b\}} \to \mathbb{R}$ is a parabolic adjoint solution of L in $D \times \{a \leqslant t \leqslant b\}$, if v satisfies the equation

$$v(y, t+1) = \sum_{e \in \Gamma} \pi(y-e, e) v(y-e, t), \qquad t = a, \ldots, b-1;\ y \in D.$$

Let v be a parabolic adjoint solution in $D \times \{a \leqslant t \leqslant b\}$; the function

$$\tilde{v}(y, t) = \frac{v(y, t)}{m(y)}, \qquad (y, t) \in \overline{D \times \{a \leqslant t \leqslant b\}},$$

where m is the global adjoint solution defined in (1.2), is called a *normalized parabolic adjoint solution* of L in $D \times \{a \leqslant t \leqslant b\}$ (cf. Mustapha 2006). It is easy to see that every $\tilde{\mathcal{L}}$-caloric function, that is, a function u such that

$$\tilde{\mathcal{L}} u(x, k) = \sum_{e \in \Gamma} \pi^*(x, e) u(x+e, k) - u(x, k+1) = 0, \qquad (x, k) \in D \times \{a \leqslant k < b\},$$

defines a normalized parabolic adjoint solution of L in $D \times \{a \leqslant t \leqslant b\}$.

The adjoint Harnack principle established in Mustapha (2006) therefore implies the following parabolic Harnack principle for $\tilde{\mathcal{L}}$-caloric functions.

Theorem 5. *Suppose that u is a non-negative $\tilde{\mathcal{L}}$-caloric function in $B_{2r}(y) \times \{s - 4r^2 \leqslant k \leqslant s\}$, where $y \in \mathbb{Z}^d$ and $r \geqslant 1$. Then there exists a constant $C > 0$, depending only on d, α, Γ, such that*

$$\sup\{u(x, k);\ x \in B_r(y),\ s - 3r^2 < k < s - 2r^2\} \leqslant C \inf\{u(x, k);\ x \in B_r(y),\ s - r^2 < k < s\}. \tag{3.1}$$

On the other hand, the following maximum principle for $\tilde{\mathcal{L}}$-caloric functions is immediate:

Theorem 6 (Maximum principle). *Let $B = A \times \{a \leqslant k \leqslant b\} \subset \mathbb{Z}^d \times \mathbb{Z}$, where $a < b \in \mathbb{Z}$ and A is a bounded domain in $\mathbb{Z}^d$, and let $u : \overline{B} \to \mathbb{R}$ be an $\tilde{\mathcal{L}}$-caloric function in B which satisfies $u \geqslant 0$ on $\partial_p B$. Then $u \geqslant 0$ in B.*

3.1. Behaviour at the boundary of non-negative $\tilde{\mathcal{L}}$-caloric functions

The proofs of Theorem 3 and estimate (5.6) of Mustapha (2006) (the Carleson estimate) are based only on the parabolic Harnack principle and the maximum principle. According to

Theorems 5 and 6, $\tilde{\mathcal{L}}$-caloric functions satisfy both of these principles. The proofs given in Mustapha (2006) therefore apply when we deal with $\tilde{\mathcal{L}}$-caloric functions.

With the notation of Section 2, we have:

Theorem 7 (Carleson estimate). *Let $Y = (y, s) \in \partial\mathbb{Z}_+^d \times \mathbb{Z}$. Assume that u is a non-negative $\tilde{\mathcal{L}}$-caloric function in $Q \cap (B_{3r}(y) \times \{s - 9r^2 \leqslant k \leqslant s + 9r^2\})$ and $u = 0$ on $(\partial\mathbb{Z}_+^d \times \mathbb{Z}) \cap (B_{2r}(y) \times \{s - 4r^2 \leqslant k \leqslant s + 4r^2\})$. Then*

$$\sup\{u(X),\ X \in Q_r(Y)\} \leqslant Cu(\overline{Y}_r), \qquad r \geqslant C, \tag{3.2}$$

where $C = C(d, \alpha, \Gamma) > 0$.

Theorem 8 (Boundary Harnack principle). *Let $Y = (y, s) \in \partial\mathbb{Z}_+^d \times \mathbb{Z}$. Assume that u and v are two non-negative $\tilde{\mathcal{L}}$-caloric functions in $Q \cap (B_{3Kr}(y) \times \{s - 9K^2r^2 \leqslant k \leqslant s + 9K^2r^2\})$ and $u = 0$ on $(\partial\mathbb{Z}_+^d \times \mathbb{Z}) \cap (B_{2Kr}(y) \times \{s - 4K^2r^2 \leqslant k \leqslant s + 4K^2r^2\})$, where K is large enough. Then*

$$\sup_{Q_r(Y)} \frac{u}{v} \leqslant C\frac{u(\overline{Y}_{Kr})}{v(\underline{Y}_{Kr})}, \qquad r \geqslant C, \tag{3.3}$$

where $C = C(d, \alpha, \Gamma) > 0$.

3.2. The $\tilde{L}$-harmonic function $u(x)$

Theorem 9. *There exists a positive $\tilde{L}$-harmonic function u defined globally in $\mathbb{Z}_+^d$ and vanishing on $\partial\mathbb{Z}_+^d$. This solution is unique up to a multiplicative constant.*

Proof. Let

$$\mathcal{U}_l(y) = \alpha_l[G_{l+1}(\mathbf{1}, y) - G_l(\mathbf{1}, y)], \qquad y \in B_{2^l}(0) \cap \mathbb{Z}_+^d,\ l = 1, 2, \ldots,$$

where $G_l(\mathbf{1}, .)$ is the Green function of $(S_j^*)_{j\in\mathbb{N}}$ in $B_{2^l}(0) \cap \mathbb{Z}_+^d$, $l = 1, 2, \ldots,$ with pole at $\mathbf{1} = (0, \ldots, 0; 1)$ and where the α_l are chosen so that

$$\mathcal{U}_l(\mathbf{1}) = 1, \qquad l = 1, 2, \ldots. \tag{3.4}$$

It is easy to see that the ellipticity condition (1.1) implies a local Harnack principle for the non-negative $\tilde{L}$-harmonic functions. This local Harnack principle, combined with the Carleson estimate (3.2) and the normalization condition (3.4), implies that the $\mathcal{U}_l$ satisfy

$$\mathcal{U}_l(y) \leqslant C, \qquad y \in B_{2^k}(0) \cap \mathbb{Z}_+^d,\ l \geqslant k,$$

with a constant $C = C(k)$ depending only on k. The diagonal process then allows us to deduce the existence of a positive $\tilde{L}$-harmonic function u defined globally on $\mathbb{Z}_+^d$ and vanishing on $\partial\mathbb{Z}_+^d$.

To prove uniqueness, let us consider u_1 and u_2, two positive $\tilde{L}$-harmonic functions in $\mathbb{Z}_+^d$ vanishing on $\partial\mathbb{Z}_+^d$. By the boundary Harnack principle (3.3) we have

$$\frac{1}{C}\frac{u_1(x)}{u_1(0;\,R)} \leqslant \frac{u_2(x)}{u_2(0;\,R)} \leqslant C\frac{u_1(x)}{u_1(0;\,R)}, \qquad x \in B_R(0)\cap \mathbb{Z}_+^d,\ R \geqslant C. \tag{3.5}$$

If we assume that u_1 and u_2 satisfy the normalization condition (3.4), then it follows that

$$\frac{1}{C}u_1(0;\,R) \leqslant u_2(0;\,R) \leqslant Cu_1(0;\,R), \qquad R \geqslant C. \tag{3.6}$$

Putting (3.5) and (3.6) together, we deduce that

$$\frac{1}{C^2} \leqslant \frac{u_1(x)}{u_2(x)} \leqslant C^2, \qquad x \in \mathbb{Z}_+^d.$$

We shall assume that $C > 1$ and define

$$u_3(x) = u_1(x) + \frac{1}{C^2-1}(u_1(x) - u_2(x)), \qquad x \in \mathbb{Z}_+^d.$$

We have

$$u_3(x) = \frac{C^2}{C^2-1}\left(u_1(x) - \frac{1}{C^2}u_2(x)\right) \geqslant 0, \qquad x \in \mathbb{Z}_+^d.$$

Clearly we also have

$$\tilde{L}u_3(x) = 0, \quad x \in \mathbb{Z}_+^d; \qquad u_3(x) = 0, \quad x \in \partial\mathbb{Z}_+^d; \qquad u_3(\mathbf{1}) = 1.$$

We can iterate and define $u_p(x)$ $(x \in \mathbb{Z}_+^d)$

$$u_p(x) = u_{p-1}(x) + \frac{1}{C^2-1}\big(u_{p-1}(x) - u_2(x)\big), \qquad p = 4, 5, \ldots,$$

with

$$u_p(x) \geqslant 0, \quad x \in \mathbb{Z}_+^d; \qquad \tilde{L}u_p(x) = 0, \quad x \in \mathbb{Z}_+^d;$$
$$u_p(x) = 0, \quad x \in \partial\mathbb{Z}_+^d; \qquad u_p(\mathbf{1}) = 1,$$

and

$$\frac{1}{C^2} \leqslant \frac{u_p(x)}{u_2(x)} \leqslant C^2, \qquad x \in \mathbb{Z}_+^d.$$

We also have

$$\frac{1}{C^2} \leqslant \frac{u_p(x)}{u_1(x)} \leqslant C^2, \qquad x \in \mathbb{Z}_+^d. \tag{3.7}$$

On the other hand, it is easy to see that

$$u_p(x) = u_1(x) + C_p(u_1(x) - u_2(x)), \qquad x \in \mathbb{Z}_+^d,\ p = 3, 4, \ldots, \tag{3.8}$$

where $C_p \to \infty$ as $p \to \infty$. Putting (3.7) and (3.8) together, we obtain

$$\frac{1}{C^2} \leqslant 1 + C_p\left(1 - \frac{u_2(x)}{u_1(x)}\right) \leqslant C^2, \qquad x \in \mathbb{Z}_+^d,\ p = 3, 4, \ldots,$$

and, letting $p \to \infty$, we deduce that

$$u_2(x) = u_1(x), \qquad x \in \mathbb{Z}_+^d.$$

$\square$

4. Proofs of the estimates

4.1. The upper Gaussian estimate (1.4)

We first observe that in the case $x_d/\sqrt{n} \geqslant C$, the upper estimate (1.4) is an immediate consequence of the estimate (2.4). This is simply due to the obvious inequality

$$p_n(x, y) \leqslant h_n(x, y), \qquad x, y \in \mathbb{Z}_+^d, \tag{4.1}$$

where $h_n(x, y)$ denotes the global transition kernel as in (2.3).

Let us assume that $x_d \leqslant C\sqrt{n}$. Let $p_n^*(x, y)$, $x, y \in \mathbb{Z}_+^d$, $n = 1, 2 \ldots,$ denote the transition kernel corresponding to the chain $(S_j^*)_{j\in\mathbb{N}}$ with killing outside of $\mathbb{Z}_+^d$. It is easy to check that this kernel satisfies

$$p_n^*(x, y) = p_n(y, x)\frac{m(y)}{m(x)}, \qquad x, y \in \mathbb{Z}_+^d. \tag{4.2}$$

Let us now fix $y = (y'; y_d) \in \mathbb{Z}_+^d$ such that

$$y_d \geqslant C\sqrt{n}. \tag{4.3}$$

By using the boundary Harnack principle (2.2) to compare the two $\mathcal{L}$-caloric functions,

$$u_1(x, s) = x_d; \qquad u_2(x, s) = p_s(x, y), \quad x \in \mathbb{Z}_+^d,\ s = 1, 2, \ldots,$$

and (2.1), we obtain

$$p_n(x, y) \leqslant \frac{Cx_d}{\sqrt{n}} \inf_{z\in B_{c\sqrt{n}}(\bar{x})} p_{Cn}(z, y), \tag{4.4}$$

where $\bar{x} = x + (0, C[\sqrt{n}])$. Substituting (4.2) into (4.4) gives

$$p_n(x, y) \leqslant \frac{Cx_d}{\sqrt{n}} \inf_{z\in B_{c\sqrt{n}}(\bar{x})} \frac{p_{Cn}^*(y, z)m(y)}{m(z)},$$

from which it follows by multiplying by $m(z)$ and taking the sum over $z \in B_{c\sqrt{n}}(\bar{x})$ that

$$p_n(x, y) \sum_{z\in B_{c\sqrt{n}}(\bar{x})} m(z) \leqslant \frac{Cx_d m(y)}{\sqrt{n}} \sum_{z\in B_{c\sqrt{n}}(\bar{x})} p_{Cn}^*(y, z).$$

Using the obvious estimate

$$\sum_{z\in B_r(x)} p_n^*(y, z) = \mathbb{P}_y[S_n^* \in B_r(x)] \leqslant 1, \qquad x, y \in \mathbb{Z}_+^d,\ r > 0,\ n = 1, 2, \ldots,$$

we deduce that

$$p_n(x, y) \leqslant \frac{Cx_d m(y)}{\sqrt{n}v(\bar{x}, c\sqrt{n})}, \qquad n = 1, 2, \ldots. \tag{4.5}$$

On the other hand, we have

$$B_{\sqrt{n}}(x) \subset B_{C/n}(\bar{x}).$$

This, combined with the doubling property (1.3), gives

$$\frac{1}{v(\bar{x}, c\sqrt{n})} \leqslant \frac{C}{v(x, \sqrt{n})}. \tag{4.6}$$

Putting (4.5) and (4.6) together, we deduce that

$$p_n(x, y) \leqslant \frac{Cx_d m(y)}{\sqrt{n}v(x, \sqrt{n})}, \qquad n = 1, 2, \ldots. \tag{4.7}$$

Let us now assume that

$$|x - y| \leqslant C\sqrt{n}.$$

This estimate implies that the Gaussian factor

$$\exp(-\frac{|x-y|^2}{Cn}) \approx 1, \tag{4.8}$$

and the estimate (4.7) can therefore be rewritten

$$p_n(x, y) \leqslant \frac{Cx_d m(y)}{\sqrt{n}v(x, \sqrt{n})} \exp\left(-\frac{|x-y|^2}{Cn}\right), \qquad n = 1, 2, \ldots.$$

It remains to consider the case

$$|x - y| \geqslant C\sqrt{n}. \tag{4.9}$$

In this case, we first use the parabolic Harnack estimate (3.1) to obtain

$$p_n^*(y, x) \leqslant C \inf_{z \in B_{c\sqrt{n}}(y)} p_{Cn}^*(z, x).$$

This is possible because of (4.3).

Using (4.2), we then deduce that

$$\frac{p_n(x, y)}{m(y)} \leqslant C \inf_{z \in B_{c\sqrt{n}}(y)} \frac{p_{Cn}(x, z)}{m(z)},$$

from which it follows that

$$\frac{p_n(x, y)}{m(y)} v(y, c\sqrt{n}) \leqslant C \sum_{z \in B_{c/n}(y)} p_{Cn}(x, z). \tag{4.10}$$

We should also observe that

$$\frac{1}{v(y,\, c\sqrt{n})} \leqslant \frac{C}{v(x,\, \sqrt{n})}(1 + \frac{|x-y|}{\sqrt{n}})^C, \qquad x,\, y \in \mathbb{Z}_+^d. \tag{4.11}$$

This is an immediate consequence of the doubling property (1.3). On the other hand, if $z \in B_{c\sqrt{n}}(y)$ then (4.9) implies that

$$|z-x| \geqslant c|x-y|. \tag{4.12}$$

Combining (4.10), (4.11) and (4.12), we obtain

$$p_n(x,\, y) \leqslant \frac{Cm(y)}{v(x,\, \sqrt{n})}\left(1 + \frac{|x-y|}{\sqrt{n}}\right)^C \sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|} p_{Cn}(x,\, z). \tag{4.13}$$

The next step is to prove the following estimate

$$\sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|} p_{Cn}(x,\, z) \leqslant \frac{Cx_d}{\sqrt{n}}\exp(-\frac{|x-y|^2}{Cn}), \qquad x \in \mathbb{Z}_+^d,\ n = 1,\, 2,\, \ldots. \tag{4.14}$$

We follow Varopoulos (2003a: Section 3). Let $m_0 = Cn$ and let $k = 1, 2 \ldots$ defined by $2^k < m_0 \leqslant 2^{k+1}$. Let $m_1 = 2^{k-1}$. We have

$$\begin{aligned}\sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|} p_{m_0}(x,\, z) &= \sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|}\left(\sum_{\zeta\in\mathbb{Z}_+^d} p_{m_1}(x,\, \zeta)p_{m_0-m_1}(\zeta,\, z)\right) \\ &= \sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|}\left(\sum_{\zeta\in\mathbb{Z}_+^d, |\zeta-x|> c|x-y|/2} p_{m_1}(x,\, \zeta)p_{m_0-m_1}(\zeta,\, z)\right) \\ &\quad + \sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|}\left(\sum_{\zeta\in\mathbb{Z}_+^d, |\zeta-x|\leqslant c|x-y|/2} p_{m_1}(x,\, \zeta)p_{m_0-m_1}(\zeta,\, z)\right).\end{aligned} \tag{4.15}$$

We estimate the first sum on the right hand side of (4.15) as follows:

$$\begin{aligned}&\sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|}\left(\sum_{\zeta\in\mathbb{Z}_+^d, |\zeta-x|> c|x-y|/2} p_{m_1}(x,\, \zeta)p_{m_0-m_1}(\zeta,\, z)\right) \\ &\leqslant \sum_{\zeta\in\mathbb{Z}_+^d, |\zeta-x|\geqslant c|x-y|/2} p_{m_1}(x,\, \zeta)\sum_{z\in\mathbb{Z}_+^d} p_{m_0-m_1}(\zeta,\, z) \\ &\leqslant \sum_{z\in\mathbb{Z}_+^d, |z-x|\geqslant c|x-y|/2} p_{m_1}(x,\, z).\end{aligned} \tag{4.16}$$

Concerning the second sum, we observe that

$$\sum_{z\in\mathbb{Z}_+^d,|z-x|\geqslant c|x-y|}\left(\sum_{\zeta\in\mathbb{Z}_+^d,|\zeta-x|\leqslant c|x-y|/2} p_{m_1}(x,\zeta)p_{m_0-m_1}(\zeta,z)\right) \tag{4.17}$$

$$\leqslant \sum_{\zeta\in\mathbb{Z}_+^d} p_{m_1}(x,\zeta) \sum_{z\in\mathbb{Z}_+^d,|z-\zeta|\geqslant c|x-y|/2} h_{m_0-m_1}(\zeta,z)$$

$$\leqslant \left(\sum_{z\in\mathbb{Z}_+^d} p_{m_1}(x,\zeta)\right)\times \sup_{\zeta\in\mathbb{Z}_+^d}\left[\sum_{z\in\mathbb{Z}_+^d,|z-\zeta|\geqslant c|x-y|/2} h_{m_0-m_1}(\zeta,z)\right],$$

where we have used the estimate (4.1).

The boundary Harnack principle (2.2) used to compare the two $\mathcal{L}$-caloric functions,

$$u_1(x,s)=x_d;\qquad u_2(x,s)=\sum_{z\in\mathbb{Z}_+^d} p_s(x,z),\quad x\in\mathbb{Z}_+^d,\quad s=1,2,\ldots,$$

gives

$$\sum_{z\in\mathbb{Z}_+^d} p_n(x,z)\leqslant\frac{Cx_d}{\sqrt{n}},\qquad x\in\mathbb{Z}_+^d,\quad n=1,2,\ldots \tag{4.18}$$

(notice that the boundary Harnack principle in needed only in the case $x_d/\sqrt{n}\leqslant C$, otherwise (4.18) is obvious). On the other hand, the upper Gaussian estimate (2.4) implies easily that

$$\sum_{z\in\mathbb{Z}_+^d,|z-\zeta|\geqslant c|x-y|/2} h_{m_0-m_1}(\zeta,z)\leqslant C\exp\left(-\frac{|x-y|^2}{C(m_0-m_1)}\right),\qquad \zeta\in\mathbb{Z}_+^d. \tag{4.19}$$

Putting (4.15)–(4.19) together, we deduce that

$$\sum_{z\in\mathbb{Z}_+^d,|z-x|\geqslant c|x-y|} p_{m_0}(x,z)\leqslant \sum_{z\in\mathbb{Z}_+^d,|z-x|\geqslant c|x-y|/2} p_{m_1}(x,z) + \frac{Cx_d}{\sqrt{m_1}}\exp\left(-\frac{|x-y|^2}{C(m_0-m_1)}\right).$$

Now set $m_2=2^{k-2}$, $m_3=2^{k-3}$, $\ldots$, $m_k=1$. Iterating the previous reasoning, we obtain:

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|} p_{m_0}(x, z) &\leqslant \sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|/2} p_{m_1}(x, z) + \frac{Cx_d}{\sqrt{m_1}}\exp\left(-\frac{|x-y|^2}{C(m_0-m_1)}\right)\\
&\leqslant \sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|/3} p_{m_2}(x, z) + \frac{Cx_d}{\sqrt{m_1}}\left(\exp\left(-\frac{|x-y|^2}{C(m_0-m_1)}\right)+\sqrt{2}\exp\left(-\frac{|x-y|^2}{4C(m_1-m_2)}\right)\right)\\
&\leqslant \sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|/4} p_{m_3}(x, z) + \frac{Cx_d}{\sqrt{m_1}}\left(\exp\left(-\frac{|x-y|^2}{C(m_0-m_1)}\right)+\sqrt{2}\exp\left(-\frac{|x-y|^2}{4C(m_1-m_2)}\right)\right.\\
&\quad\left. + 2\exp\left(-\frac{|x-y|^2}{9C(m_2-m_3)}\right)\right)\\
&\leqslant \dots\\
&\leqslant \frac{Cx_d}{\sqrt{n}}\sum_{j=0}^{k-1} 2^{j/2}\exp\left(-\frac{|x-y|^2}{C(j+1)^2(m_j-m_{j+1})}\right)\\
&\quad + \sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|/\log(n)} p_1(x, z).
\end{aligned}
$$

On the other hand, by using (4.9) and the bounded increments assumption on $(S_j)_{j\in\mathbb{N}}$, we see that

$$
\sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|/\log(n)} p_1(x, z) = 0.
$$

We finally obtain

$$
\begin{aligned}
\sum_{z\in\mathbb{Z}^d_+, |z-x|\geqslant c|x-y|} p_{m_0}(x, z) &\leqslant \frac{Cx_d}{\sqrt{n}}\sum_{j=0}^{\infty} 2^{j/2}\exp\left(-\frac{2^j|x-y|^2}{Cn(j+1)^2}\right)\\
&\leqslant \frac{Cx_d}{\sqrt{n}}\exp\left(-\frac{|x-y|^2}{Cn}\right).
\end{aligned}
$$

This completes the proof of (4.14). The upper estimate (1.4) in the case $|x-y|\geqslant C\sqrt{n}$ is an immediate consequence of (4.13) and (4.14).

4.2. The upper Gaussian estimate (1.5)

Let us fix $x\in\mathbb{Z}^d_+$ and $y=(y';\, y_d)\in\mathbb{Z}^d_+$ such that

$$
y_d \leqslant C\sqrt{n}.
$$

By the boundary Harnack estimate (3.3) we have

$$\frac{p_n^*(y, x)}{p_{Cn}^*(\bar{y}, x)} \leqslant C \frac{u(y)}{u(y';\ C[\sqrt{n}])},$$

where $\bar{y} = (y',\ y_d + C[\sqrt{n}])$. This means that

$$p_n^*(y, x) \leqslant C \frac{u(y)}{u(y';\ C[\sqrt{n}])} p_{Cn}^*(\bar{y}, x).$$

Hence, by (3.1),

$$p_n^*(y, x) \leqslant C \frac{u(y)}{u(y';\ C[\sqrt{n}])} \inf_{z \in B_{c\sqrt{n}}(\bar{y})} p_{Cn}^*(z, x).$$

It follows that

$$\frac{p_n(x, y)}{m(y)} v(\bar{y}, c\sqrt{n}) \leqslant C \frac{u(y)}{u(y';\ C[\sqrt{n}])} \sum_{z \in B_{c\sqrt{n}}(\bar{y})} p_{Cn}(x, z). \tag{4.20}$$

Let us assume that

$$|x - y| \leqslant C\sqrt{n}. \tag{4.21}$$

It follows that

$$B_{\sqrt{n}}(x) \subset B_{C/n}(\bar{y}).$$

This, combined with the doubling property (1.3), gives

$$\frac{1}{v(\bar{y}, c\sqrt{n})} \leqslant \frac{C}{v(x, \sqrt{n})}.$$

Substituting this into (4.20) gives

$$p_n(x, y) \leqslant C \frac{m(y)u(y)}{v(x, \sqrt{n})u(y';\ C[\sqrt{n}])} \sum_{z \in B_{c\sqrt{n}}(\bar{y})} p_{Cn}(x, z).$$

We now use (4.18) to obtain

$$p_n(x, y) \leqslant \frac{Cx_d m(y)u(y)}{\sqrt{n}v(x, \sqrt{n})u(y';\ C[\sqrt{n}])}. \tag{4.22}$$

Combining (4.22) and (4.8) (which holds because of (4.21)) gives

$$p_n(x, y) \leqslant \frac{Cx_d m(y)u(y)}{\sqrt{n}v(x, \sqrt{n})u(y';\ C[\sqrt{n}])} \exp\left(-\frac{|x - y|^2}{Cn}\right), \qquad |x - y| \leqslant C\sqrt{n}. \tag{4.23}$$

Let us now assume that

$$|x - y| \geqslant C\sqrt{n}. \tag{4.24}$$

Observing that

$$B_{\sqrt{n}}(y) \subset B_{C\sqrt{n}}(\bar{y}),$$

we deduce

$$\frac{1}{v(\bar{y},\, c\sqrt{n})} \leqslant \frac{C}{v(y,\, C\sqrt{n})},$$

and, by using the doubling property,

$$\frac{1}{v(\bar{y},\, c\sqrt{n})} \leqslant \frac{C}{v(x,\, \sqrt{n})}\left(1 + \frac{|x-y|}{\sqrt{n}}\right)^{C}. \tag{4.25}$$

We should also observe that if $z \in B_{c\sqrt{n}}(\bar{y})$ then

$$|z - x| \geqslant c|x - y|, \tag{4.26}$$

provided that the constant $C > 0$ that appears in (4.24) is chosen large enough. Combining (4.20), (4.25) and (4.26), we deduce

$$p_n(x,\, y) \leqslant \frac{Cm(y)u(y)}{v(x,\, \sqrt{n})u(y',\, C[\sqrt{n}])}\left(1 + \frac{|x-y|}{\sqrt{n}}\right)^{C} \sum_{z \in \mathbb{Z}^d_+, |z-x| \geqslant c|x-y|} p_{Cn}(x,\, z).$$

We now use (4.14) to obtain

$$p_n(x,\, y) \leqslant \frac{Cx_d\, m(y)u(y)}{\sqrt{n}v(x,\, \sqrt{n})u(y';\, C[\sqrt{n}])}\exp\left(-\frac{|x-y|^2}{Cn}\right), \qquad |x - y| \geqslant C\sqrt{n}. \tag{4.27}$$

The upper estimate (1.5) is an immediate consequence of (4.23) and (4.27). This completes the proof of Theorem 1.

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References

Bauman, P.E. (1984) Positive solutions of elliptic equations in nondivergence form and their adjoints. *Ark. Mat.*, **22**, 153–173.

Bass, R.F. and Burdzy, K. (1994) The boundary Harnack principle for nondivergence form elliptic operators. *J. London Math. Soc. (2)*, **50**, 157–169.

Fabes, E.B., Safonov, M.V. and Yuan, Y. (1999) Behavior near the boundary of positive solutions of second order parabolic equations. II. *Trans. Amer. Mat. Soc.*, **351**, 4947–4961.

Feller, W. (1966) *An Introduction to Probability Theory and its Applications*, Vol. II. New York: Wiley.

Kozlov S.M. (1985) The method of averaging and random walks in inhomogeneous environments. *Russian Math. Surveys*, **40**, 73–145.

Kuo, H.J. and Trudinger, N.S. (1998) Evolving monotone difference operators on general space-time meshes. *Duke Math. J.*, **91**, 587–607.

Lawler, G.F. (1991) Estimates for differences and Harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proc. London Math. Soc.*, **63**, 552–568.

Mustapha, S. (2006) Gaussian estimates for spatially inhomogeneous random walks on $\mathbb{Z}^d$. *Ann. Probab.*, **34**, 264–283.

Spitzer, Y. (1976) *Principles of Random Walks* (2nd edn). New York: Springer-Verlag.

Varopoulos, N.Th. (2000) Marches aléatoires et diffusions dans les domaines lipschitziens. *C. R. Acad. Sci. Paris Sér. I Math.*, **330**, 317–320.

Varopoulos, N.Th. (2001) Marches aléatoires et diffusions dans les domaines lipschitziens. *C. R. Acad. Sci. Paris Sér. I Math.*, **332**, 357–360.

Varopoulos, N.Th. (2003a) Gaussian estimates in Lipschitz domains. *Canad. J. Math.*, **55**, 401–431.

Varopoulos, N.Th. (2003b) Marches aléatoires et théorie du potentiel dans les domaines lipschitziens. *C. R. Acad. Sci. Paris Sér. I Math.*, **337**, 615–618.

Varopoulos, N.Th. (2005) The central limit theorem in Lipschitz (an overview and the conformal invariance). *Math. Proc. Cambridge Philos. Soc.*, **139**, 161–180.

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