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CONFORMAL AND PROJECTIVE CHARACTERIZATIONS OF AN ODD DIMENSIONAL UNIT SPHERE

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Abstract

We obtain two characterizations of an odd-dimensional unit sphere of dimension > 3 by proving the following two results: (i) If a complete connected η-Einstein K-contact manifold M of dimension > 3 admits a conformal vector field V, then either M is isometric to a unit sphere, or V is an infinitesimal automorphism of M. (ii) If V was a projective vector field in (i), then the same conclusions would hold, except in the first case, M would be locally isometric to a unit sphere.

1. Introduction

It is known (Yano [17]) that an m-dimensional Riemannian manifold (M, g) admitting a maximal, i.e. an $\dfrac{(m+1)(m+2)}{2}$-parameter group of conformal transformations is conformally flat. In [9], Okumura showed that a conformally flat Sasakian manifold has constant curvature 1. This result holds more generally on a K-contact manifold, as shown by Tanno [16]. Thus the existence of a maximal conformal group places a severe restriction on Sasakian (more generally, K-contact) manifolds, and so it would be interesting to see the effect of a single 1-parameter conformal group generated by a conformal vector field on those manifolds. This was accomplished by Okumura in [10], who proved that a non-Killing conformal vector field on a Sasakian manifold M of dimension > 3 is special concircular and hence if, in addition, M is complete and connected then it is isometric to a unit sphere. The proof of the last part of this result uses Obata's theorem ([7]): A complete and connected Riemannian manifold M of dimension > 1 admits a non-trivial solution ρ of the system of partial differential equations $\nabla\nabla\rho = -c^2\rho g$ if and only if M is isometric to a sphere of radius $\dfrac{1}{c}$. Conformal vector fields on 3-dimensional Sasakian manifolds were studied by Sharma and Blair [13] and Sharma [12] using certain conditions on the scalar

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curvature. Okumura's result motivates us to examine the effect of a conformal vector field on K-contact manifolds, which seems to be a formidable task. On the other hand, we note the following result of Yano and Nagano [19]: A complete Einstein manifold admitting a complete non-homothetic conformal vector field is isometric to a round sphere. Inspired by the aforementioned results, we study a conformal vector field on a class of K-contact manifolds that are η-Einstein (a generalization of Einstein condition on K-contact manifold), and prove the following result.

THEOREM 1. *If a complete connected η-Einstein K-contact manifold M of dimension* > 3 *admits a conformal vector field V, then either* (i) *M is isometric to a unit sphere, or* (ii) *V is an infinitesimal automorphism of the contact metric structure on M.*

This result provides the following characterization of a unit odd-dimensional sphere.

COROLLARY 1. *Among all complete simply connected η-Einstein K-contact manifolds of dimension* > 3, *only the unit sphere admits a non-isometric conformal vector field.*

Remark 1. An example of an η-Einstein K-contact manifold is the Sasakian space form, i.e. a Sasakian manifold with constant φ-sectional curvature, and which is known to be η-Einstein. Theorem 1 generalizes and improves the corresponding result of Okumura [10]. In this context, we also point out the following generalization of Okumura's result by Mizusawa [5]: A conformal vector field on a $(2n+1)$-dimensional $(n > 1)$ Sasakian manifold of constant scalar curvature $\neq 2n(2n+1)$ is Killing.

Next, we recall [17] that an m-dimensional Riemannian manifold (M, g) admitting a maximal, i.e. $m(m+2)$-parameter group of projective transformations is projectively flat, and hence has constant curvature. A Sasakian (more generally, K-contact) manifold of constant curvature has constant curvature 1. In [10], Okumura proved that a projective vector field on a non-Einstein η-Einstein Sasakian manifold is Killing and an infinitesimal strict contact transformation. Generalizing and improving this result we prove the following result.

THEOREM 2. *Let V be a projective vector field on a connected η-Einstein K-contact manifold M of dimension* > 3. *Then, either* (i) *M is Einstein, or* (ii) *V is an infinitesimal automorphism of the contact metric structure on M. In case* (i), *if M is complete then it has constant curvature* 1, *and if it is also simply connected then it is isometric to a unit sphere.*

This result provides the following characterization of a unit odd-dimensional sphere.

Corollary 2. *Among all complete simply connected η-Einstein K-contact manifolds of dimension > 3, only the unit sphere admits a non-isometric projective vector field.*

Remark 2. For dimension 3, a K-contact manifold is Sasakian for which we have the following result of Ghosh and Sharma [3]: If V is a projective vector field on a 3-dimensional Sasakian manifold M, then either V is Killing or M has constant curvature 1.

2. A Brief review of contact geometry

A $(2n+1)$-dimensional smooth manifold is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \eta(X) = g(X, \xi), \quad \varphi^2 = -I + \eta \otimes \xi. \tag{1}$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. A contact metric structure is said to be K-contact if ξ is Killing with respect to g. The contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler. Sasakian manifolds are K-contact and K-contact 3-manifolds are Sasakian. We have the following formulas for a K-contact manifold.

$$\nabla_X \xi = -\varphi X \tag{2}$$

$$(\nabla_X \varphi) Y = R(\xi, X) Y \tag{3}$$

$$Ric(X, \xi) = 2n\eta(X) \tag{4}$$

$$R(X, \xi)\xi = X - \eta(X)\xi \tag{5}$$

where ∇, R, and Ric denote respectively, the Riemannian connection, curvature tensor, and Ricci tensor of g. For details we refer to the standard monograph of Blair [1].

A vector field V on a contact metric manifold M is said to be an infinitesimal contact transformation if $\pounds_V \eta = \sigma\eta$ for some smooth function σ on M. In particular, for $\sigma = 0$, V is called an infinitesimal strict contact transformation. V is said to be an infinitesimal automorphism of the contact metric structure on M if it leaves all the structure tensors η, ξ, g, φ invariant (see Tanno [15]).

A contact metric manifold M is said to be η-Einstein in the wider sense, if the Ricci tensor can be written as

$$Ric(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{6}$$

for some smooth functions a and b on M. It is well-known (Yano and Kon [18]) that a and b are constant if M is K-contact, and has dimension greater than 3. For $n > 1$, it follows through a straightforward computation that

$$a = \frac{r}{2n} - 1, \quad b = 2n + 1 - \frac{r}{2n} \tag{7}$$

Boyer, Galicki and Matzeu [2] have proved the existence of η-Einstein Sasakian structures on many different compact manifolds, and have shown the relation between a K-contact η-Einstein manifold and an Einstein-Weyl structure. Motivated by equations (6) and (7), Hasegawa and Nakane [4] studied η-Einstein tensor S defined on a Sasakian (more generally, K-contact) manifold by

$$S = Ric - \left(\frac{r}{2n} - 1\right)g - \left(2n + 1 - \frac{r}{2n}\right)\eta \otimes \eta$$

where r is the scalar curvature (not necessarily constant). Thus a K-contact manifold is η-Einstein if and only if $S = 0$. A straightforward computation shows

$$|S|^2 = |Ric^0|^2 - \frac{1}{2n(2n+1)}[r - 2n(2n+1)]^2$$

where Ric^0 denotes the trace-free Ricci tensor $Ric - \frac{r}{2n+1}g$. The above equation implies $|S| \leq |Ric^0|$, equality holds if and only if $r = 2n(2n+1)$.

3. Conformal and projective vector fields

A vector field V on an m-dimensional Riemannian manifold (M, g) is said to be a conformal vector field if it satisfies

$$L_V g = 2\rho g \tag{8}$$

for a smooth function ρ on M, and where L_V denotes the Lie derivative operator along V. In particular, the conformal vector field V is homothetic (respectively, Killing) when ρ is constant (respectively, zero). Let us denote the gradient vector field of ρ by $D\rho$, and the Laplacian of ρ by $\Delta\rho = Div.D\rho = g^{ij}\nabla_i\nabla_j\rho$. Then we have the following integrability conditions for the conformal vector field V (Yano [17]):

$$(L_V\nabla)(X, Y) = (X\rho)Y + (Y\rho)X - g(X, Y)D\rho, \tag{9}$$

$$\begin{aligned}(L_V R)(X, Y, Z) = {} & g(\nabla_X D\rho, Z)Y - g(\nabla_Y D\rho, Z)X \\ & + g(X, Z)\nabla_Y D\rho - g(Y, Z)\nabla_X D\rho,\end{aligned} \tag{10}$$

$$(L_V Ric)(X, Y) = -(m-2)g(\nabla_X D\rho, Y) - (\Delta\rho)g(X, Y), \tag{11}$$

$$L_V r = -2(m-1)\Delta\rho - 2r\rho \tag{12}$$

where r denotes the scalar curvature of g.

A vector field V on an n-dimensional Riemannian manifold (M,g) is said to be a projective vector field if there is a 1-form π on M such that

$$(L_V\nabla)(X,Y) = \pi(X)Y + \pi(Y)X. \tag{13}$$

For $\pi = 0$, V becomes an affine Killing vector field. In a local coordinate system, equation (13) becomes

$$L_V\Gamma^k_{ij} = \pi_i\delta^k_j + \pi_j\delta^k_i.$$

Its contraction at k and j shows that $\pi = df$ for a smooth function f on M. The projective vector field V satisfies the following integrability conditions:

$$(L_V R)(X,Y,Z) = g(\nabla_X Df, Z)Y - g(\nabla_Y Df, Z)X, \tag{14}$$

$$(L_V Ric)(X,Y) = -(m-1)g(\nabla_X Df, Y). \tag{15}$$

4. Proofs of the results

We need the following lemmas.

Lemma 1 ([10], [14]). *A conformal vector field V on a contact metric manifold, with associated conformal function ρ, has the following properties*: (i) $(L_V\eta)(\xi) = \rho$ *and* (ii) $\eta(L_V\xi) = -\rho$.

Lemma 2 ([14]). *Let V be a conformal vector field on a contact metric manifold M. If V is an infinitesimal contact transformation, then it is an infinitesimal automorphism of M.*

Lemma 3 ([12]). *A homothetic vector field on a K-contact manifold is Killing.*

Lemma 4 ([11]). *A second order symmetric parallel tensor on a K-contact manifold (M,g) is a constant multiple of g.*

Proof of Theorem 1. Here $m = 2n+1$. Taking the Lie derivative of (6) with a and b given by (7), along the conformal vector field V, and using (11) gives the following equation

$$\begin{aligned}(1-2n)g(\nabla_X D\rho, Y) - (\Delta\rho)g(X,Y) &= 2\left(\frac{r}{2n}-1\right)\rho g(X,Y)\\ &\quad + \left(2n+1-\frac{r}{2n}\right)[(L_V\eta)(X)\eta(Y) + \eta(X)(L_V\eta)(Y)].\end{aligned} \tag{16}$$

Now Lie differentiating the second equation in (1) along V we have

$$(L_V\eta)X = 2\rho\eta(X) + g(L_V\xi, X). \tag{17}$$

As r is constant, we have from (12) that

$$\Delta\rho = -\frac{r}{2n}\rho. \tag{18}$$

The use of equations (17) and (18) in (16) provides

$$\nabla_X D\rho = \alpha\rho X + \beta[g(U,X)\xi + 4\rho\eta(X)\xi + \eta(X)U]. \tag{19}$$

where we have set $U = L_V\xi$, and

$$\alpha = \frac{r-4n}{2n(1-2n)}, \quad \beta = \frac{2n(2n+1)-r}{2n(1-2n)}. \tag{20}$$

Now, using (19) we compute $R(X,Y)D\rho$ and find that

$$\begin{aligned} R(X,Y)D\rho = {} & \alpha((X\rho)Y - (Y\rho)X) + \beta[(g(\nabla_X U, Y) - g(\nabla_Y U, X))\xi \\ & + g(U,X)\varphi Y - g(U,Y)\varphi X + 4((X\rho)\eta(Y) - (Y\rho)\eta(X))\xi \\ & + 8\rho g(X,\varphi Y)\xi + 4\rho(\eta(X)\varphi Y - \eta(Y)\varphi X) \\ & + 2g(X,\varphi Y)U + \eta(Y)\nabla_X U - \eta(X)\nabla_Y U]. \end{aligned} \tag{21}$$

Substituting ξ for Y in (21), taking inner product with ξ, and using (5) yields

$$\begin{aligned} 0 = {} & (\alpha+1)(X\rho - (\xi\rho)\eta(X)) + \beta[2g(\nabla_X U,\xi) \\ & - g(\nabla_\xi U, X) + 4X\rho - 4(\xi\rho)\eta(X) - \eta(X)g(\nabla_\xi U,\xi)]. \end{aligned} \tag{22}$$

Recalling the commutation formula [17]:

$$L_V\nabla_X Y - \nabla_X L_V Y - \nabla_{[V,X]}Y = (L_V\nabla)(X,Y),$$

substituting $X = Y = \xi$ and using (9) we find

$$\nabla_\xi U = \varphi U - 2(\xi\rho)\xi + D\rho. \tag{23}$$

At this point, we notice from Lemma 1, that $g(U,\xi) = g(L_V\xi,\xi) = -\rho$. Differentiating it and using (2) gives $g(\nabla_X U,\xi) = g(U,\varphi X) - X\rho$. Using this in conjunction with (23) we obtain

$$(\alpha+\beta+1)(D\rho - (\xi\rho)\xi) - 3\beta(\varphi U) = 0. \tag{24}$$

But we see from (20) that $\alpha+\beta+1 = 0$. Hence equation (24) implies that, either (i) $\beta = 0$, or (ii) $U = \eta(U)\xi$. In case (i) (M,g) is Einstein, i.e. $Ric = 2ng$, and (19) reduces to $\nabla\nabla\rho = -\rho g$. So, if (M,g) is complete and connected, then by Obata's theorem mentioned in Section 1, (M,g) is isometric to a unit sphere.

For case (ii), applying Lemma 1 provides $L_V\xi = -\rho\xi$. Lie differentiating $\eta(X) = g(\xi,X)$ along V and using the conformal Killing equation (8) we obtain $L_V\eta = \rho\eta$. Thus V is an infinitesimal contact transformation. As V is also conformal, by Lemma 2 we conclude that V is an infinitesimal automorphism of the contact metric structure, completing the proof.

Remark. In case (i) of Theorem 1, as shown in the proof, we see that $L_{V+D\rho}g = L_V g + 2\nabla\nabla\rho = 2\rho g - 2\rho g = 0$. So $V = -D\rho + K$, where K is a Killing vector field.

Proof of Theorem 2. Let us denote the strain tensor $L_V g$ along the projective vector field V by h and the corresponding $(1,1)$-tensor field by H such that $g(HX, Y) = h(X, Y)$. Using the commutation formula [17]:

$$(L_V\nabla_X g - \nabla_X L_V g - \nabla_{[V,X]}g)(Y,Z) = -g((L_V\nabla)(X,Y),Z) - g((L_V\nabla)(X,Z),Y)$$

and equation (13) with $\pi = df$ shows

$$(\nabla_X H)Y = 2(Xf)Y + (Yf)X + g(X,Y)(Df). \tag{25}$$

Now, Lie differentiating the η-Einstein condition (6) along the projective vector field V and using the integrability condition (15) with $m = 2n+1$ we get

$$-2ng(\nabla_X Df, Y) = a(L_V g)(X,Y) + b[(L_V\eta)(X)\eta(Y) + \eta(X)(L_V\eta)Y]. \tag{26}$$

Substituting ξ for Y in the above equation and noting $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$ (which follows from the Poincare lemma: $d^2 = 0$) we get

$$-2ng(\nabla_\xi Df, X) = ah(X,\xi) + b[(L_V\eta)X + (L_V\eta)(\xi)\eta(X)]. \tag{27}$$

Next, Lie differentiating formula (5) along V and using integrability condition (14) we have

$$\begin{aligned} g(\nabla_X Df,\xi)\xi - g(\nabla_\xi Df,\xi)X + R(X, L_V\xi)\xi + R(X,\xi)L_V\xi \\ = -(L_V\eta)(X)\xi - \eta(X)L_V\xi. \end{aligned} \tag{28}$$

Taking its inner product with ξ and using formula (5) we get

$$\begin{aligned} g(\nabla_\xi Df, X) - g(\nabla_\xi Df,\xi)\eta(X) - g(L_V\xi, X) \\ = -(L_V\eta)(X) + 2(L_V\eta)(\xi)\eta(X). \end{aligned} \tag{29}$$

Let (e_i) $(i = 1,\dots,2n+1)$ be a local orthonormal frame on M. Substituting e_i for X in (28), taking inner product with e_i and summing over i we find

$$g(\nabla_\xi Df,\xi) = 2g(L_V\xi,\xi).$$

The Lie derivative of $g(\xi,\xi) = 1$ along V yields $h(\xi,\xi) + 2g(L_V\xi,\xi) = 0$. Consequently,

$$g(\nabla_\xi Df,\xi) = -h(\xi,\xi). \tag{30}$$

Now, the Lie derivative of the second equation in (1) along V provides

$$(L_V\eta)X = g(H\xi + L_V\xi, X). \tag{31}$$

The use of (31) in equation (27) and the relation $a + b = 2n$ (which follows from (7)) shows

$$\nabla_\xi Df = -H\xi - \frac{b}{2n}(L_V\xi + (L_V\eta)(\xi)\xi). \tag{32}$$

Further, using (31) and (30) in (29) we find

$$\nabla_\xi Df = -H\xi + (h(\xi,\xi) + 2\eta(L_V\xi))\xi. \tag{33}$$

Comparing the above two equations we obtain

$$\frac{b}{2n}[L_V\xi - \eta(L_V\xi)\xi] + [h(\xi,\xi) + 2\eta(L_V\xi)]\xi = 0. \tag{34}$$

We consider the cases (i) $b = 0$ and (ii) $b \neq 0$. For case (i) $a = 2n$, and (M, g) is Einstein with Einstein constant $2n$. So, if (M, g) is complete, then it is compact, and by a result of Nagano [6], is of constant curvature 1. Equations (25) and (26) imply

$$(\nabla\nabla\nabla f)(X, Y, Z) + 2(Xf)g(Y, Z) + (Yf)g(X, Z) + (Zf)g(X, Y) = 0. \tag{35}$$

Hence, if (M, g) is complete and simply connected, then by a result of Obata [8], it is isometric to a unit sphere.

For case (ii), equation (34) gives

$$L_V\xi = \left[-\frac{2n}{b}h(\xi,\xi) + \left(1 - \frac{4n}{b}\right)\eta(L_V\xi)\right]\xi.$$

Its inner product with ξ gives $\eta(L_V\xi) = -\dfrac{h(\xi,\xi)}{2}$, and therefore

$$L_V\xi = -\frac{h(\xi,\xi)}{2}\xi. \tag{36}$$

The use of (31) in conjunction with (36) transforms the equation (26) into

$$-2n\nabla_X Df = aHX + b[g(H\xi, X)\xi + \eta(X)H\xi - h(\xi,\xi)\eta(X)\xi]. \tag{37}$$

We compute $-2nR(Y, X)Df$ through equations (37) and (25), and subsequently contract it with respect to Y, in order to obtain

$$\begin{aligned}-2nRic(X, Df) = -2naXf + b[&(2n+3)(\xi f)\eta(X) - 3Xf \\ &+ 5g(H\xi, \varphi X) + X(h(\xi,\xi)) - \xi(h(\xi,\xi))\eta(X)].\end{aligned}$$

As $b \neq 0$, equation (6) reduces the preceding equation to

$$\begin{aligned}&(4n+3)(\xi f)\eta(X) - 3Xf + 5g(H\xi, \varphi X) \\ &\quad + X(h(\xi,\xi)) - \xi(h(\xi,\xi))\eta(X) = 0.\end{aligned} \tag{38}$$

Substituting ξ for X in (38) immediately yields $\xi f = 0$. Differentiating it along ξ and noting the property $\nabla_\xi\xi = 0$ (a consequence of (2)) we have $g(\nabla_\xi Df, \xi) = 0$. It follows immediately from (30) that $h(\xi,\xi) = 0$. Thus (38) reduces to

$$3Xf = 5g(H\xi, \varphi X). \tag{39}$$

At this stage, differentiating $h(\xi,\xi) = 0$ and using (25) we find $Xf = g(H\xi, \varphi X)$. Using it back in (39) shows $Xf = 0$, i.e. f is constant. Thus V becomes affine

Killing, and so $L_V g$ is parallel. By Lemma 4, V is homothetic, and hence, by Lemma 3, becomes Killing. Further, we note from (36) that $L_V\xi = 0$. As f is constant, equation (39) reduces to $H\xi = 0$ and so (31) provides $L_V\eta = 0$. Finally, using this in the Lie derivative of the first equation in (1) and noting that Lie and exterior derivations commute, we obtain $L_V\varphi = 0$. Thus V is an infinitesimal automorphism of the contact metric structure, completing the proof.

5. Concluding remarks

(1) In case (i) of Theorem 2, equation (26) reduces to $L_V g + \nabla\nabla f = 0$, i.e. $L_{V+(1/2)Df}\, g = 0$. So, $V = -\frac{1}{2}Df$ up to the addition of a Killing vector field.

(2) In cases (ii) of Theorems 1 and 2, V is an infinitesimal strict contact transformation and hence by a result of Blair ([1], p. 72), can be expressed as $V = -\frac{1}{2}\varphi D\psi + \psi\xi$, where ψ is a smooth function on M such that $\xi\psi = 0$. So, it is an open question whether there would be any further restriction on V due to the full condition of its being infinitesimal automorphism.

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