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THE EFFECT OF FENCHEL-NIELSEN COORDINATES UNDER ELEMENTARY MOVES

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Abstract

We describe the effect of Fenchel-Nielsen coordinates under elementary move for hyperbolic surfaces with geodesic boundaries, punctures and cone points, which generalize Okai's result for surfaces with geodesic boundaries. The proof relies on the parametrization of the Teichmüller space of surface of type $(1,1)$ or $(0,4)$ as a sub-locus of an algebraic equation in $\mathbf{R}^3$. As an application, we show that the hyperbolic length functions of closed curves are asymptotically piecewise linear functions with respect to the Fenchel Nielsen coordinates in the Teichmüller spaces of surfaces with cone points.

1. Introduction

Let S be an oriented surface of genus $g \geq 2$ and $\mathscr{T}(S)$ be the Teichmüller space of hyperbolic structures on S.

A *pants decomposition* of S is a maximal set of mutually disjoint simple closed curves which decompose the surface into pairs of pants. Let $\mathscr{P} = \{\alpha_i\}_{i=1}^{3g-3}$ be a pants decomposition of S. Associated with $\mathscr{P}$ is a homeomorphism $\Psi_{\mathscr{P}}$ [1]:

$$\begin{aligned} \Psi_{\mathscr{P}} : \mathscr{T}(S) &\to (\mathbf{R}_+ \times \mathbf{R})^{3g-3} \\ X &\mapsto (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)), \end{aligned}$$

where ℓ_{α_i} is the hyperbolic length function of α_i and τ_{α_i} is the *twist coordinate* of α_i. The parametrization of $\mathscr{T}(S)$ under $\Psi_{\mathscr{P}}$ is called the *Fenchel-Nielsen coordinates* of $\mathscr{T}(S)$ (associated with $\mathscr{P}$).

By work of Wolpert [10], the Kähler form of Weil-Petersson metric on $\mathscr{T}(S)$ has a simple expression in terms of the Fenchel-Nielsen coordinates:

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

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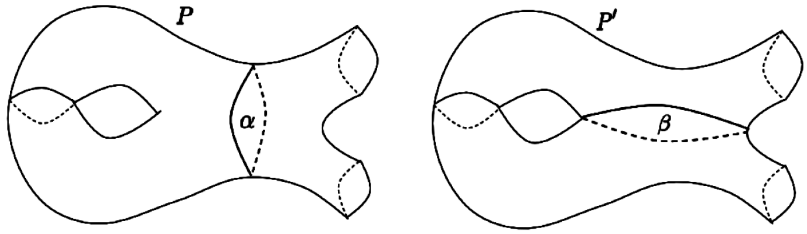

FIGURE 1. α and β intersect *minimally*: $i(\alpha, \beta) = 2$ if α separates the surface and $i(\alpha, \beta) = 1$ if α does not separate the surface.

Let $\mathscr{P}$ and $\mathscr{P}'$ be two distinct pants decompositions of S. An interesting question is to understand the transition relation:

$$\Psi_{\mathscr{P}'} \circ \Psi_{\mathscr{P}}^{-1}.$$

Recall that two pants decompositions $\mathscr{P}$ and $\mathscr{P}'$ are differed by an *elementary move* if there exists $\alpha \in \mathscr{P}$, $\beta \in \mathscr{P}'$ such that

(1) $\mathscr{P}' = (\mathscr{P} \backslash \{\alpha\}) \cup \{\beta\}$;

(2) α and β intersect *minimally* (see Figure 1).

THEOREM 1.1 (Hatcher-Thurston [2]). *There exists a finite sequence of pants decompositions* $\{\mathscr{P}_0, \dots, \mathscr{P}_n\}$ *such that* $\mathscr{P}_0 = \mathscr{P}$, $\mathscr{P}_n = \mathscr{P}'$, *and* $\mathscr{P}_i$ *and* $\mathscr{P}_{i+1}$, $0 \leq i \leq n-1$ *are differed by an elementary move.*

By Theorem 1, we have

$$\Psi_{\mathscr{P}'} \circ \Psi_{\mathscr{P}}^{-1} = (\Psi_{\mathscr{P}_n} \circ \Psi_{\mathscr{P}_{n-1}}^{-1}) \circ \cdots \circ (\Psi_{\mathscr{P}_1} \circ \Psi_{\mathscr{P}_0}^{-1}).$$

As a result, the question reduces to:

Understand the transition relation $\Psi_{\mathscr{P}'} \circ \Psi_{\mathscr{P}}^{-1}$ *in the case that* $\mathscr{P}$ *and* $\mathscr{P}'$ *are differed by an elementary move.*

Assuming that $\mathscr{P}' = (\mathscr{P} \backslash \{\alpha\}) \cup \{\beta\}$ as above. Note that $\Psi_{\mathscr{P}}$ and $\Psi_{\mathscr{P}'}$ only differ on the coordinates $(\ell_\alpha, \tau_\alpha)$ and (ℓ_β, τ_β). As a result, one may assume that S is a hyperbolic surface with geodesic boundary of type $(1,1)$ or $(0,4)$. Then the above question is studied by Okai [5] where the equation for $\Psi_{\mathscr{P}'} \circ \Psi_{\mathscr{P}}^{-1}$ is given explicitly.

In this paper, we shall first introduce the language of [4], where the Teichmüller space of surface of type $(1,1)$ or $(0,4)$ is parameterized in $\mathbf{R}^3$ as a sub-locus of a equation, and the length and twist coordinates are expressed by the above parameters. Then by a detailed analysis, we describe the effect of Fenchel-Nielsen coordinates under elementary move for hyperbolic surfaces with geodesic boundaries, punctures and cone points. The precise result which is the main result of this paper will be stated in §3. The result generalize Okai's result [5] for surfaces with geodesic boundaries. And also this give a new proof of

Okai's result. Okai's result is useful to understand the locally structure of quasiconformal Teichmüller space [6]. An interesting question is to extend the result to complex Fenchel-Nilesen coordinates [7].

As an application of the main result, we show that the hyperbolic length functions of closed curves are asymptotically piecewise linear functions with respect to the Fenchel Nielsen coordinates in the Teichmüller spaces of surfaces with cones. This result is a generalization of Mirzakhani's result [3] for closed surfaces.

2. Preliminaries

2.1. Definitions and notations. Let $S = S_{g,n}$ be a oriented surface of genus g with n boundary components. A *hyperbolic structures* on S is a metric of constant curvature -1 such that each boundary component of S is a totally geodesic closed curve, a puncture or a cone-point.

In polar co-ordinates around a cone-point, the metric has the form $dr^2 + \sinh^2 r\, d\eta^2$ where r is the distance from the cone-point, η is the angular measure around the cone-point, which is measured modulo θ for some $\theta \in (0, 2\pi)$. There is some $r_0 > 0$ such that the quotient

$$\{(r,\eta) \mid 0 < r < r_0, 0 \le \eta \le \theta\}/(r,0) \sim (r,\theta)$$

is isometric to a neighborhood of the cone-point. This θ is called the *angle* around the cone-point.

Endow S with a hyperbolic structure R. We shall assign a number to a cone-point, a puncture or a geodesic boundary component of R in the following way: for a cone-point, we let the number be the angle; for a puncture, we let the number be 0; for a geodesic boundary component of length L, we let the number be iL. Let $\theta_1, \dots, \theta_n$ be the numbers assigned to the cone-points, punctures and geodesic boundary components. We call the tuple $(g; \theta_1, \dots, \theta_n)$ the *signature* of R.

The Teichmüller space $\mathscr{T}(g; \theta_1, \dots, \theta_n)$ parameterizes hyperbolic structures of signature $(g; \theta_1, \dots, \theta_n)$ on S up to isotopy. Points in $\mathscr{T}(g; \theta_1, \dots, \theta_n)$ are pairs (f, X), where X is a hyperbolic structures on S of signature $(g; \theta_1, \dots, \theta_n)$ equipped with a homeomorphism $f : S \to X$, up to the equivalence $(f, X) \sim (g, Y)$ if there is an isometry $\phi : X \to Y$ such that $\phi \circ f \simeq g$.

A simple closed curve on S is *essential* if it is not homotopic to a boundary component of S or to a point in the interior of S. Let γ be an essential simple closed curve on S. Given any hyperbolic structure X on S, there is a unique simple closed geodesic γ^X isotopic to γ. Denote by $\ell_\gamma(X)$ the length of γ^X in X. $\ell_\gamma(X)$ is called the *hyperbolic length* of γ in X. The definition only depends on the isotopy class of γ and the isotopy class of X in Teichmüller space.

A *pants decomposition* of S is a maximal set of mutually disjoint essential simple closed curves which decompose S into pairs of pants. Let $\mathscr{P} = \{\alpha_i\}_{i=1}^{3g-3+n}$ be a pants decomposition of S. Endow S with a hyperbolic structure X of signature $(g; \theta_1, \dots, \theta_n)$. Without loss of generality, we shall assume that α_i is its geodesic representation in X. Along each α_i, there is a *twist parameter* $\tau_{\alpha_i}(X)$ measures the relative twist amount between the two generalized hyperbolic pair of pants (which might be the same) having α_i in common.

More precisely, we follow the convention in [9]. In this description, we fix a small tubular neighborhood N_i for every geodesic α_i and an orientation of α_i, and we also fix two points x_i, y_i on α_i. For every hyperbolic pairs of pants $P \subset X \setminus \bigcup_{i=1}^{3g-3} \alpha_i$, we consider three disjoint arcs that join the boundary components, with endpoints on the chosen points. Every pair of distinct boundary components of P are joined by a unique geodesic (called a seam) that is perpendicular to the boundary components. By performing an isotopy, we can deform the chosen arcs such that they coincide with the corresponding seams outside the union of the neighborhoods N_i, and such that in every neighborhood N_i they just spin around the cylinder (see [9], Figure 4.19). Using the orientation of α_i, we can then compute the amount of spinning of each of these arcs, as in Figure 4.20 of [9]. For every curve α_i, the twist parameter $\tau_{\alpha_i}(X)$ is then defined as the difference between the amount of spinning of two of the chosen arcs from the two sides of α_i (we need to use the orientation of α_i to choose the order of subtraction).

A pants decomposition $\mathscr{P} = \{\alpha_i\}_{i=1}^{3g-3+n}$ determines a *Fenchel-Nielsen coordinates*

$$\begin{aligned} \mathscr{T}(g; \theta_1, \dots, \theta_n) &\to (\mathbf{R}_+ \times \mathbf{R})^{3g-3+n} \\ X &\mapsto (\ell_{\alpha_i}(X), \tau_{\alpha_i}(X)). \end{aligned}$$

2.2. Parameterizing the space $\mathscr{T}(2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$. Next we state the parameterization of the space $\mathscr{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$. For reference we refer to [4].

We assume that $S = S_{0,4}$ is a sphere with four holes. Let $\theta_1, \theta_2, \theta_3, \theta_4 \in [0, \pi/2] \cup i\mathbf{R}_+$ and denote by

$$\mathscr{T}(2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4) = \mathscr{T}(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$$

the Teichmüller space of hyperbolic structures of signature $(0; 2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ on S. We only treat the case where θ_1, θ_2, θ_3, θ_4 are numbers in $[0, \pi/2]$. Other cases where some $\theta_i \in i\mathbf{R}_+$ follow the same way by slight modifications.

Let α be an essential simple closed curve on S. Note that $\{\alpha\}$ is a pants decomposition of S. Denote by $(\ell_\alpha, \tau_\alpha)$ the Fenchel-Nielsen coordinates associated to α.

Let $X \in \mathscr{T}(2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ be a hyperbolic structure with $\tau_X(\alpha) = 0$. Denote by $\ell = \ell_\alpha(X)$. Consider a fundamental domain of X in $\mathbf{H}^2$ (see Figure 2). We may assume that $i\mathbf{R}_+$ is a lift of α. Let L_0 and L_1 be the geodesics obtained by the circles $|z| = 1$ and $|z| = e^{\ell/2}$, respectively. Choose a point $P = -u + iv \in L_0$, $0 \le u \le 1$, $v = \sqrt{1 - u^2}$. Let L_2 be the geodesic that intersects with L_0 at P and also intersects with L_1 at some point.

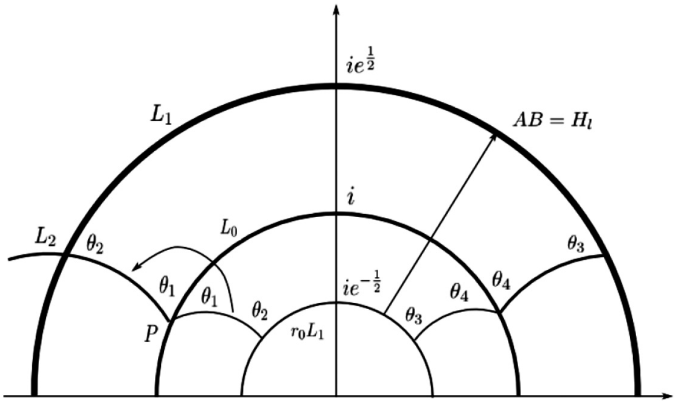

FIGURE 2. Fundamental domain.

We claim that L_2 does not meet L_1 in the right half plane. Note that if we denote the angle from L_0 to L_2 by θ_1 and the angle from L_2 to L_1 by θ_2, then this claim is equivalent to the following inequality:

$$\cos\theta_1 + \cosh\frac{\ell}{2}\cos\theta_2 \geq 0. \tag{1}$$

As we can not find a reference for this result, we give a proof of it.

Let us consider the case that L_2 meets $i\mathbf{R}_+$ at the point $ie^{\ell/2}$. There is a geodesic triangle surrounded by L_0, L_2 and $i\mathbf{R}_+$. The angles of the triangle are θ_1, $\pi/2$ and $\theta_2 - \pi/2$. Moreover, the length of the opposite side of θ_1 is equal to $\ell/2$. By formula of hyperbolic triangle (ref. [1]), we have

$$\cosh\ell/2 = \frac{\cos\theta_1}{\sin(\theta_2 - \pi/2)} = \frac{\cos\theta_1}{-\cos\theta_2}.$$

Hence

$$\cos\theta_1 + \cosh\frac{\ell}{2}\cos\theta_2 = 0.$$

Note that $\cos\theta_1 + \cosh\dfrac{\ell}{2}\cos\theta_2$ is a strictly decreasing function of θ_1 and θ_2. θ_2 is increasing as L_2 move from the left to the right. As a result, if L_2 meets L_1 in the right half plane, then

$$\cos\theta_1 + \cosh\frac{\ell}{2}\cos\theta_2 < 0.$$

If L_2 meets L_1 in the left half plane, then

$$\cos\theta_1 + \cosh\frac{\ell}{2}\cos\theta_2 > 0.$$

By our assumption, $\theta_1, \theta_2, \theta_3, \theta_4 \in [0, \pi/2]$. So (1) is always satisfied.

Note that each geodesic line in the upper half plane gives rise a reflection, which is an isometry of the Poincaré metric. We denote by r_0, r_1, r_2 the reflections correspond to L_0, L_1, L_2, respectively.

Let $A = r_2 r_0$ and $H_\ell = r_1 r_0$. Then A is the Möbius transformation corresponds the rotation at P with rotation angle $2\theta_1$, and H_ℓ corresponds to the hyperbolic transformation $z \to e^\ell z$. Let us write down the matrix representation of A and H_ℓ in $\mathrm{SL}_2(\mathbf{R})$ (recall that $P = -u + iv$):

$$A = \begin{pmatrix} \cos\theta_1 - uv^{-1}\sin\theta_1 & v^{-1}\sin\theta_1 \\ -v^{-1}\sin\theta_1 & \cos\theta_1 + uv^{-1}\sin\theta_1 \end{pmatrix}, \tag{2}$$

$$H_l = \begin{pmatrix} -e^{\ell/2} & 0 \\ 0 & -e^{\ell/2} \end{pmatrix}. \tag{3}$$

Set $B = A^{-1} H_l$, the matrix for B is

$$B = \begin{pmatrix} -e^{\ell/2}\cos\theta_1 + uv^{-1}\sin\theta_1 & e^{-\ell/2}v^{-1}\sin\theta_1 \\ -e^{\ell/2}v^{-1}\sin\theta_1 & -e^{-\ell/2}\cos\theta_1 - uv^{-1}\sin\theta_1 \end{pmatrix}. \tag{4}$$

where

$$u = \frac{\left|\cos\theta_2 + \cosh\frac{l}{2}\cos\theta_1\right|}{\left(\cosh^2\frac{l}{2} + 2\cosh\frac{l}{2}\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2 - 1\right)^{1/2}}, \tag{5}$$

$$v = \frac{\sinh\frac{l}{2}\sin\theta_1}{\left(\cosh^2\frac{l}{2} + 2\cosh\frac{l}{2}\cos\theta_1\cos\theta_2 + \cos^2\theta_1 + \cos^2\theta_2 - 1\right)^{1/2}}. \tag{6}$$

By (1), thus the numerator of the first expression in (5) does not need absolute-values.

Similarly, by replacing (θ_1, θ_2) by (θ_4, θ_3), we may define $\tilde{A}$, $\tilde{B}$ in the same way as we defined A, B through equations (2)–(4). These $\tilde{A}$, $\tilde{B}$ correspond to (negative) rotations at the cone-points with angles $2\theta_4$ and $2\theta_3$, respectively.

Let $X_s \in \mathscr{T}(2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ be the hyperbolic surface obtained from X by the s-twist along α, that is, $\ell_\alpha(X_s) = \ell$, $\tau_\alpha(X_s) = s$. Note that under the s-twist deformation, L_0 (and L_1) becomes a piece-wise geodesic arc. See Figure 3.

Let $E(z) = -1/z$, $H_s(z) = e^s z$. We denote D and C by the conjugation of $\tilde{A}^{-1}$ and $\tilde{B}^{-1}$ by $H_s r_0 E$, respectively. Then we have

$$C = \begin{pmatrix} -e^{-l/2}(\cos\theta_4 - \xi\eta^{-1}\sin\theta_4) & e^s e^{-l/2}\eta^{-1}\sin\theta_4 \\ -e^s e^{l/2}\eta^{-1}\sin\theta_4 & -e^{l/2}(\cos\theta_4 + \xi\eta^{-1}\sin\theta_4) \end{pmatrix},$$

$$D = \begin{pmatrix} \cos\theta_4 + \xi\eta^{-1}\sin\theta_4 & e^s\eta^{-1}\sin\theta_4 \\ -e^{-s}\eta^{-1}\sin\theta_4 & \cos\theta_4 - \xi\eta^{-1}\sin\theta_4 \end{pmatrix},$$

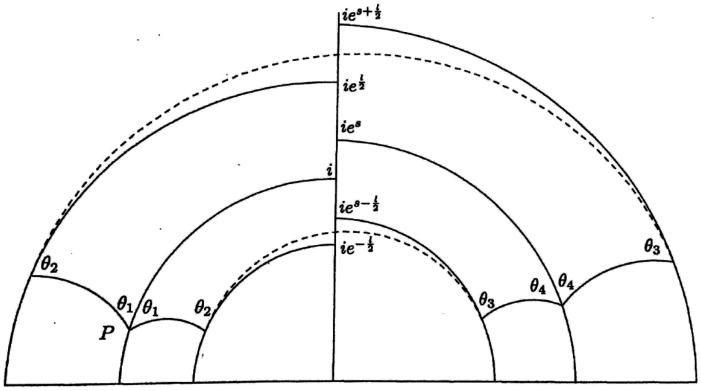

FIGURE 3. Twist.

where

$$\xi = \frac{\left|\cos\theta_3 + \cosh\frac{\ell}{2}\cos\theta_4\right|}{\left(\cosh^2\frac{\ell}{2} + 2\cosh\frac{\ell}{2}\cos\theta_3\cos\theta_4 + \cos^2\theta_3 + \cos^2\theta_4 - 1\right)^{1/2}},$$

$$\eta = \frac{\sinh\frac{\ell}{2}\sin\theta_4}{\left(\cosh^2\frac{\ell}{2} + 2\cosh\frac{\ell}{2}\cos\theta_3\cos\theta_4 + \cos^2\theta_3 + \cos^2\theta_4 - 1\right)^{1/2}}.$$

Note that D and C correspond to the rotations at the cone-points with angles $2\theta_4$ and $2\theta_3$, respectively. In particular, when $s = 0$, $D = \tilde{A}^{-1}$, $C = \tilde{B}^{-1}$.

In conclusion, A, B, C, D correspond to the rotations around the cone-points and each of which can be considered as a matrix-function of the Fenchel-Nielsen coordinates $(\ell_\alpha, \tau_\alpha)$.

Denote $x = -\frac{1}{2}\,tr\,BC$, $y = -\frac{1}{2}\,tr\,CA$, $z = -\frac{1}{2}\,tr\,AB$. Let $a = \cos\theta_1\cos\theta_4 + \cos\theta_2\cos\theta_3$, $b = \cos\theta_2\cos\theta_4 + \cos\theta_1\cos\theta_3$, $c = \cos\theta_3\cos\theta_4 + \cos\theta_1\cos\theta_2$ and $d = 4\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 - 1$.

PROPOSITION 2.1 ([4]). *With the above notations, X_s satisfies the following algebraic equation in $\mathbf{R}^3$:*

$$x^2 + y^2 + z^2 - 2xyz + 2ax + 2by + 2cz + d = 0, \tag{7}$$

and $x > 1$, $y > 1$, $z > 1$. Moreover, the set T represents the Teichmüller space $\mathscr{T}(2\theta_1, 2\theta_2, 2\theta_3, 2\theta_4)$ where

$$T = \{(x, y, z) \in \mathbf{R}^3 \mid x > 1, y > 1, z > 1 \ \textit{and satisfy}\ (7)\}.$$

If we denote

$$L = \frac{x + \cos\theta_1 \cos\theta_4 - u\xi^{-1}v^{-1}\eta^{-1}\sin\theta_1 \sin\theta_4}{v^{-1}\eta^{-1}\sin\theta_1\sin\theta_4},$$

From [4], we know that

$$\ell_\alpha = \ell = 2\log(z + \sqrt{z^2 - 1}), \quad \tau_\alpha = s = \log(L + \sqrt{L^2 - 1}). \tag{8}$$

Remark 2.1. In the above discussion, we have assumed that θ_1, θ_2, θ_3, θ_4 are numbers in $[0, \pi/2]$. To include the cases where some $\theta_i \in i\mathbf{R}_+$, we could follow the same way by slight modifications. Note that $\cos(i\theta) = \cosh\theta$ and we don't need to change the notations in the above equations.

3. Statement and proof of the main result

In this section, we will prove the formula for the effect of Fenchel-Nielsen coordinates under elementary move. We will assume that $S = S_{0,4}$ and apply the notations and results in Section 3.

Suppose that α' is an essential simple closed curve on S differs from α by an elementary move. Denote by (ℓ', τ') the Fenchel-Nielsen coordinates of X_τ associated to α'. Recall that X_τ is the hyperbolic surface obtained from X by τ-twist deformation along α.

Note that α' separates the cone-points of angles $2\theta_1$, $2\theta_4$ with the cone-points of angles $2\theta_2$, $2\theta_3$ (recall Figure 3). The hyperbolic transformation corresponding to α' is given by BC. We can replace $(\theta_1, \theta_2, \theta_3, \theta_4)$ by $(\theta_4, \theta_1, \theta_2, \theta_3)$ (and replace x, y, z, a, b, c, d by x', y', z', a', b', c', d') and follow the same proof as in Section 3 to show that

$$x'^2 + y'^2 + z'^2 - 2x'y'z' + 2a'x' + 2b'y' + 2c'z' + d' = 0. \tag{9}$$

By the corresponding between cone angles, we have $x' = z$, $y' = y$, $z' = x$ and $a' = c$, $b' = b$, $c' = a$, $d' = d$. As a result, (9) is equivalent to (7). Similar to (8), we have

$$\begin{aligned} \ell' &= 2\log(z' + \sqrt{(z')^2 - 1}) = 2\log(x + \sqrt{x^2 - 1}), \\ \tau' &= \log(L' + \sqrt{(L')^2 - 1}). \end{aligned} \tag{10}$$

Theorem 3.1 ([Main result]). *We have the following formulae for the transition map $\Phi_{\alpha'}^{-1} \circ \Phi_\alpha$:*

(i) *ℓ' satisfies*

$$\begin{aligned} \cosh(\ell'/2) = \sinh(\ell/2)^{-2}\{&\cos\theta_1\cos\theta_4 + \cos\theta_2\cos\theta_3 \\ &+ \cosh(\ell/2)(\cos\theta_1\cos\theta_3 + \cos\theta_2\cos\theta_4) + \cosh(\tau)(\cosh^2(\ell/2) \end{aligned}$$

$$+ 2\cos\theta_1\cos\theta_2\cosh(\ell/2) + \cos^2\theta_1 + \cos^2\theta_2 - 1)^{1/2}(\cosh(\ell/2)^2$$
$$+ 2\cos\theta_3\cos\theta_4\cosh(\ell/2) + \cos^2\theta_3 + \cos^2\theta_4 - 1)^{1/2}\}.$$

(ii) $|\tau'|$ *satisfies*

$$\begin{aligned}\cosh(\tau') = {} & \{\cos^2\theta_1 + \cos^2\theta_4 + 2\cos\theta_1\cos\theta_4\cosh(\ell'/2) + \sinh^2(\ell'/2)\}^{-1/2} \\ & \times \{\cos^2\theta_2 + \cos^2\theta_3 + 2\cos\theta_2\cos\theta_3\cosh(\ell'/2) + \sinh^2(\ell'/2)\}^{-1/2} \\ & \times \{\sinh(\ell'/2)^2\cosh(\ell/2) - \cos\theta_1\cos\theta_2 - \cos\theta_3\cos\theta_4 \\ & \quad - (\cos\theta_1\cos\theta_3 + \cos\theta_2\cos\theta_4)\cosh(\ell'/2)\}.\end{aligned}$$

Proof. By (8), $L = \cosh(\tau)$. Using the equation of L, we get

$$x = \cos\theta_1\cos\theta_4 - u\xi v^{-1}\eta^{-1}\sin\theta_1\sin\theta_4 + v^{-1}\eta^{-1}\sin\theta_1\sin\theta_4\cosh(s).$$

Note that (10) implies that $x = \cosh(l'/2)$. Hence

$$\cosh(l'/2) = \cos\theta_1\cos\theta_4 - u\xi v^{-1}\eta^{-1}\sin\theta_1\sin\theta_4 + v^{-1}\eta^{-1}\sin\theta_1\sin\theta_4\cosh(s).$$

Combining the above equation with the equations of u, v, ξ, η, a detailed calculation shows that

$$\begin{aligned}\cosh(\ell'/2) = {} & \sinh(\ell/2)^{-2}\{\cos\theta_1\cos\theta_4 + \cos\theta_2\cos\theta_3 \\ & + \cosh(\ell/2)(\cos\theta_1\cos\theta_3 + \cos\theta_2\cos\theta_4) + \cosh(\tau)(\cosh^2(\ell/2) \\ & + 2\cos\theta_1\cos\theta_2\cosh(\ell/2) + \cos^2\theta_1 + \cos^2\theta_2 - 1)^{1/2}(\cosh^2(\ell/2) \\ & + 2\cos\theta_3\cos\theta_4\cosh(\ell/2) + \cos^2\theta_3 + \cos^2\theta_4 - 1)^{1/2}\}.\end{aligned}$$

This proves (i). To prove (ii), we only need to replace ℓ, τ in (i) with l', τ' and exchange θ_2 with θ_4. □

Our main result is a slight generalization of Okai's result. In the case that the boundary of $S_{0,4}$ are totally geodesic boundary with length ℓ_1, ℓ_2, ℓ_3, ℓ_4, we let $\theta_i = \frac{i\ell_i}{2}$. Then $\cos\theta_i = \cosh(\ell_i/2)$ and we get the original formulae of Okai [5] by the above theorem:

$$\begin{aligned}\cosh(\ell'/2) = {} & \sinh(\ell/2)^{-2}\{\cosh(\ell_1/2)\cosh(\ell_4/2) + \cosh(\ell_2/2)\cosh(\ell_3/2) \\ & + \cosh(\ell/2)(\cosh(\ell_1/2)\cosh(\ell_3/2) + \cosh(\ell_2/2)\cosh(\ell_4/2)) \\ & + \cosh(\tau)(\cosh^2(\ell/2) + 2\cosh(\ell_1/2)\cosh(\ell_2/2)\cosh(\ell/2) \\ & + \cosh^2(\ell_1/2) + \cosh^2(\ell_2/2) - 1)^{1/2} \\ & \times (\cosh^2(\ell/2) + 2\cosh(\ell_3/2)\cosh(\ell_4/2)\cosh(\ell/2) \\ & + \cosh^2(\ell_3/2) + \cosh^2(\ell_4/2) - 1)^{1/2}\},\end{aligned}$$

$$\begin{aligned}\cosh(\tau') = {} & \{\cosh^2(\ell_1/2) + \cosh^2(\ell_4/2) + 2\cosh(\ell_1/2)\cosh(\ell_4/2)\cosh(\ell'/2) \\ & + \sinh^2(\ell'/2)\}^{-1/2}\{\cosh^2(\ell_2/2) + \cosh^2(\ell_3/2) \\ & + 2\cosh(\ell_2/2)\cosh(\ell_3/2)\cosh(\ell'/2) + \sinh^2(\ell'/2)\}^{-1/2} \\ & \times \{\sinh^2(\ell'/2)\cosh(\ell/2) - \cosh(\ell_1/2)\cosh(\ell_2/2) \\ & - \cosh(\ell_3/2)\cosh(\ell_4/2) - (\cosh(\ell_1/2)\cosh(\ell_3/2) \\ & + \cosh(\ell_2/2)\cosh(\ell_4/2)\cosh(\ell'/2)\}.\end{aligned}$$

Remark 3.1. Our proof is different from Okai's proof. So we give a new proof of Okai's result. Okai [5] also proved that $sign(\tau) = -sign(\tau')$.

We make a few comment about the case that $S = S_{1,1}$ is a torus with one hole. Let $\mathcal{T}(2\theta)$ be the Teichmüller space of hyperbolic structures on $S_{1,1}$ with cone angle $2\theta \in [0, \pi] \cup i\mathbf{R}_+$. There is a natural identification of $\mathcal{T}(2\theta)$ with $\mathcal{T}(\pi, \pi, \pi, \theta)$ (ref. [4]). Apply $\mathcal{T}(\pi, \pi, \pi, \theta)$ to Theorem 3.1, we have

$$\cosh(\ell'/2) = \sinh(\ell/2)^{-1}\cosh(\tau/2)\left\{\frac{\cosh(\ell) + \cos\theta}{2}\right\}^{1/2}$$

and

$$\begin{aligned}\cosh(\tau'/2) = {} & \cosh(\ell/2)\{\cosh^2(\tau/2)(\cosh(\ell) + \cos\theta) - 2\sinh^2(\ell/2)\}^{1/2} \\ & \times \{\cosh^2(\tau/2)(\cosh^2(\ell/2) + \cos\theta) + \sinh^2(\ell/2)\cos\theta\}^{-1/2}.\end{aligned}$$

In case that $\theta = i\dfrac{\ell_0}{2}$, we have

$$\cosh(\ell'/2) = \sinh(\ell/2)^{-1}\cosh(\tau/2)\left\{\frac{\cosh(\ell) + \cosh(\ell_0/2)}{2}\right\}^{1/2}$$

and

$$\begin{aligned}\cosh(\tau'/2) = {} & \cosh(\ell/2)\{\cosh^2(\tau/2)(\cosh(\ell) + \cos(\ell_0/2)) - 2\sinh^2(\ell/2)\}^{1/2} \\ & \times \{\cosh^2(\tau/2)(\cosh^2(\ell/2) + \cos(\ell_0/2)) + \sinh^2(\ell/2)\cos(\ell_0/2)\}^{-1/2}.\end{aligned}$$

The above formula is also given by Okai [5].

As an application, we consider the 2-form $d\ell \wedge d\tau$. Using (8), we have

$$d\ell \wedge d\tau = \frac{4dz \wedge dx}{\sqrt{x^2z^2 - x^2 - z^2 - 2bxz - 2ax - 2cx + b^2 - d}}.$$

By (7), the denominator is equal to $xz - y - b$, and then

$$d\ell \wedge d\tau = \frac{4\, dz \wedge dx}{xz - y - b}.$$

By (7) again, we know that

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy, \quad dx = \frac{\partial x}{\partial y}\,dy + \frac{\partial x}{\partial z}\,dz,$$

where

$$\frac{\partial z}{\partial y} = \frac{xz - y - b}{z - xy + c}, \quad \frac{\partial x}{\partial y} = \frac{xz - y - b}{x - yz + a}.$$

Hence

$$\frac{4\,dz \wedge dx}{xz - y - b} = \frac{4\,dx \wedge dy}{xy - z - c} = \frac{4\,dy \wedge dz}{yz - x - a}.$$

Since $\dfrac{4\,dx \wedge dy}{xy - z - c}$ is equal to $dl' \wedge d\tau'$, we have

$$d\ell \wedge d\tau = d\ell' \wedge d\tau'.$$

As a result, the two form $d\ell_\alpha \wedge d\tau_\alpha$ is independent of the choice of α. It is a generalization of the Weil-Petersson symplectic form studied by Wolpert [10].

4. An application

Mirzakhani [3] proved the following theorem:

THEOREM 4.1. *Let γ be a closed curve on $S_{g,n}$. The length function*

$$l_\gamma : \mathscr{T}(g; 0, \ldots, 0) \to \mathbf{R}_+$$
$$X \mapsto l_\gamma(X)$$

is an asymptotically piecewise linear function of rational type with respect to the Fenchel-Nielsen coordinates.

Our goal is to generalize the above theorem to surfaces with cone points by using Theorem 3.1. We'll prove the following theorem.

THEOREM 4.2. *Let γ be a closed curve on $S_{g,n}$ and $0 \le \theta_i < \pi$, $1 \le i \le n$. Then the length function*

$$l_\gamma : \mathscr{T}(g; \theta_1, \ldots, \theta_n) \to \mathbf{R}_+$$
$$X \mapsto l_\gamma(X)$$

is an asymptotically piecewise linear function with respect to the Fenchel-Nielsen coordinates.

Remark 4.1. In the case the surfaces without cone points, the length function of a closed curve is an asymptotically piecewise linear function of rational type with respect to the Fenchel-Nielsen coordinates. Then in the case the surfaces with cone points, the length function of a closed curve is an asymptotically piecewise linear function with respect to the Fenchel-Nielsen coordinates but it's not necessarily of rational type. Because the cone points can cause some irrational coefficients.

4.1. Definitions and lemmas. We give the following definitions and lemmas that appeared in [3] for completeness. Let $\mathscr{C}$ denote a closed cone in $\mathbf{R}^m$.

Definition 4.1 (Asymptotically linear [3]). Let $F : \mathscr{C} \to \mathbf{R}$ be a function, we say F is asymptotically linear with respect to coordinates $x_1, \dots, x_m$ iff there are linear functions $\mathscr{R}_1, \dots, \mathscr{R}_{m'}$, $\mathscr{L} : \mathbf{R}^m \to \mathbf{R}$ and $c \in \mathbf{R}^+$ such that

$$F(x_1, \dots, x_m) - \mathscr{L}(x_1, \dots, x_m) \to c$$

uniformly as $\min_{1 \le i \le m'} \{\mathscr{R}_i(x_1, \dots, x_m)\}_{i=1}^{m'} \to \infty$.

Definition 4.2 (Asymptotically piecewise linear [3]). Let $F : \mathscr{C} \to \mathbf{R}$ be a function, we say F is asymptotically piecewise linear iff there are linear functions $\mathscr{W}_1, \dots, \mathscr{W}_k$ such that for any $\epsilon = (\epsilon_1, \dots, \epsilon_m)$, $\epsilon_i = 1$ or -1. The restriction of F on each sub-cone defined by $\mathscr{C}_\epsilon = \{x \mid \mathrm{Sign}(\mathscr{W}_i(x)) = \epsilon\}$ is asymptotically linear.
Moreover, if $\mathscr{R}_1, \dots, \mathscr{R}_{m'}$, $\mathscr{W}_1, \dots, \mathscr{W}_k$ and $\mathscr{L}$ can be chosen to all having rational coefficients, then we call F is of rational type.

For $x = (x_1, \dots, x_k) \in \mathbf{R}^k$, we define $e_i : \mathbf{R}^k \to \mathbf{R}$ by $e_i(x) = e^{x_i}$. Let $\mathscr{F}_k^*$ to be the smallest family of functions $\mathbf{R}^k \to \mathbf{R}$ containing the functions e_i for $1 \le i \le k$ such that the following holds: for any two $f, g \in F_k^*$ and $m \in N$, $c \in R$ we have

$$\left\{ f + g, f - g, f \times g, \frac{f}{g}, f^{1/m}, cf \right\} \subset \mathscr{F}_k^*$$

Moreover we define

$$\mathscr{F}^* = \bigcup_{k \ge 1} \mathscr{F}_k^*.$$

Then for any $P \in \mathscr{F}_k^*$, we have

$$F(x) = \mathrm{Arccosh}(P(x))$$

is asymptotically piecewise linear on any cone where it is defined.
Let $F_1, \dots, F_k$ in $\mathscr{F}_k^*$. For $P \in \mathscr{F}_k^*$, we define a function

$$G(x) = P(F_1'(x), \dots, F_k'(x)),$$

where $F_i'(x) = \mathrm{Arcsinh}(F_i(x))$ or $F_i'(x) = \mathrm{Arccosh}(F_i(x))$. Then $G \in \mathscr{F}_k^*$ and $\mathrm{Arccosh}(G(x))$ is also asymptotically piecewise linear on any cone where it is defined.

Let

$$\mathscr{A}^* = \{G : \mathbf{R}^m \to \mathbf{R}^m \mid \cosh(\pi_i(G)) \in \mathscr{F}^* \text{ or } \sinh(\pi_i(G)) \in \mathscr{F}^*\}$$

where $\pi_i : \mathbf{R}^m \to \mathbf{R}$ is the natural projection map to the ith coordinate.
With the above notations, the following lemma is from [3]:

LEMMA 4.3.
- *Any function $G \in \mathscr{A}^*$ is asymptotically piecewise linear.*
- *The composition $G_1 \circ G_2$ of any two maps in $\mathscr{A}^*$ is again in $\mathscr{A}^*$.*

4.2. An application. Now we give an application of the main Theorem 3.1. For convenience we let

$$A_1 = \cos\theta_1 \cos\theta_4 + \cos\theta_2 \cos\theta_3, \quad A_2 = \cos\theta_1 \cos\theta_3 + \cos\theta_2 \cos\theta_4.$$

$$A_3 = 2\cos\theta_1 \cos\theta_2, \quad A_4 = \cos^2\theta_1 + \cos^2\theta_2 - 1.$$

$$A_5 = 2\cos\theta_3 \cos\theta_4, \quad A_6 = \cos^2\theta_3 + \cos^2\theta_4 - 1.$$

$$B_1 = \cos^2\theta_1 + \cos^2\theta_4, \quad B_2 = 2\cos\theta_1 \cos\theta_4.$$

$$B_3 = \cos^2\theta_2 + \cos^2\theta_3, \quad B_4 = 2\cos\theta_2 \cos\theta_3.$$

$$B_5 = \cos\theta_1 \cos\theta_2 + \cos\theta_3 \cos\theta_4, \quad B_6 = \cos\theta_1 \cos\theta_3 + \cos\theta_2 \cos\theta_4.$$

It is clear that $A_i, B_i \in R$, $i = 1, \dots, 6$. And the formulas from Theorem 3.1 can be simplified as:

$$\begin{aligned}\cosh(\ell'/2) = \sinh(\ell/2)^{-2}\{&A_1 + A_2\cosh(\ell/2) + \cosh(\tau)\\ &\times [\cosh^2(\ell/2) + A_3\cosh(\ell/2) + A_4]^{1/2}\\ &\times [\cosh(\ell/2)^2 + A_5\cosh(\ell/2) + A_6]^{1/2}\}.\end{aligned}$$

and

$$\begin{aligned}\cosh(\tau') = \{&B_1 + B_2\cosh(\ell'/2) + \sinh^2(\ell'/2)\}^{-1/2}\\ &\times \{B_3 + B_4\cosh(\ell'/2) + \sinh^2(\ell'/2)\}^{-1/2}\\ &\times \{\sinh(\ell'/2)^2\cosh(\ell/2) - B_5 - B_6\cosh(\ell'/2)\}.\end{aligned}$$

It is clear that $\cosh(\ell'/2) \in \mathscr{F}^*$ and $\cosh(\tau') \in \mathscr{F}^*$. By Lemma 4.3 we know that $\ell'/2$ and τ' are asymptotically piecewise linear. This implies that the asymptotic piecewise linearity does not depend on the choice of the Fenchel-Nielsen coordinates. Then the Theorem 4.2 is clearly true for the case that γ is a simple closed curve because one can take Fenchel-Nielsen coordinates by using γ.

To prove the Theorem 4.2 for the case that γ has self-intersections (γ is a closed curve but is not a simple closed curve), we need to generalize the Lemma 3.11 in [8] from hyperbolic surface without cone points to hyperbolic surface with cone points. We express this generalization as following.

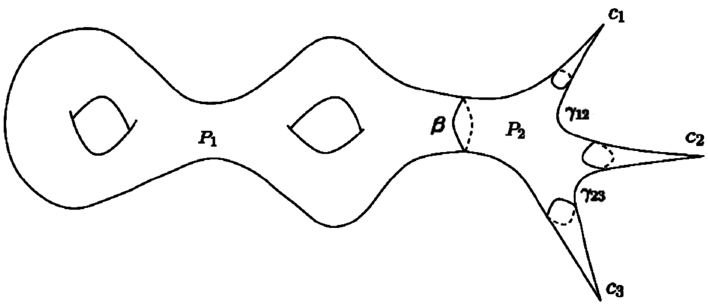

FIGURE 4. Hyperbolic cone surface.

LEMMA 4.4. *Let α be a closed curve on $S_{g,n}$ with $i(\alpha,\alpha) > 0$. There are closed curves α_1, α_2, α_3 on $S_{g,n}$ such that $i(\alpha_j,\alpha_j) < i(\alpha,\alpha)$ for $1 \le j \le 3$ and for any $X \in \mathcal{T}(g;\theta_1,\dots,\theta_n)$ we have*:

$$\cosh\frac{l_\alpha(X)}{2} = 2\cosh\frac{l_{\alpha_1}(X)}{2}\cosh\frac{l_{\alpha_2}(X)}{2} + \varepsilon\cosh\frac{l_{\alpha_3}(X)}{2}.$$

where $\varepsilon \in \{1,-1\}$.

Remark 4.2. For each intersection $p \in \alpha$, we have two operations: separating resolution and non-separation resolution. Let α_1 and α_2 respectively denote the two components of the separation of α at p, and let α_3 denote the component of the non-separating resolution of α at p. See [8] for more details.

We can not give a proof of the above lemma directly by using the idea of the proof of the Lemma 3.11 in [8], since we do not know the metric structure of the universal cover of hyperbolic surface with cones. But we can view the hyperbolic cone surface as the subsurface of the corresponding complete surface in some senses. For this purpose, we need to do some preparations.

First we can find a simple closed geodesic β separating the surface into two parts as show in Figure 4: one of them is a hyperbolic surface P_1 with punctures and geodesic boundaries; the other is a hyperbolic surface P_2 with cone points and geodesic boundaries, and the genus of P_2 is zero. Let $c_1,\dots,c_m$ denote the cone points. We can find some geodesic arcs $\gamma_{i,i+1}$ $i = 1,\dots,m-1$, satisfy the following conditions:

- $\gamma_{i,i+1}$ connects the cone points c_i and c_{i+1}.
- $\gamma_{i,i+1} \cap \gamma_{j,j+1} = \emptyset$, $i \ne j$.

Here the surface S endows with a hyperbolic structure R. For simplicity, in the following we denote S as a surface with hyperbolic structure. That is, S is a hyperbolic surface. Then we cut the hyperbolic surface S along the geodesic arcs $\gamma_{i,i+1}$ $i = 1,\dots,m-1$, and obtain a new hyperbolic surface S^* with boundaries. Here the boundary component $\gamma'_{1,2} \cup \gamma'_{2,3} \cup \cdots \cup \gamma'_{m-1,m} \cup \gamma''_{m-1,m} \cup \cdots \cup \gamma''_{1,2}$ is

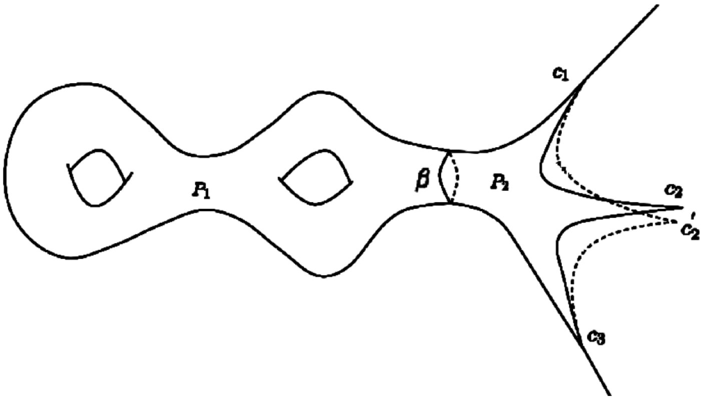

FIGURE 5. S^* as a subsurface of S'.

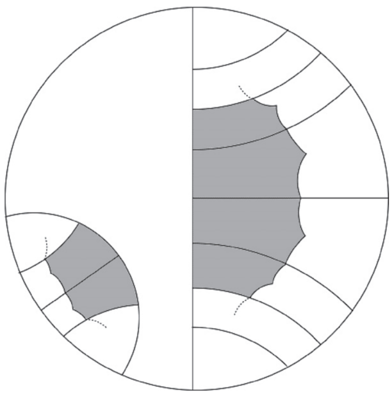

FIGURE 6. Universal cover of S'.

piecewise geodesic arcs. Let S' denote the completion of P_1. It is clearly that the hyperbolic surface S^* can be isometrically embedded into S' as show in Figure 5.

Then we consider the universal cover of S' as show in Figure 6, here the grey shaded parts denote the lifts of P_2 as the subsurface of S'. We will need the following lemma about the hyperbolic surface with cones.

LEMMA 4.5. *With the above notations and* $0 \le \theta_i < \pi$, $1 \le i \le n$, *we have*:

- *Let* α *be a closed geodesic of* $X \in \mathscr{T}(g; \theta_1, \dots, \theta_n)$. *If* $\alpha \cap \gamma_{i,i+1} = \emptyset$ *for all* $\gamma_{i,i+1}$ $i = 1, \dots, m-1$. *Then* $\alpha \subset P_1$.
- *Let* α *be a closed geodesic of* $X \in \mathscr{T}(g; \theta_1, \dots, \theta_n)$. *Then* $c_i \notin \alpha$ *for any cone point* c_i.

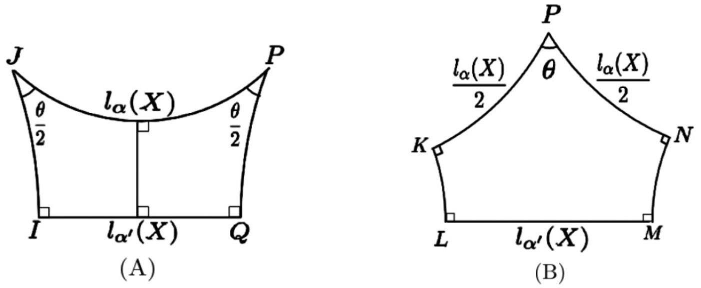

FIGURE 7. Generalized triangle.

LEMMA 4.6. *Let α be a closed curve of $X \in \mathcal{T}(g; \theta_1, \dots, \theta_n)$, and $p \in \alpha$. Assume α also satisfies the following two conditions*:

- *$\alpha \backslash p$ is a smooth geodesic arc,*
- *The angle of α at p is θ.*

Let α' denote the geodesic representation of α in the homotopy class. Then we have:

$$\cos(\theta) = \frac{-\cosh l_{\alpha'}(X) + \sinh^2 \frac{l_\alpha(X)}{4}}{\cosh^2 \frac{l_\alpha(X)}{4}}.$$

In case that α is isotopic to a cone point with cone angle φ, we denote $l_{\alpha'}(X) = i\varphi$.

Proof. If $\alpha \cap \gamma_{i,i+1} = \emptyset$ for all $\gamma_{i,i+1}$ $i = 1, \dots, m-1$. By using the above lemma we know that $\alpha \subset P_1$. Then the proof is done by applying the cosine law on the generalized triangles $PJIQ$ or $PKLMN$ as in the Figure 7.

If $\alpha \cap \gamma_{i,i+1} \neq \emptyset$ for some $\gamma_{i,i+1}$. Let $\tilde{\alpha}$ and $\tilde{\alpha}'$ be the lifts of α and α' in the universal cover $\tilde{S}'$ of S', respectively. Let ρ denote the hyperbolic metric of $\tilde{S}'$. Then the lifts $\tilde{\alpha}$ and $\tilde{\alpha}'$ are divided into some geodesic arcs $\widetilde{\alpha_i}$, $i = 1, \dots, n$ and $\widetilde{\alpha_i}'$, $i = 1, \dots, n$ by the lifts of $\gamma_{i,i+1}$. We also know that $\widetilde{\alpha_i}$ and $\widetilde{\alpha_i}'$ are homotopy relative to some $\widetilde{\gamma_{i,j}}$.

Let P be a lift of p in the universal cover, and let θ be the angle at P of $\tilde{\alpha}$. Let the solid geodesic arcs $\widetilde{\alpha_1} = PA \cup BC$, $\widetilde{\alpha_2} = DP$ denote the lift of α and the solid geodesic arcs $\widetilde{\alpha_1}' = HE$, $\widetilde{\alpha_2}' = FG$ denote the lift of α', as shown in Figure 8. Then there are two cases:

- α is not isotopic to a cone point. Let us choose a point Q belong to the geodesic HE such that PQ is vertical to HE. We extend PA to obtain PJ satisfies $l_\rho(PJ) = l_\alpha(X)$, and extend QE to obtain QI satisfies $l_\rho(QI) = l_{\alpha'}(X)$. Hence we obtain (A) of Figure 7. It is clear that $l_\rho(PQ) = l_\rho(IJ)$. Moreover we obtain (B) of Figure 7 by a simple geometry operation.

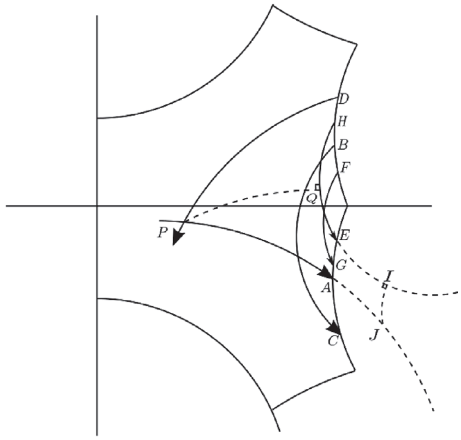

FIGURE 8.

Applying the cosine law to the generalized triangles $PJIQ$ or $PKLMN$, we have:

$$\cos(\theta) = \frac{-\cosh l_X(\alpha') + \sinh^2\left(\frac{l_X(\alpha)}{4}\right)}{\cosh^2\left(\frac{l_X(\alpha)}{4}\right)}.$$

- α is isotopic to a cone point with cone angle φ, then we obtain the Figure 9. Applying the cosine law to the generalized triangles $PKMN$ in Figure 9,

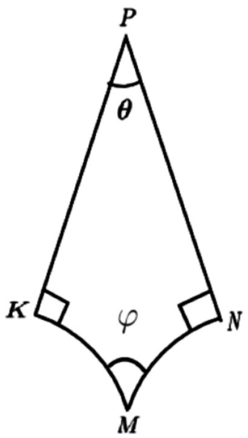

FIGURE 9.

we have:

$$\sin\left(\frac{\theta}{2}\right) = \frac{\cos\left(\frac{\varphi}{2}\right)}{\cosh\left(\frac{l_p(\alpha)}{4}\right)}.$$

$$\cos(\theta) = \frac{-2\cos^2\left(\frac{\varphi}{2}\right) + \cosh^2\left(\frac{l_p(\alpha)}{4}\right)}{\cosh^2\left(\frac{l_p(\alpha)}{4}\right)} = \frac{-\cos(\varphi) + \sinh^2\left(\frac{l_X(\alpha)}{4}\right)}{\cosh^2\left(\frac{l_X(\alpha)}{4}\right)}. \qquad \square$$

Basic on the above lemma, the proof of Lemma 4.4 is similar to the proof of Lemma 3.11 in [8]. For the sake of simplicity, we omit the details.

COROLLARY 4.7. *For any closed geodesic $\alpha \in X \in \mathscr{T}(g; \theta_1, \dots, \theta_n)$, there exist finite simple closed geodesics β_i, $i = 1, \dots, m$ such that*

$$\begin{aligned}\cosh\left(\frac{l_\alpha(X)}{2}\right) &= k_1 \cosh^{t_{1,1}}\left(\frac{l_{\alpha_1(X)}}{2}\right) \cdots \cosh^{t_{1,m}}\left(\frac{l_{\alpha_m(X)}}{2}\right) \\ &\quad + k_s \cosh^{t_{s,1}}\left(\frac{l_{\alpha_1(X)}}{2}\right) \cdots \cosh^{t_{s,m}}\left(\frac{l_{\alpha_m(X)}}{2}\right).\end{aligned}$$

where $t_{i,j} \in N$ and $k_i = \pm 2^r$, $r \in N$.

Proof. It sufficces to prove the corollary in the case that α has self-intersections. Let we assume p be one of self-intersection point of α. Applying the Lemma 4.4 to α at p, we have

$$\cosh\left(\frac{l_\alpha(X)}{2}\right) = 2\cosh\left(\frac{l_{\alpha_1}(X)}{2}\right)\cosh\left(\frac{l_{\alpha_2}(X)}{2}\right) + \varepsilon\cosh\left(\frac{l_{\alpha_3}(X)}{2}\right).$$

where $\varepsilon \in \{1, -1\}$, and α_1, α_2 denote the two components of the separation of α at p, and α_3 denote the component of the non-separating resolution of α at p. Moreover, $i(\alpha_j, \alpha_j) < i(\alpha, \alpha)$, $j = 1, 2, 3$. If $i(\gamma_j, \gamma_j) > 0$ for some $j = 1, 2, 3$, we apply the Lemma 4.4 to γ_j again. Then the corollary will be proved in finite steps by applying Lemma 4.4 repeatedly. $\square$

Then the proof of Theorem 4.2 is followed directly by Lemma 4.3 and Corollary 4.7.

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