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ON FALTINGS' LOCAL-GLOBAL PRINCIPLE OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

Let R be a commutative Noetherian ring, I an ideal of R and M, N finitely generated R-modules. Let $0 \le n \in \mathbf{Z}$. This note shows that the least integer i such that $\dim \operatorname{Supp}(H_I^i(M,N)/K) \ge n$ for any finitely generated submodule K of $H_I^i(M,N)$ equal to the number $\inf\{f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}(N/I_M N), \dim R/\mathfrak{p} \ge n\}$, where $f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is the least integer i such that $H_{I_\mathfrak{p}}^i(M_\mathfrak{p}, N_\mathfrak{p})$ is not finitely generated, and $I_M = \operatorname{ann}(M/IM)$. This extends the main result of Asadollahi-Naghipour [1] and Mehrvarz-Naghipour-Sedghi [8] for generalized local cohomology modules by a short proof.

1. Introduction

Let R be a commutative Noetherian ring, I an ideal of R, and M, N finitely generated R-modules. The Local-global Principle of Faltings for the finiteness of local cohomology modules [4, Satz 1] which states that for a given positive integer r, the $R_\mathfrak{p}$-module $H_{IR_\mathfrak{p}}^i(N_\mathfrak{p})$ is finitely generated for all $i < r$ and for all $\mathfrak{p} \in \operatorname{Spec} R$ if and only if the R-module $H_I^i(N)$ is finitely generated for all $i < r$. Another statement of Faltings' local-global principle, particularly relevant for this paper, is in terms of the finiteness dimension $f_I(N)$ of N relative to I, where $f_I(N) = \inf\{0 \le i \in \mathbf{Z} \mid H_I^i(N) \text{ is not finitely generated}\}$, with the usual convention that the infimum of the empty set of integers is interpreted as ∞. K. Bahmanpour et al., in [2], introduced the notion of the *n-th* finiteness dimension $f_I^n(N)$ of N relative to I by $f_I^n(N) = \inf\{f_{IR_\mathfrak{p}}(N_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}(N/IN), \dim(R/\mathfrak{p}) \ge n\}$. In [1], Asadollahi-Naghipour introduced the class of in dimension $< n$ modules, and they showed that if $(R, \mathfrak{m})$ is a complete local ring, I an ideal of R and N a finitely generated R-module, then $f_I^n(N) = \inf\{0 \le i \in \mathbf{Z} \mid H_I^i(N) \text{ is not in dimension} < n\}$ for all n (see [1, Thm. 2.5]). Recently, in [8], they obtained this result without the condition that $(R, \mathfrak{m})$ is

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complete local ring. The purpose of this note is to extend the main result of Mehrvarz-Naghipour-Sedghi (see [8, Thm. 2.10]) to the case of generalized local cohomology modules as follows: *Let R be a commutative Noetherian ring, I an ideal of R, and M, N finitely generated R-modules. Then* $f_I^n(M,N) = \inf\{0 \le i \in \mathbf{Z} \mid H_I^i(M,N)$ *is not in dimension* $< n\}$, *where* $f_I^n(M,N) = \inf\{f_{IR_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}(N/I_M N), \dim(R/\mathfrak{p}) \ge n\}$ *and* $f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \inf\{0 \le i \in \mathbf{Z} \mid H^i_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ *is not finitely generated*$\}$ (see Theorem 2.4). But we here use a new proof method (compare with [1] and [8]). Recall that the generalized local cohomology module $H_I^j(M,N)$ is introduced by J. Herzog [5] as $H_I^j(M,N) = \varinjlim_n \operatorname{Ext}_R^j(M/I^nM, N)$, and for generalized local cohomology refer to [5], [6], [7], and [3].

2. Main result

Let $0 \le n \in \mathbf{Z}$. We first recall that an R-module T is called *in dimension* $< n$ if there exists a finitely generated submodule K of T such that $\dim \operatorname{Supp}(T/K) < n$ (see [1, Def. 2.1]). Moreover, an R-module T is said to be minimax, if there exists a finitely generated submodule K of T such that T/K is Artinian (cf. [11] and [1]). Note that the class of minimax modules and the class of in dimension $< n$ modules are Serre subcategories, i.e., it is closed under taking submodules, quotients and extensions (cf. [9, Sect. 4] and [8, Cor. 2.13]).

Lemma 2.1. *Let $0 \le t \in \mathbf{Z}$. Assume that $H_I^j(M,N)$ is in dimension $< n$ for all $j < t$. Then we have*

i) $\operatorname{Hom}(R/I, H_I^t(M,N))$ *is in dimension* $< n$.

ii) $\operatorname{Ass}(H_I^t(M,N)/K)_{\ge n}$ *is finite for any minimax submodule K of $H_I^t(M,N)$, where we set $S_{\ge n} = \{\mathfrak{p} \in S \mid \dim(R/\mathfrak{p}) \ge n\}$ for any subset S of* $\operatorname{Spec} R$.

Proof. i) We process by induction on t. The case $t = 0$ is trivial because $\operatorname{Hom}(R/I, H_I^0(M,N)) \subseteq \Gamma_I(\operatorname{Hom}(M,N))$. Assume $t > 0$ and the lemma is true for $t-1$. Set $\overline{N} = N/\Gamma_I(N)$. From the short exact sequence $0 \to \Gamma_I(N) \to N \to \overline{N} \to 0$ we get the following exact sequences

$$\operatorname{Ext}_R^i(M, \Gamma_I(N)) \xrightarrow{f_i} H_I^i(M,N) \xrightarrow{g_i} H_I^i(M,\overline{N}) \xrightarrow{h_i} \operatorname{Ext}_R^{i+1}(M, \Gamma_I(N)),\ \forall i,$$

$$0 \to \operatorname{Im} f_t \to H_I^t(M,N) \to \operatorname{Im} g_t \to 0, \quad 0 \to \operatorname{Im} g_t \to H_I^t(M,\overline{N}) \to \operatorname{Im} h_t \to 0.$$

Hence $H_I^i(M,\overline{N})$ is in dimension $< n$ for all $i < t$ by the hypothesis; we also have $\operatorname{Hom}(R/I, H_I^t(M,N))$ is in dimension $< n$ if only if so is $\operatorname{Hom}(R/I, H_I^t(M,\overline{N}))$ since $\operatorname{Im} f_t$ and $\operatorname{Im} h_t$ are finitely generated. By replacing N by $\overline{N}$, we may henceforth assume that $\Gamma_I(N) = 0$. Thus there exists $x \in I$ such that x is an N-regular element. So, we have the short exact sequence $0 \to N \xrightarrow{x} N \to N/xN \to 0$. This yields the exact sequence

$$H_I^i(M,N) \to H_I^i(M,N/xN) \to H_I^{i+1}(M,N)$$

for all i. Hence $H_I^i(M,N/xN)$ is in dimension $< n$ for all $i < t-1$. So by the induction asumption, we get that $\operatorname{Hom}(R/I, H_I^{t-1}(M,N/xN))$ is in dimension $< n$.

Consider the long exact sequence

$$H_I^{t-1}(M,N) \xrightarrow{x} H_I^{t-1}(M,N) \xrightarrow{u} H_I^{t-1}(M,N/xN) \xrightarrow{v} H_I^t(M,N) \xrightarrow{x} H_I^t(M,N).$$

We split the above sequence into two the following exact sequences

$$(*) \qquad 0 \to \operatorname{Im} u \to H_I^{t-1}(M,N/xN) \to \operatorname{Im} v \to 0,$$

$$(**) \qquad 0 \to \operatorname{Im} v \to H_I^t(M,N) \xrightarrow{x} H_I^t(M,N),$$

where $\operatorname{Im} u \cong H_I^{t-1}(M,N)/xH_I^{t-1}(M,N)$ is in dimesion $< n$. The following sequence

$$\operatorname{Hom}(R/I, H_I^{t-1}(M,N/xN)) \to \operatorname{Hom}(R/I, \operatorname{Im} v) \to \operatorname{Ext}_R^1(R/I, \operatorname{Im} u)$$

is exact by (*). The left-most module is in dimension $< n$. Moreover, we get by [8, Cor. 2.16] that $\operatorname{Ext}_R^1(R/I, \operatorname{Im} u)$ is in dimension $< n$, then so is $\operatorname{Hom}(R/I, \operatorname{Im} v)$. By (**) we get the following exact sequence

$$0 \to \operatorname{Hom}(R/I, \operatorname{Im} v) \to \operatorname{Hom}(R/I, H_I^t(M,N)) \xrightarrow{x} \operatorname{Hom}(R/I, H_I^t(M,N)).$$

Thus $\operatorname{Hom}(R/I, H_I^t(M,N)) \cong \operatorname{Hom}(R/I, \operatorname{Im} v)$ is in dimension $< n$ by the fact that $x \in I \subseteq \operatorname{ann}_R(\operatorname{Hom}(R/I, H_I^t(M,N)))$, as required.

ii) Let K be a minimax submodule of $H_I^t(M,N)$. We get the following exact sequence

$$\operatorname{Hom}(R/I, H_I^t(M,N)) \xrightarrow{f} \operatorname{Hom}(R/I, H_I^t(M,N)/K) \to \operatorname{Ext}_R^1(R/I, K).$$

Hence the set $\operatorname{Ass}(\operatorname{Hom}(R/I, H_I^t(M,N)/K))_{\geq n}$ is contained in $\operatorname{Ass}(\operatorname{Im} f)_{\geq n} \cup \operatorname{Ass}(\operatorname{Ext}_R^1(R/I,K))_{\geq n}$. Note that $\operatorname{Im} f$ is a quotient of $\operatorname{Hom}(R/I, H_I^t(M,N))$, so $\operatorname{Im} f$ is in dimension $< n$ by i); thus $\operatorname{Ass}(\operatorname{Im} f)_{\geq n}$ is a finite set by [8, Lem. 2.6]. Moreover $\operatorname{Ass} \operatorname{Ext}_R^1(R/I,K)_{\geq n}$ is finite by [8, Rmk. 2.2]. Therefore $\operatorname{Ass}(H_I^t(M,N)/K))_{\geq n} = \operatorname{Ass}(\operatorname{Hom}(R/I, H_I^t(M,N)/K))_{\geq n}$ is finite. □

THEOREM 2.2. *If $H_I^0(M,N), \dots, H_I^t(M,N)$ is finitely generated locally at every $\mathfrak{p} \in \operatorname{Supp}(N/I_M N)_{\geq n}$ (i.e. $H_I^0(M,N)_{\mathfrak{p}}, \dots, H_I^t(M,N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}(N/I_M N)_{\geq n}$, where $I_M = \operatorname{ann}(M/IM)$) then $H_I^0(M,N), \dots, H_I^t(M,N)$ is in dimension $< n$.*

Proof. We prove the theorem by induction on t. The case of $t = 0$ is trivial since $H_I^0(M,N)$ is finitely generated. We assume that $t > 0$ and the theorem is true for $t-1$. By induction, $H_I^0(M,N), \dots, H_I^{t-1}(M,N)$ are in dimension $< n$. So we have to show that $H_I^t(M,N)$ is in dimension $< n$. We get by Lemma 2.1 ii) that $\operatorname{Ass}(H_I^t(M,N))_{\geq n}$ is a finite set. For convention we set $H = H_I^t(M,N)$. For any $\mathfrak{p} \in \operatorname{Ass}(H)_{\geq n}$, we obtain that $H_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ by the hypothesis. Since $H_{\mathfrak{p}}$ is $I_{\mathfrak{p}}$-torsion, so there exists $n_{\mathfrak{p}} \in \mathbf{N}$ such that $I^{n_{\mathfrak{p}}} H_{\mathfrak{p}} = 0$. Set $m = \max\{n_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Ass}(H)_{\geq n}\}$.

For any $\omega \in H$, we get that $I^m \omega = \langle \omega_1, \dots, \omega_r \rangle$ is a finitely generated R-module for some $\omega_1, \dots, \omega_r \in H$. For each $\mathfrak{p} \in \operatorname{Ass}(H)_{\geq n}$, $(I^m \omega)_{\mathfrak{p}} \subseteq (I^m H)_{\mathfrak{p}} = 0$. It follows that $\omega_i/1 = 0$ in $H_{\mathfrak{p}}$ for all $i = 1, \dots, r$. Thus there exists $s_i \in R \backslash \mathfrak{p}$

such that $s_i\omega_i = 0$ for all $i = 1, \ldots, r$. Set $s_\mathfrak{p} = s_1 s_2 \cdots s_r$. Then $s_\mathfrak{p} \in R \backslash \mathfrak{p}$ and $s_\mathfrak{p}(I^m\omega) = 0$. Let J be the ideal of R generated by the set $\{s_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Ass}(H)_{\geq n}\}$. Hence $J(I^m\omega) = 0$. Note that $s_\mathfrak{p} \in J$ and $s_\mathfrak{p} \notin \mathfrak{p}$, and so $J \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}(H)_{\geq n}$. It yields that there exists $y \in J$ such that $y \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}(H)_{\geq n}} \mathfrak{p}$. Thus $\dim \mathrm{Supp}(0 : y)_H < n$ (indeed, for any $\mathfrak{q} \in \mathrm{Ass}((0 : y)_H) \subseteq \mathrm{Ass}(H)$, so $\mathfrak{q} \supseteq \mathrm{ann}((0 : y)_H) \ni y$. Hence $\dim(R/\mathfrak{q}) < n$). Since $y(I^m\omega) \subseteq J(I^m\omega) = 0$, it holds that $I^m\omega \subseteq (0 : y)_H$. It implies that $\dim(I^m\omega) < n$ (***). Keep in mind that the following sequence

$$\mathrm{Ext}_R^j(M, \Gamma_I(N)) \to H_I^j(M, N) \to H_I^j(M, \overline{N}) \to \mathrm{Ext}_R^{j+1}(M, \Gamma_I(N))$$

is exact for all j, where $\overline{N} = N/\Gamma_I(N)$. Therefore we may assume that $\Gamma_I(N) = 0$. Hence we can choose $x \in I$ such that x is a regular element of N. Thus $x^m \in I^m$ is also a regular element of N. We then have $\dim \mathrm{Supp}(x^m H) < n$ (indeed, for any $\mathfrak{p} \in \min \mathrm{Ass}(x^m H)$, then there is an element $\omega \in H$ such that $\mathfrak{p} = (0 : x^m\omega) = \mathrm{ann}(x^m\omega)$. Thus we have by (***) that $\dim(R/\mathfrak{p}) = \dim(x^m\omega) \leq \dim(I^m\omega) < n$). We get the following exact sequences

$$H_I^{j-1}(M, N) \to H_I^{j-1}(M, N/x^m N) \to H_I^j(M, N) \xrightarrow{x^m} H_I^j(M, N).$$

Then we get by the hypothesis that $H_I^{j-1}(M, N/x^m N)$ is finitely generated locally at every $\mathfrak{p} \in \mathrm{Supp}(N/I_M N)_{\geq n}$ for all $j \leq t$. From this we obtain by the inductive hypothesis that $H_I^{j-1}(M, N/x^m N)$ is in dimension $< n$ for all $j \leq t$. In particular, $H_I^{t-1}(M, N/x^m N)$ is in dimension $< n$. Therefore we get by the exact sequence $H_I^{t-1}(M, N/x^m N) \to (0 : x^m)_H \to 0$ that $(0 : x^m)_H$ is in dimension $< n$, where $H = H_I^t(M, N)$. We now consider the short exact sequence

$$0 \to (0 : x^m)_H \to H \to x^m H \to 0.$$

Since $(0 : x^m)_H$ is in dimension $< n$ and $\dim \mathrm{Supp}(x^m H) < n$, we obtain by the above exact sequence that H is in dimension $< n$, as required. $\square$

We next introduce an extension of the notion *n-th finiteness dimension* $f_I^n(N)$ *of* N *with respect to* I of K. Bahmanpour et al., in [2].

Definition 2.3. For any $0 \leq n \in \mathbf{Z}$, we set

$$f_I^n(M, N) = \inf\{f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Supp}(N/I_M N)_{\geq n}\},$$

where $f_{I_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \inf\{0 \leq i \in \mathbf{Z} \mid H_{I_\mathfrak{p}}^i(M_\mathfrak{p}, N_\mathfrak{p})$ is not finitely generated$\}$. The number $f_I^n(M, N)$ is called *n-th finiteness dimension* $f_I^n(M, N)$ *of* M *and* N *with respect to* I. Note that

$$f_I^0(M, N) = f_I(M, N) = \inf\{0 \leq i \in \mathbf{Z} \mid H_I^i(M, N) \text{ is not finitely generated}\}.$$

Theorem 2.4. *Let* R *be a commutative Noetherian ring,* I *an ideal of* R*, and* M*,* N *finitely generated* R*-modules. Then we have*

$$f_I^n(M, N) = \inf\{0 \leq i \in \mathbf{Z} \mid H_I^i(M, N) \textit{ is not in dimension} < n\}.$$

Proof. Set $h_I^n(M,N) = \inf\{0 \le i \in \mathbf{Z} \mid H_I^i(M,N)$ is not in dimension $< n\}$. It is clear that $f_I^n(M,N) \ge h_I^n(M,N)$, since if $H_I^j(M,N)$ is in dimension $< n$ for all $j < h_I^n(M,N)$ then $H_I^j(M,N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $j < h_I^n(M,N)$ and all $\mathfrak{p} \in \operatorname{Supp}(N/I_M N)_{\ge n}$. Conversely, if $H_I^j(M,N)_{\mathfrak{p}}$ is finitely generated over $R_{\mathfrak{p}}$ for all $j < f_I^n(M,N)$ and all $\mathfrak{p} \in \operatorname{Supp}(N/I_M N)_{\ge n}$ then we get by Theorem 2.2 that $H_I^j(M,N)$ is in dimension $< n$ for all $j < f_I^n(M,N)$. Therefore $f_I^n(M,N) \le h_I^n(M,N)$, as required. □

COROLLARY 2.5 ([1, Thm. 2.5], [8, Thm. 2.10]). *Let R be a commutative Noetherian ring, I an ideal of R, and N finitely generated R-modules. Then we have $f_I^n(N) = \inf\{i \in \mathbf{N} \mid H_I^i(N)$ is not in dimension $< n\}$.*

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