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KODAI MATH. J.
32 (2009), 521–529

A CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS IN COMPLEX EUCLIDEAN SPACES WITH LEGENDRE CURVES

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Abstract

In [1], B. Y. Chen provided a new method to construct Lagrangian surfaces in $\mathbf{C}^2$ by using Legendre curves in $S^3(1) \subset \mathbf{C}^2$. In this paper, we investigate the similar methods to construct some Lagrangian submanifolds in complex Euclidean spaces $\mathbf{C}^n$ $(n \geq 3)$.

1. Introduction

A regular curve $z : I \to S^{2n-1}(r) \subset \mathbf{C}^\mathbf{n}$ in the hypersphere $S^{2n-1}(r)$ of radius r centered at the origin of $\mathbf{C}^\mathbf{n}$ is called a Legendre curve if $\langle z'(t), iz(t) \rangle = 0$ identically. The idea of special Legendre curves was introduced by B. Y. Chen in his paper [2]. The similar notion was introduced by the author, in [6], to find the explicit construction of Lagrangian isometric immersion of a real-space-form $M^n(c)$ into a complex-space-form $\tilde{M}^n(4c)$. For $l = 1, \dots, k$, let $z : I_1 \times \cdots \times I_k \to S^{2n-1}(1) \hookrightarrow \mathbf{C}^\mathbf{n}$ be a sum of l unit speed Legendre curves $z_i : I_i \to S^{2n-1}(r_i) \in \mathbf{C}^\mathbf{n}$ and a $\mathbf{C}^\mathbf{n}$-valued function z_{l+1} of the variables $t_{l+1}, \dots, t_k$, i.e.

$$z = z_1(t_1) + z_2(t_2) + \cdots + z_l(t_l) + z_{l+1}(t_{l+1}, t_{l+2}, \dots, t_k)$$

such that $\left\{ z_1', \dots, z_l', \dfrac{\partial z_{l+1}}{\partial t_{l+1}}, \dots, \dfrac{\partial z_{l+1}}{\partial t_k} \right\}$ spans tangent space satisfying

$$\langle z, iz_1' \rangle = \langle z, iz_2' \rangle = \cdots = \langle z, iz_l' \rangle = 0.$$

We define that z is an l-th Legendre translation submanifold in $S^{2n-1}(1) \subset \mathbf{C}^\mathbf{n}$.

Since $\{z_j'(t_j)\}_{j=1}^{l}$ are orthonormal tangent vector fields, we can choose $k - l$ orthonormal vector fields $\tilde{z}_{l+1}, \dots, \tilde{z}_k$ by taking Gram-Schmidt process to the tangent vector fields $\left\{ \dfrac{\partial z_{l+1}}{\partial t_{l+1}}, \dots, \dfrac{\partial z_{l+1}}{\partial t_k} \right\}$ and thus $\{z_1', \dots, z_l', \tilde{z}_{l+1}, \dots, \tilde{z}_k\}$ are orthonormal tangent frame fields.

2000 *Mathematics Subject Classification*. Primary 53C42; Secondary 53C40.

Key words and phrases. Lagrangian submanifolds, Legendre curves, special Legendre translation submanifolds.

Received February 24, 2009.

Hence, z, iz, $z_1', \dots, z_l'$, $\tilde{z}_{l+1}, \dots, \tilde{z}_k$, $iz_1', \dots, iz_l'$, $i\tilde{z}_{l+1}, \dots, i\tilde{z}_k$ form orthonormal vector fields defined on $I_1 \times \cdots \times I_k$. Thus there exist orthonormal normal vector fields $A_{k+2}, \dots, A_n$ defined along z such that z, iz, $z_1', \dots, z_l'$, $\tilde{z}_{l+1}, \dots, \tilde{z}_k$, $iz_1', \dots, iz_l'$, $i\tilde{z}_{l+1}, \dots, i\tilde{z}_k$, $A_{k+2}, \dots, A_n$, $iA_{k+2}, \dots, iA_n$ form an orthonormal basis on $\mathbf{C^n}$. With respect to the above orthonormal frame fields,

$$z_j'' = -z_j + i\lambda_j z_j' - \sum_{\alpha} a_\alpha A_\alpha + \sum_{\beta} b_\beta i A_\beta,$$

where $a_\alpha = a_\alpha(t_1, \dots, t_k)$, $b_\beta = b_\beta(t_1, \dots, t_k)$ are real valued functions.

If this expression can be reduced to

$$z_j'' = -z_j + i\lambda_j z_j' - \sum_{\alpha} a_{j,\alpha} A_{j,\alpha}, \tag{1.1}$$

where $A_\alpha = A_{1,\alpha}(t_1) + \cdots + A_{k,\alpha}(t_k)$ are associated orthonormal normal vector fields and $A_{j,\alpha}' = a_{j,\alpha}(t_j) z_j'(t_j)$ for some real valued functions $a_{j,\alpha} = a_{j,\alpha}(t_j)$ $(j = 1, \dots, l)$, then $z = z(t_1, \dots, t_k)$ is called an lth special Legendre translation submanifold in $S^{2n-1}(1) \hookrightarrow \mathbf{C^n}$. The curvature of the curves z_j are defined as $\lambda_j = \langle z_j'', iz_j' \rangle$. Here, we can easily notice that if $k = 1$, then $z(t)$ becomes a special Legendre curve as in [2]. Also, we note here that every $(n-1)$-th Legendre translation submanifold in $S^{2n-1} \subset \mathbf{C^n}$ is special.

We need the following lemma.

Lemma 1 [1]. *Let A, B be two vectors in $\mathbf{C^n}$ and z, w be two complex numbers. Then we have*

$$\begin{aligned} \langle zA, wB \rangle &= \langle z, w \rangle \langle A, B \rangle + \langle iz, w \rangle \langle A, iB \rangle, \\ \langle zA, iwB \rangle &= \langle z, w \rangle \langle A, iB \rangle + \langle z, iw \rangle \langle A, B \rangle, \end{aligned} \tag{1.2}$$

where $\langle z, w \rangle = Real(z\overline{w})$ denote the real part of the complex number $z\overline{w}$, $\overline{w}$ the complex conjugate of w, and $\langle A, B \rangle$ denotes the canonical inner product of the vector A and B in the complex Euclidean n-plane $\mathbf{C^n}$.

As a converse of the definition of Legendre translation submanifold, we have the following proposition.

Proposition 1. *If $z = z_1(t_1) + \cdots + z_{n-1}(t_{n-1}) : I_1 \times \cdots \times I_{n-1} \to S^{2n-1}(1) \subset \mathbf{C^n}$ is a sum of unit speed Legendre curves satisfying*

$$z_j'' = i\lambda_j z_j' - z \tag{1.3}$$

for some nonzero real valued functions λ_j, $j = 1, \dots, n-1$, then z is an $(n-1)$-th Legendre translation submanifold.

Proof. It suffices to show that $\langle z, iz_j'\rangle = 0$, $j = 1, \dots, n-1$. Since $\langle z, z_j'\rangle = 0$, and $\langle z, z_j''\rangle = -1$, we have $\langle z, i\lambda_j z_j' - z\rangle = -1$ and then $\lambda_j\langle z, iz_j'\rangle = 0$. Since λ_j is nonzero, $\langle z, iz_j'\rangle = 0$ identically for all $j = 1, \dots, n-1$. □

Next, we provide the example of a k-th special Legendre translation submanifold in $S^{2n+1} \subset \mathbf{C}^{\mathbf{n+1}}$. Let $\lambda_1, \dots, \lambda_k$ $(k \geq 2)$, $a_{1,k+2}, \dots, a_{1,n+1}, \dots, a_{k,k+2}, \dots, a_{k,n+1}$ be real numbers such that

$$\lambda_j > 0, \quad 1 + \sum_{\alpha=k+2}^{n+1} a_{i,\alpha} a_{j,\alpha} = 0,$$

$$1 = \frac{1}{\gamma_1} + \cdots + \frac{1}{\gamma_k} \quad \text{for } \gamma_j = 1 + \sum_{\alpha=k+2}^{n+1} a_{j,\alpha}^2,$$

$$\mu_j^2 = \lambda_j^2 + 4\gamma_j \quad \text{for } 1 \leq j \neq i \leq k.$$

Then

$$\begin{aligned} z &= z_1(t_1) + \cdots + z_k(t_k) \\ &= \frac{\mu_1 - \lambda_1}{2\mu_1\gamma_1}\left(\frac{2\gamma_1}{\mu_1 - \lambda_1}, 0, \dots, 1, a_{1,k+2}, \dots, a_{1,n+1}\right) e^{((\lambda_1+\mu_1)/2)it_1} \\ &\quad + \frac{\mu_1 + \lambda_1}{2\mu_1\gamma_1}\left(\frac{-2\gamma_1}{\mu_1 + \lambda_1}, 0, \dots, 1, a_{1,k+2}, \dots, a_{1,n+1}\right) e^{((\lambda_1-\mu_1)/2)it_1} \\ &\quad + \cdots + \frac{\mu_j - \lambda_j}{2\mu_j\gamma_j}\left(0, \dots, \frac{2\gamma_j}{\mu_j - \lambda_j}, 0, \dots, 1, a_{j,k+2}, \dots, a_{j,n+1}\right) e^{((\lambda_j+\mu_j)/2)it_j} \\ &\quad + \frac{\mu_j + \lambda_j}{2\mu_j\gamma_j}\left(0, \dots, \frac{-2\gamma_j}{\mu_j + \lambda_j}, 0, \dots, 1, a_{j,k+2}, \dots, a_{j,n+1}\right) e^{((\lambda_j-\mu_j)/2)it_j} \\ &\quad + \cdots + \frac{\mu_k - \lambda_k}{2\mu_k\gamma_k}\left(0, \dots, \frac{2\gamma_k}{\mu_k - \lambda_k}, 1, a_{k,k+2}, \dots, a_{k,n+1}\right) e^{((\lambda_k+\mu_k)/2)it_k} \\ &\quad + \frac{\mu_k + \lambda_k}{2\mu_k\gamma_k}\left(0, \dots, \frac{-2\gamma_k}{\mu_k + \lambda_k}, 1, a_{k,k+2}, \dots, a_{k,n+1}\right) e^{((\lambda_k-\mu_k)/2)it_k} \end{aligned}$$

defines a special Legendre translation submanifold in $S^{2n+1}(1) \hookrightarrow \mathbf{C}^{\mathbf{n+1}}$, that is, it satisfies

$$z_j'' = -z_j + i\lambda_j z_j' - \sum a_{j,\alpha} A_{j,\alpha}$$
$$\langle z, iz_j'\rangle = 0 \quad \text{for } j = 1, \dots, k,$$

where $A_\alpha = A_{1,\alpha} + \cdots + A_{k,\alpha}$ are some associated orthonormal normal vector fields satisfying $A_{j,\alpha} = a_{j,\alpha} z_j$ for $j = 1, \dots, k$ and $\alpha = k+2, \dots, n+1$.

2. Main results

In this chapter, we construct some Lagrangian submanifolds in complex Euclidean space $\mathbf{C^n}$ using an $(n-1)$ special Legendre translation submanifold.

THEOREM 1. *Let* $f : I_1 \to \mathbf{C}^*$ *be a regular curve defined on an open interval* I_1 *and* $z : I_2 \times \cdots \times I_n \to S^{2n-1}(r) \subset \mathbf{C}^n$ *be a sum of unit speed Legendre curves* z_j *in* $S^{2n-1}(r_j)$ *defined on an open interval* I_j. *Then we have the following.*

(a) *If* $z = z_2(t_2) + z_3(t_3) + \cdots + z_n(t_n)$ *is an* $(n-1)$ *Legendre translation submanifold, then, for any function* $p_j : I_j \to \mathbf{C}$, $j = 2, \dots, n$ *the map*

$$L(s, t_2, \dots, t_n) = f(s) z(t_2, \dots, t_n) - \sum_{j=2}^{n} \int_0^{t_j} p_j(t) z_j'(t)\, dt \tag{2.1}$$

is a Lagrangian isometric immersion of $M^n = (U, g)$ *into* $\mathbf{C}^n$, *where the set* U *is defined as*

$$U := \{(s, t_2, \dots, t_n) \in I_1 \times I_2 \cdots \times I_n : f(s) \neq p_j(t_j) \text{ for all } j = 2, \dots, n\}$$

and the metric g *is the induced metric given by*

$$g = r^2 |f'(s)|^2\, ds^2 + \sum_{j=2}^{n} |f(s) - p_j(t_j)|^2\, dt_j \tag{2.2}$$

(b) *Conversely, if* f *does not contain any circular arcs, and if* L *as in* (2.1) *is a Lagrangian immersion, then* $z = z_2(t_2) + \cdots + z_n(t_n) : I_2 \times \cdots \times I_n \to S^{2n-1}(r) \subset \mathbf{C}^n$ *is an* $(n-1)$ *Legendre translation submanifold.*

Proof. Let $f : I_1 \to \mathbf{C}^*$ be a regular curve defined on an open interval I_1 and $z : I_2 \times \cdots \times I_n \to S^{2n-1}(r) \subset \mathbf{C}^n$ be a smooth $\mathbf{C}^n$-valued map defined on a product of open intervals $I_2, \dots, I_n$. Using (2.1), we have

$$L_s = \frac{\partial L}{\partial s} = f'(s) z(t_2, \dots, t_n), \quad L_{t_j} = (f(s) - p_j(t_j)) z_j'(t_j), \quad j = 2, \dots, n. \tag{2.3}$$

Now, applying Lemma 1 and using (2.3) yield

$$\begin{cases} \langle L_s, L_s \rangle = r^2 |f'(s)|^2, \\ \langle L_s, L_{t_j} \rangle = \langle i f', f(s) - p_j(t_j) \rangle \langle z, i z_j'(t_j) \rangle, \\ \langle L_{t_j}, L_{t_j} \rangle = |f(s) - p_j(t_j)|^2, \quad j = 2, \dots, n \\ \langle L_{t_j}, L_{t_k} \rangle = \langle i(f(s) - p_j), f(s) - p_k(t_k) \rangle \langle z_j, i z_k' \rangle, \quad j \neq k = 2, \dots, n \end{cases} \tag{2.4}$$

Since z is an $(n-1)$ Legendre translation submanifold, we can find the induced metric g on U given as in (2.2) from (2.4) and also, $\langle L_s, i L_{t_j} \rangle = \langle L_{t_j}, i L_{t_k} \rangle = 0$ for all $j, k = 2, \dots, n$ which imply that the map L is Lagrangian.

For (b), suppose L, defined in (2.2) is a Lagrangian isometric immersion. The similar computation shows that $\langle L_s, iL_{t_j}\rangle = \langle f'(s), f(s) - p_j(t_j)\rangle\langle z, iz_j'(t_j)\rangle \equiv 0$ identically. If there exist one j such that $\langle f'(s), f(s) - p_j(t_j)\rangle = 0$ for all s in an open subinterval $I_0 \subset I_1$, then $\frac{d}{ds}|f(s) - p_j(t_j)|^2 = 0$ which means for each $t_j \in I_j$, the curve f is contained in a circle centered at $p_j(t_j)$. It is impossible so that if f does not contain any circular arcs, then we have $\langle z, iz_j'(t_j)\rangle \equiv 0$ for all $j = 2, \dots, n$ and thus z becomes an $(n-1)$ Legendre translation submanifold. □

The next theorem shows the extrinsic properties of the immersion.

THEOREM 2. *Let $f : I_1 \to \mathbf{C}^*$ be a unit speed curve, $z : I_2 \times \cdots \times I_n \to S^{2n-1}(1) \subset \mathbf{C}^n$ an $(n-1)$th Legendre translation submanifold, $p_j : I_j \to \mathbf{C}$, $j = 2, \dots, n$ complex valued functions, and $L : (U, g) \to \mathbf{C}^\mathbf{n}$ be the Lagrangian isometric immersion defined by*

$$L(s, t_2, \dots, t_n) = f(s)z(t_2, \dots, t_n) - \sum_{j=2}^{n} \int_0^{t_j} p_j(t)z_j'(t)\, dt. \tag{2.5}$$

Then we find

(a) *L_s is an eigenvector of the shape operator A_{JL_s} with eigenvalue κ, where κ is the curvature function of f.*

(b) *For $j = 2, \dots, n$, L_{t_j} is an eigenvector of the shape operator $A_{JL_{t_j}}$ if and only if $p_2 = \cdots = p_n = p$ are constants and $f(s) = cs + p$ for some $c \in \mathbf{C}$ with $|c| = 1$.*

(c) *L is totally geodesic if and only if $n = 2$, $p_2 = p$ is a constant, $f(s) = cs + p$, $|c| = 1$, z is a great circle in $S^3(1)$ and $L = (f - p)z = csz$ for a constant c.*

Proof. From (2.5), we have

$$\begin{aligned} &L_{ss} = f''(s)z, \quad L_{st_j} = f'(s)z_j', \\ &L_{t_jt_j} = -p_j'z_j' + (f - p_j)z_j'', \quad L_{t_it_j} = 0, \quad i \neq j = 2, \dots, n, \end{aligned} \tag{2.6}$$

By applying Lemma 1 to (2.6), we obtain

$$\begin{aligned} &\langle L_{ss}, iL_s\rangle = \langle f''(s)z, if'z\rangle = \kappa, \\ &\langle L_{ss}, iL_{t_j}\rangle = \langle L_{st_j}, iL_s\rangle = \langle L_{st_j}, iL_{tk}\rangle = 0, \quad k \neq j \\ &\langle L_{t_jt_j}, iL_s\rangle = \langle L_{st_j}, iL_{t_j}\rangle = \langle f', i(f - p_j)\rangle, \\ &\langle L_{t_jt_j}, iL_{t_j}\rangle = k_j(t_j)\|f - p_j\|^2 - \langle p_j', i(f - p_j)\rangle, \quad j = 2, \dots, n, \end{aligned} \tag{2.7}$$

where $k_j(t_j) = \langle z_j'', iz_j'\rangle$ is the curvature function of the curve z_j. Let $e_1 = L_s$, $e_j = \frac{L_{t_j}}{|f - p_j|}$, $j = 2, \dots, n$. Then $e_1, e_2, \dots, e_n$ are orthonormal frame fields.

Therefore, the second fundamental form h is

$$(2.8)\qquad \begin{aligned} &h(e_1, e_1) = \kappa(s)Je_1, \\ &h(e_1, e_j) = \mu_j Je_j, \\ &h(e_j, e_j) = \mu_j Je_1 + \alpha_j Je_j, \\ &h(e_i, e_j) = 0, \quad \text{for } i \neq j = 2, \dots, n, \end{aligned}$$

where $\mu_j = \dfrac{\langle f', i(f - p_j)\rangle}{|f - p_j|^2}$, $\alpha_j = \dfrac{1}{|f - p_j|^3}(\kappa_j(t_j)|f - p_j|^2 - \langle p_j', i(f - p_j)\rangle)$ and $\kappa_j = \langle z_j'', iz_j'\rangle$ for $j = 2, \dots, n$. We can easily see that L_s is an eigenvector of A_{JL_s} with eigenvalue κ which is the curvature function of f.

For (b), L_{t_j} is an eigenvector of $A_{JL_{t_j}}$ if and only if $\langle f', i(f - p_j)\rangle = 0$ for $j = 2, \dots, n$ which implies that the position vector $\gamma_j(s) = f(s) - p_j(t_j)$ is always tangent to the curve γ_j for any fixed t_j. Thus, for each t_j, γ_j is a part of a line through 0 of $\mathbf{C}$. Therefore, there exist unit vector fields c_j in $\mathbf{C}$ such that $\gamma_j(s) = f(s) - p_j(t_j) = c_j(t_j)s$ which yields $c_2(t_2) = \cdots = c_n(t_n) = c$, $p_2 = \cdots = p_n = p$, where c and p are constants in $\mathbf{C}$. Thus, $f(s) = cs + p$ and $|c| = 1$.

Suppose L is totally geodesic. From the second statement (b), we know that $\kappa = 0$. By (2.8), it suffices to show that $\alpha_j = 0$ for $j = 2, \dots, n$ which is equivalent to $\kappa_j(t_j) = 0$. It is impossible unless $n = 2$ which means that $z : I_2 \to S^3(1) \subset \mathbf{C}^2$ is a great circle in $S^3(1)$. □

3. Application

The following result shows examples of Lagrangian submanifolds in complex Euclidean space using the main results discussed before.

THEOREM 3. *Let $f : I_1 \to \mathbf{C}^*$ be a unit speed curve, $z : I_2 \times \cdots \times I_n \to S^{2n-1}(1) \subset \mathbf{C}^n$ a $(n-1)$-th special Legendre translation submanifold, $p_j : I_j \to \mathbf{C}$, $j = 2, \dots, n$ complex valued functions. Then $L : (U, g) \to \mathbf{C}^n$, defined as in* (2.5), *is minimal if and only if, up to rigid motions of $\mathbf{C}^n$, one of the following holds*:

(a) *If $n = 2$, then L is either a totally geodesic immersion or an open portion of the Lagrangian catenoid, up to dilations and rigid motions.*

(b) *If $n \geq 3$, then $L = (f - p) \otimes z$ is a complex extensor where $f(s(x)) = p + x + iy(x)$ is a unit speed curve satisfying a differential equation*

$$(y - xy')^{n+1} = c(y'')^{n-1}(1 + y'(x)^2)^{2-n}$$

for a constant c and $z = z_2 + \cdots + z_n$ is a sum of circles z_j's in $\mathbf{C}^n$.

Proof. The induced metric on U is

$$g = ds^2 + \sum_2^n |f(s) - p_j(t_j)|^2\, dt_j^2.$$

Then $e_1 = L_s$, $e_j = \dfrac{L_{t_j}}{|f - p_j|}$, $j = 2, \dots, n$ are orthonormal frame fields and using these frames and the induced metric g, we have

$$\begin{aligned}
&\nabla_{e_1} e_1 = \nabla_{e_1} e_j = \nabla_{e_i} e_j = 0, \quad \text{for } i \neq j = 2, \dots, n,\\
&\nabla_{e_j} e_j = \frac{-\langle f', f - p_j\rangle}{|f - p_j|^2} e_1,\\
&\nabla_{e_j} e_1 = \frac{\langle f', f - p_j\rangle}{|f - p_j|^2} e_j, \quad j = 2, \dots, n.
\end{aligned} \tag{3.1}$$

Using the Codazzi equation and (2.8), we have, for $j = 2, \dots, n$, $(\nabla_{e_j} h)(e_1, e_j) = (\nabla_{e_1} h)(e_j, e_j)$ which gives

$$e_1(\mu_j) = (\kappa - 2\mu_j) \frac{\langle f', f - p_j\rangle}{|f - p_j|^2} \tag{3.2}$$

$$e_1(\alpha_j) = e_j(\mu_j) - \alpha_j \frac{\langle f', f - p_j\rangle}{|f - p_j|^2} \tag{3.3}$$

Because of the minimality condition, (2.8) implies

$$\kappa + \mu_2 + \cdots + \mu_n = 0, \quad e_j(\mu_j) = 0, \quad \alpha_j = 0, \quad j = 2, \dots, n \tag{3.4}$$

and then

$$\kappa_j(t_j)|f(s) - p_j|^2 - \langle p_j', i(f - p_j)\rangle = 0. \tag{3.5}$$

$$(\kappa - 2\mu_j)\langle p_j', f'\rangle = 0 \tag{3.6}$$

By differentiating the equation (3.5) with respect to s, we obtain

$$\langle if', p_j'\rangle = 2\kappa_j \langle f', f(s) - p_j\rangle \tag{3.7}$$

Another differentiating the equation (3.7) with respect to s and replacing f'' by $i\kappa f'$ yields

$$\kappa\langle f', p_j'\rangle + 2\kappa_j + 2\kappa\kappa_j \langle if', f(s) - p_j\rangle = 0. \tag{3.8}$$

Now, we can consider two cases.

Case (a) Suppose $\kappa = 0$. It is immediate to know that $\kappa_j = 0$ for all $j = 2, \dots n$ from (3.8). Since $\alpha_j = 0$, $j = 2, \dots, n$, (3.3) and (3.7) imply that

$$0 = \frac{\partial}{\partial t_j}\left(\frac{\langle f', i(f - p_j)\rangle}{|f - p_j|^2}\right) = \mu_j \frac{2\langle f - p_j, p_j'\rangle}{|f - p_j|^2}$$

If there exist $\mu_j \neq 0$, then the above equation implies $\langle f - p_j, p_j' \rangle = 0$ and thus $|f(s) - p_j(t_j)|^2_{t_j} = 0$ which means that for each s, every curve p_j is contained in a circle centered at $f(s)$ which is impossible. Therefore, we can conclude that for all $j = 2, \ldots, n$, $\mu_j = 0$. Therefore, L is totally geodesic and then by theorem 2, n must be 2.

Case (b) Assume that $\kappa \neq 0$. Differentiating (3.8) with respect to s and replacing f'' by $i\kappa f'$, we find that

$$0 = \kappa^2 \langle if', p_j' \rangle - 2\kappa^2 \kappa_j \langle f', f - p_j \rangle + \kappa' \langle f', p_j' \rangle + 2\kappa' \kappa_j \langle if', f - p_j \rangle$$

By applying (3.7) and (3.8), the above becomes that

$$\begin{aligned} 2\kappa^2 \kappa_j \langle f', f - p_j \rangle &+ 2\kappa' \kappa_j \langle f', i(f - p_j) \rangle \\ &= \kappa^2 \langle if', p_j' \rangle + \kappa' \langle f', p_j' \rangle \\ &= 2\kappa^2 \kappa_j \langle f', f - p_j \rangle + 2\kappa' \kappa_j \left(\langle f', i(f - p_j) \rangle - \frac{1}{\kappa} \right) \\ &= 2\kappa^2 \kappa_j \langle f', f - p_j \rangle - \frac{2\kappa' \kappa_j}{\kappa} + 2\kappa' \kappa_j \langle f', i(f - p_j) \rangle \end{aligned}$$

Thus, we have $\kappa' \kappa_j = 0$. Suppose there exits a j such that $\kappa_j \neq 0$. Then κ must be a nonzero constant which implies f is a circle in $\mathbf{C}^*$. For each s, and t_j, $f(s) - p_j(t_j) = \langle f(s) - p_j, f' \rangle f' + \langle f(s) - p_j, if' \rangle if'$ which implies that $p_2 = \cdots = p_n$ equal to a constant p since f is a unit circle. Then (3.5) implies that $\kappa_j = 0$ for all $j = 2, \ldots, n$ which is a contradiction to our assumption. Therefore, $\kappa_j = 0$ for all $j = 2, \ldots, n$. Now, each z_j is a circle in $\mathbf{C^n}$ and by (3.5), and (3.8), $\langle f', ip_j' \rangle = \langle f', p_j' \rangle = 0$, $j = 2, \ldots, n$ which implies that for each s and t_j, $f(s) - p_j(t_j) = c_1(s) f'(s) + c_2(s) if'(s)$ for some two functions c_1 and c_2 of s, yielding $p_2(t_2) = \cdots = p_n(t_n) = p$ become a constant. Therefore, $\mu_2 = \cdots = \mu_n = \dfrac{\langle f', i(f - p) \rangle}{|f - p|^2} = \mu(s)$. Now, then (2.5) becomes $L(s, t_2, \ldots, t_n) = (f - p) \cdot (z_2 + \cdots + z_n)$. Therefore, from (3.4), we obtain

$$|f - p|^2 \kappa(s) + (n - 1) \langle f', i(f - p) \rangle = 0 \tag{3.9}$$

By differentiating the equation (3.9) with respect to s, we get

$$\kappa'(s) |f - p|^2 + (n + 1) \kappa \langle f', f - p \rangle = 0$$

and then it yields

$$|f - p|^2 = \left(\frac{a^2}{\kappa(s)} \right)^{2/(n+1)} \tag{3.10}$$

for a real constant a. By substituting (3.10) into (3.9), we get

$$\langle if', f - p \rangle = \frac{1}{n - 1} a^{4/(n+1)} \kappa(s)^{(n-1)/(n+1)} \tag{3.11}$$

Let's reparametrize $f(s(x)) = p + x + iy(x)$. Then

$$\kappa = \langle f'', if' \rangle = \frac{y''(x)}{(1 + y'(x)^2)^{3/2}}, \quad \langle if', f - p \rangle = \frac{y - xy'}{(1 + y'(x)^2)^{1/2}} \tag{3.12}$$

From (3.11) and (3.12), we obtain that

$$(y - xy')^{n+1} = c(y'')^{n-1}(1 + y'(x)^2)^{2-n}, \quad c = \left(\frac{1}{n-1}\right)^{n+1} a^4 \tag{3.13}$$

If $n = 2$, then this equation was completely solved by Bang-Yen Chen in his paper [1] which is that up to dilations and rigid motions, the Lagrangian immersion L is an open portion of the Lagrangian catenoid.

Now, we assume that $n \geq 3$. Put $y_1 = y$, $y_2 = y'$ and $y_3 = y''$. Then the above equation is equivalent to the system:

$$\begin{aligned} &y_1' = y_2, \quad y_2' = y_3, \\ &y_3' = \frac{-(1 + y_2^2)^{n-2}((n+1)xy_3(y_1 - xy_2)^n + 2cy_3^{n-1}(2-n)(1 + y_2^2)^{1-n} y_2 y_3)}{c(n-1)y_3^{n-2}} \end{aligned} \tag{3.14}$$

It follows from Picard's theorem that, for a given initial conditions: $y_3(s_0) > 0$, $y_2(s_0) = y_2^o$, $y_1(s_0) = y_1^o$ for any constants y_1^o and y_2^o, the initial value problem has a unique solution in some open interval around s_0. □

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