A. RAFIQ
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IMPLICIT FIXED POINT ITERATIONS FOR PSEUDOCONTRACTIVE MAPPINGS

Arif Rafiq

Abstract

In this paper, we prove some results for implicit fixed point iterations associated with pseudocontractive mappings.

1. Introduction

Let H be a Hilbert space. A mapping $T : H \to H$ is said to be *pseudocontractive* (see e.g., [1, 2]) if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H \tag{1.1}$$

and is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Let $F(T) := \{x \in H : Tx = x\}$ and let K be a nonempty subset of H. A map $T : K \to K$ is called *hemicontractive* if $F(T) \ne \emptyset$ and

$$\|Tx - x^*\|^2 \le \|x - x^*\|^2 + \|x - Tx\|^2 \quad \forall x \in K,\ x^* \in F(T). \tag{1.3}$$

It is easy to see that the class of pseudocontractive maps with fixed points is a subclass of the class of hemicontractions. The following example, due to Rhoades [18], shows that the inclusion is proper. For $x \in [0, 1]$, define $T : [0, 1] \to [0, 1]$ by $Tx = (1 - x^{2/3})^{3/2}$. It is shown in [18] that T is not Lipschitz and so cannot be nonexpansive. A straightforward computation (see e.g., [19]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractions the reader may consult [1].

In the last ten years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive (and correspondingly Lipschitz *strongly* accretive) maps using the *Mann iteration process* (see e.g., [11]). In 1974, Ishikawa [8] introduced an iteration process which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T. He proved the following theorem.

Key words: Banach space, Hilbert Space, Implicit Mann type iteration process with errors, Pseudocontractive mappings

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THEOREM 1. *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive map and x_0 is any point in K, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 0$ by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$\text{(i)}\ \ 0 \leq \alpha_n \leq \beta_n < 1; \quad \text{(ii)}\ \ \lim_{n\to\infty} \beta_n = 0; \quad \text{(iii)}\ \ \sum_{n\geq 0} \alpha_n \beta_n = \infty.$$

Since its publication in 1974, Theorem 1, as far as we know, has never been extended to more general Banach spaces. The iteration process (1.4) is generally referred to as the *Ishikawa iteration process* in light of [8].

Another iteration process which has been studied extensively in connection with fixed points of pseudocontractive maps is the following:

For a nonempty convex subset K of a Banach space E, and $T : K \to K$, the sequence $\{x_n\}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, \quad n \geq 1, \tag{1.5}$$

where $\{c_n\}$ is a real sequence satisfying the following conditions:

$$\text{(iv)}\ \ 0 \leq c_n < 1; \quad \text{(v)}\ \ \lim_{n\to\infty} c_n = 0; \quad \text{(vi)}\ \ \sum_{n=1}^{\infty} c_n = \infty.$$

The iteration process (1.5) is generally referred to as the *Mann iteration process* in light of [11].

In 1995, Liu [10] introduced what he called *Ishikawa and Mann iteration processes with errors* as follows:

(1-a) For a nonempty convex subset K of E and $T : K \to E$, the sequence $\{x_n\}$ defined by

$$\begin{aligned} & x_1 \in K, \\ & x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, \\ & y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \quad n \geq 1, \end{aligned} \tag{1.6}$$

where, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty$, $\sum \|v_n\| < \infty$ is called the *Ishikawa Iteration process with errors*.

(1-b) With K, E and T as in part (1-a), the sequence $\{x_n\}$ defined by

$$\begin{aligned} & x_1 \in K, \\ & x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, \quad n \geq 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying appropriate conditions and $\sum \|u_n\| < \infty$, is called the *Mann iteration process with errors*.

While it is known that consideration of error terms in iterative processes is an important part of the theory, it is also clear that the iteration processes with errors introduced by Liu in (1-a) and (1-b) are unsatisfactory. The occurrence of errors is random so that the conditions imposed on the error terms in (1-a) and (1-b) which imply, in particular, that they tend to zero as n tends to infinity are, therefore, unreasonable.

In 1997, Y. Xu [23] introduced the following more satisfactory definitions.

(1-c) Let K be a nonempty convex subset of E and $T : K \to K$ a mapping. For any given $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T x_n + c_n' v_n, \quad n \geq 1, \end{aligned} \tag{1.8}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in K and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$ and $\{c_n'\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n = a_n' + b_n' + c_n' = 1$ $\forall n \geq 1$ is called the *Ishikawa iteration sequence with errors in the sense of Xu*.

(1-d) If, with the same notations and definitions as in (1-c), $b_n' = c_n' = 0$, for all integers $n \geq 1$, then the sequence $\{x_n\}$ now defined by

$$\begin{aligned} &x_1 \in K \\ &x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

is called the *Mann iteration sequence with errors in the sense of Xu*. We remark that if K is bounded (as is generally the case), the error terms u_n, v_n are *arbitrary* in K.

In [3], Chidume and Chika Moore proved the following theorem.

Theorem 2. *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$ and $\{c_n'\}$ be real sequences in $[0,1]$ satisfying the following conditions*:

(vii) $a_n + b_n + c_n = 1 = a_n' + b_n' + c_n'$ $\forall n \geq 1$;

(viii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_n' = 0$;

(ix) $\sum c_n < \infty$; $\sum c_n' < \infty$;

(x) $\sum \alpha_n \beta_n = \infty$; $\sum \alpha_n \beta_n \delta_n < \infty$, *where* $\delta_n := \|Tx_n - Ty_n\|^2$;

(xi) $0 \leq \alpha_n \leq \beta_n < 1$ $\forall n \geq 1$, *where* $\alpha_n := b_n + c_n$; $\beta_n := b_n' + c_n'$.

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ iteratively by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n, \\ y_n &= a_n' x_n + b_n' T x_n + c_n' v_n, \quad n \geq 1, \end{aligned}$$

where $\{u_n\}$, $\{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}$ converges strongly to a fixed point of T.

They also gave the following remark in [3].

Remark 3. 1. In connection with the iterative approximation of fixed points of pseudocontractions, the following question is still open. Does the Mann iteration process always converge for continuous pseudocontractions, or for even Lipschitz pseudocontractions?

2. Let E be a Banach space and K be a nonempty compact convex subset of E. Let $T: K \to K$ be a Lipschitz pseudocontractive map. Under this setting, even for $E = H$, a Hilbert space, the answer to the above question is not known. There is, however, an example [7] of a discontinuous pseudocontractive map T with a unique fixed point for which the Mann iteration process does not always converge to the fixed point of T. Let H be the complex plane and $K := \{z \in H : |z| \leq 1\}$. Define $T: K \to K$ by

$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta+\pi/3)}, & \text{for } 0 \leq r \leq \frac{1}{2}, \\ e^{i(\theta+2\pi/3)}, & \text{for } \frac{1}{2} < r \leq 1. \end{cases}$$

Then, zero is the only fixed point of T. It is shown in [5] that T is pseudocontractive and that with $c_n = \dfrac{1}{n+1}$, the sequence $\{z_n\}$ defined by $z_{n+1} = (1 - c_n)z_n + c_nTz_n$, $z_0 \in K$, $n \geq 1$, does not converge to zero. Since the T in this example is not continuous, the above question remains open.

In [4], Chidume and Mutangadura, provide an example of a Lipschitz pseudocontractive map with a unique fixed point for which the Mann iteration sequence failed to converge and they stated there "This resolves a long standing open problem".

In [16], the author proved the following theorem.

THEOREM 4. *Let K be a compact convex subset of a real Hilbert space H, $T: K \to K$ a hemicontractive mapping. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$. For arbitrary $x_0 \in K$, the sequence $\{x_n\}$ is defined by*

$$\begin{aligned} &x_0 \in K, \\ &x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n, \quad n \geq 1. \end{aligned} \tag{1.10}$$

Then $\{x_n\}$ converges strongly to a fixed point of T.

Very recently, Yao et al. [24] introduced the iterative scheme (1.11) below and extended the Theorem 4 to more general Banach spaces.

Let C be a closed convex subset of a real Banach space E and $T: C \to C$ be a mapping. Define $\{x_n\}$ in the following way:

$$\begin{aligned} &x_0 \in C, \\ &x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n Tx_n, \quad n \geq 1, \end{aligned} \tag{1.11}$$

where u is an anchor and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $(0,1)$ satisfying some appropriate conditions.

They proved the following theorem.

THEOREM 5. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0,1)$ satisfying the following conditions*:

(xii) $\alpha_n + \beta_n + \gamma_n = 1$;
(xiii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\lim_{n\to\infty} \dfrac{\alpha_n}{\beta_n} = 0$;
(xiv) $\sum_{n=0}^{\infty} \dfrac{\alpha_n}{\alpha_n + \beta_n} = \infty$.

For arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by (1.11). *Then $\{x_n\}$ converges strongly to a fixed point of T.*

In this paper, we introduce some implicit Mann type iteration processes with errors associated with pseudocontractive mappings to have the strong convergences in the setting of Hilbert and Banach spaces respectively.

2. Preliminaries

We shall make use of the following results.

LEMMA 6 [21]. *Suppose that $\{\rho_n\}$, $\{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$,*

$$\rho_{n+1} \leq \rho_n + \sigma_n \quad \forall n \geq N_0.$$

(a) *If $\sum \sigma_n < \infty$, then, $\lim \rho_n$ exists.*
(b) *If $\sum \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.*

LEMMA 7 [22]. *Let β_n be a nonnegative sequence satisfying*

$$\beta_{n+1} \leq (1 - \delta_n)\beta_n + \sigma_n,$$

with $\delta_n \in [0,1]$, $\sum_{i=1}^{\infty} \delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

LEMMA 8 [8]. *Let H be a Hilbert space, for all $x, y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds*:

$$\|(1-\lambda)x + \lambda y\|^2 = (1-\lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2.$$

LEMMA 9 [12]. *Let H be a Hilbert space, then for all $x, y, z \in H$*

$$\|ax + by + cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2,$$

where $a, b, c \in [0,1]$ and $a + b + c = 1$.

Remark 10. *For $c = 0$ in Lemma 9, we get the results of Lemma 8.*

3. Mann-type iteration process for pseudocontractive mappings in Hilbert spaces

Now we prove our main results.

THEOREM 11. *Let K be a compact convex subset of a real Hilbert space H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n + c_n = 1$ and satisfying*

(xv) $\{b_n\} \subset [\delta, 1-\delta]$ *for some* $\delta \in \left(0, \frac{1}{2}\right)$,

(xvi) $\sum c_n < \infty$.

For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by

$$\begin{aligned} &x_0 \in K, \\ &x_n = a_n x_{n-1} + b_n T x_n + c_n u_n, \quad n \geq 1, \end{aligned} \tag{3.1}$$

where $\{u_n\}$ is an arbitrary sequence in K. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $T : K \to K$ is a continuous mapping, then for every fixed $u \in K$ and $t \in (0,1)$, the operator $S_t : K \to K$ defined for all $x \in K$ by

$$S_t x = tu + (1-t)Tx,$$

is also continuous, so that S_t has a fixed point x_t in K (by Schauder's fixed point theorem (Let K be a compact convex subset of a normed linear space E and let T be a continuous mapping of K into itself. Then T has a fixed point in K.)), i.e.,

$$x_t = tu + (1-t)Tx_t.$$

Thus the implicit iteration process (1.10) is defined in K for the continuous self-mappings of a nonempty convex subset K of a Hilbert space provided that $\alpha_n \in (0,1)$ for all $n \geq 1$. Since $\{u_n\}$ is just the bounded sequence of error terms, it can be easily seen that the implicit iteration process (3.1) is also well defined.

Let $x^* \in K$ be a fixed point of T and $M = \dim(K)$. Using the fact that T is hemicontractive we obtain

$$\|Tx_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2. \tag{3.2}$$

With the help of (1.3), Lemma 9 and (3.2), we obtain the following estimates:

$$\begin{aligned} \|x_n - x^*\|^2 &= \|a_n x_{n-1} + b_n T x_n + c_n u_n - x^*\|^2 \\ &= \|a_n(x_{n-1} - x^*) + b_n(Tx_n - x^*) + c_n(u_n - x^*)\|^2 \\ &= a_n\|x_{n-1} - x^*\|^2 + b_n\|Tx_n - x^*\|^2 + c_n\|u_n - x^*\|^2 \\ &\quad - a_n b_n\|x_{n-1} - Tx_n\|^2 - b_n c_n\|Tx_n - u_n\|^2 - a_n c_n\|x_{n-1} - u_n\|^2 \end{aligned} \tag{3.3}$$

$$\begin{aligned}
&\leq a_n\|x_{n-1}-x^*\|^2+b_n\|Tx_n-x^*\|^2+c_n\|u_n-x^*\|^2\\
&\quad -a_nb_n\|x_{n-1}-Tx_n\|^2\\
&\leq (1-b_n)\|x_{n-1}-x^*\|^2+b_n\|Tx_n-x^*\|^2+M^2c_n\\
&\quad -a_nb_n\|x_{n-1}-Tx_n\|^2\\
&\leq (1-b_n)\|x_{n-1}-x^*\|^2+b_n(\|x_n-x^*\|^2+\|x_n-Tx_n\|^2)\\
&\quad +M^2c_n-a_nb_n\|x_{n-1}-Tx_n\|^2\\
&= (1-b_n)\|x_{n-1}-x^*\|^2+b_n\|x_n-x^*\|^2+b_n\|x_n-Tx_n\|^2\\
&\quad +M^2c_n-a_nb_n\|x_{n-1}-Tx_n\|^2.
\end{aligned}$$

Also

$$\begin{aligned}
\|x_n-Tx_n\|^2 &= \|a_nx_{n-1}+b_nTx_n+c_nu_n-Tx_n\|^2\\
&= \|(1-b_n)(x_{n-1}-Tx_n)+c_n(u_n-x_{n-1})\|^2\\
&\leq [(1-b_n)\|x_{n-1}-Tx_n\|+c_n\|u_n-x_{n-1}\|]^2\\
&\leq [(1-b_n)\|x_{n-1}-Tx_n\|+Mc_n]^2\\
&\leq (1-b_n)^2\|x_{n-1}-Tx_n\|^2+3M^2c_n.
\end{aligned}\tag{3.4}$$

Substituting (3.4) in (3.3), we get

$$\begin{aligned}
\|x_n-x^*\|^2 &\leq (1-b_n)\|x_{n-1}-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +3M^2b_nc_n+M^2c_n-b_n[a_n-(1-b_n)^2]\|x_{n-1}-Tx_n\|^2.
\end{aligned}\tag{3.5}$$

By $\sum c_n<\infty$, $\lim_{n\to\infty}c_n=0$, so there exists a natural number $n_0\in\mathbf{N}$ such that for all $n\geq n_0$, we have $c_n\leq\eta$; $\eta\in(0,\delta^2)$ and $\{b_n\}\subset[\delta,1-\delta]$ for some $\delta\in\left(0,\frac{1}{2}\right)$ lead to $a_n-(1-b_n)^2\geq\delta^2-\eta$, and (3.5) gives us

$$\begin{aligned}
\|x_n-x^*\|^2 &\leq (1-b_n)\|x_{n-1}-x^*\|^2+b_n\|x_n-x^*\|^2\\
&\quad +3M^2b_nc_n+M^2c_n-\delta(\delta^2-\eta)\|x_{n-1}-Tx_n\|^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
\|x_n-x^*\|^2 &\leq \|x_{n-1}-x^*\|^2+M^2\frac{3b_n+1}{1-b_n}c_n-\delta(\delta^2-\eta)\|x_{n-1}-Tx_n\|^2\\
&\leq \|x_{n-1}-x^*\|^2+M^2\frac{\gamma}{\delta}c_n-\delta(\delta^2-\eta)\|x_{n-1}-Tx_n\|^2;
\end{aligned}\tag{3.6}$$

$\gamma=3(1-\delta)+1$, holds for all fixed points x^* of T. Hence

$$\delta(\delta^2-\eta)\|x_{n-1}-Tx_n\|^2\leq\|x_{n-1}-x^*\|^2-\|x_n-x^*\|^2+M^2\frac{\gamma c_n}{\delta},$$

and thus

$$\delta(\delta^2-\eta)\sum_{j=n_0}^{\infty}\|x_{j-1}-Tx_j\|^2 \le \frac{M^2\gamma}{\delta}\sum_{j=n_0}^{\infty}c_j+\sum_{j=n_0}^{\infty}(\|x_{j-1}-x^*\|^2-\|x_j-x^*\|^2)$$
$$=\frac{M^2\gamma}{\delta}\sum_{j=n_0}^{\infty}c_j+\|x_{n_0-1}-x^*\|^2.$$

Hence

$$\sum_{j=n_0}^{\infty}\|x_{j-1}-Tx_j\|^2<+\infty. \tag{3.7}$$

It implies that

$$\lim_{n\to\infty}\|x_{n-1}-Tx_n\|=0.$$

From $\|x_n-Tx_n\|\le(1-b_n)\|x_{n-1}-Tx_n\|+Mc_n$ and the condition (xvi) it further implies that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

By compactness of K this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a fixed point of T, say y^*. Since (3.6) holds for all fixed points of T we have

$$\|x_n-y^*\|^2\le\|x_{n-1}-y^*\|^2+M^2\frac{\gamma}{\delta}c_n-\delta(\delta^2-\eta)\|x_{n-1}-Tx_n\|^2,$$

and in view of (3.7) and Lemma 6 we conclude that $\|x_n-y^*\|\to 0$ as $n\to\infty$, i.e., $x_n\to y^*$ as $n\to\infty$. The proof is complete.

Corollary 12 [16]. *Let K be a compact convex subset of a real Hilbert space H; $T: K\to K$ a continuous hemicontractive map. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ satisfying $\{\alpha_n\}\subset[\delta,1-\delta]$ for some $\delta\in\left(0,\frac{1}{2}\right)$. For arbitrary $x_0\in K$, define the sequence $\{x_n\}$ by (1.10). Then $\{x_n\}$ converges strongly to a fixed point of T.*

4. Mann-type iteration process for pseudocontractive mappings in Banach spaces

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J: E\to E^*$ is defined as

$$J(x):=\{x^*\in E^*;\langle x,x^*\rangle=\|x\|^2=\|x^*\|^2\}.$$

Let C be a closed convex subset of E. The mapping $T: C\to C$ is called pseudocontractive if

$$\|x-y\|\le\|x-y+t((I-T)x-(I-T)y)\|, \tag{4.1}$$

holds for every $x, y \in C$ and $t > 0$. An equivalent definition of pseudocontractive mappings is due to Kato [9],

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2, \tag{4.2}$$

for $x, y \in C$ and $j(x - y) \in J(x - y)$.

Let $U = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The norm on E is said to be G'ateaux differentiable if the

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{4.3}$$

exists for each $x, y \in U$ and in this case E is said to be smooth. E is said to have a uniformly Fr'echet differentiable norm if the limit (4.3) is attained uniformly for $x, y \in U$ and in this case E is said to be uniformly smooth. It is well known that if E is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on all bounded subsets of E.

In this section, we modified the iteration process (1.11) due to Yao et al. for pseudocontractive mappings under the setting of uniformly smooth Banach spaces. It is worth to mention here, that our proof is different from Yao et al. and is independent of interest.

We need the following lemma for the proof of our results.

Lemma 13 [17]. *Let E be a real uniformly smooth Banach space. Then there exists a non-decreasing continuous function $b : [0, \infty) \to [0, \infty)$ satisfying*:
(i) $b(ct) = cb(t)$;
(ii) $\lim_{t \to 0^+} b(t) = 0$;
(iii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|)$, *for all* $x, y \in E$.

Theorem 14. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences, satisfying the following conditions*:
(xvii) $0 \le \alpha_n, \beta_n, \delta_n \le 1,\ 0 < \gamma_n < 1$;
(xviii) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;
(xix) $\lim_{n\to\infty} \beta_n = 0 = \lim_{n\to\infty} \alpha_n$;
(xx) $\sum_{n=0}^{\infty} \dfrac{\alpha_n}{\alpha_n + \beta_n + \delta_n} = \infty$;
(xxi) $\delta_n = o(\alpha_n)$.
For arbitrary initial value $x_0 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by

$$\text{(YA)} \qquad \begin{aligned} & x_0 \in C, \\ & x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n + \delta_n u_n, \quad n \ge 1, \end{aligned}$$

where $\{u_n\}$ is a bounded sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The existence of a fixed point of T follows from Schauder's fixed point theorem, so $F(T) \neq \phi$.

Indeed, suppose we take a fixed point p of T. Since $\{u_n\}$ is a bounded sequence of error terms, set $M_2 = \sup_{n\geq 1}\|u_n - p\|$.

First, we show that $\{x_n\}$ is bounded. Consider

$$\begin{aligned} x_n - p &= (1-\gamma_n)\left(\frac{\alpha_n}{1-\gamma_n}u + \frac{\beta_n}{1-\gamma_n}x_{n-1} + \frac{\delta_n}{1-\gamma_n}u_n\right) + \gamma_n T x_n - p \\ &= (1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\delta_n}{1-\gamma_n}(u_n - p)\right] \\ &\quad + \gamma_n(Tx_n - p). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\ &= \left\langle (1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right]\right. \\ &\quad \left. + \gamma_n(Tx_n - p), j(x_n - p)\right\rangle \\ &= (1-\gamma_n)\left\langle \frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p)\right. \\ &\quad \left. + \frac{\delta_n}{1-\gamma_n}(u_n - p), j(x_n-p)\right\rangle + \gamma_n\langle Tx_n - p, j(x_n - p)\rangle \\ &\leq (1-\gamma_n)\left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\| \\ &\quad \times \|x_n - p\| + \gamma_n\|x_n - p\|^2, \end{aligned}$$

so

$$\begin{aligned} \|x_n - p\| &\leq \left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\| \\ &\leq \frac{\alpha_n}{1-\gamma_n}\|u-p\| + \frac{\beta_n}{1-\gamma_n}\|x_{n-1}-p\| + \frac{\delta_n}{1-\gamma_n}\|u_n - p\| \\ &\leq \frac{\alpha_n}{1-\gamma_n}\|u-p\| + \frac{\beta_n}{1-\gamma_n}\|x_{n-1}-p\| + M_2\frac{\delta_n}{1-\gamma_n} \\ &\leq \max\{\|u-p\|, \|x_{n-1}-p\|, M_2\}\left(\frac{\alpha_n}{1-\gamma_n} + \frac{\beta_n}{1-\gamma_n} + \frac{\delta_n}{1-\gamma_n}\right) \\ &= \max\{\|u-p\|, \|x_{n-1}-p\|, M_2\}. \end{aligned} \tag{4.4}$$

Now, induction yields

$$\|x_n - p\| \leq \max\{\|u-p\|, \|x_0 - p\|, M_2\}.$$

This implies $\{x_n\}$ is bounded and so is $\{Tx_n\}$. Let

$$N = \sup_{n\geq 1}\|x_n - p\| + \sup_{n\geq 1}\|Tx_n - p\| + M_2.$$

Finally, we prove that $x_n \to p$. From Lemma 13 and (4.4), we have

$$\begin{aligned}
\|x_n - p\|^2 &\leq \left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\|^2 \\
&\leq \left(\frac{\beta_n}{1-\gamma_n}\right)^2\|x_{n-1}-p\|^2 + 2\frac{\alpha_n\beta_n}{(1-\gamma_n)^2}\langle u-p, j(x_{n-1}-p)\rangle \\
&\quad + 2\frac{\delta_n\beta_n}{(1-\gamma_n)^2}\langle u_n-p, j(x_{n-1}-p)\rangle + \max\left\{\frac{\beta_n}{1-\gamma_n}\|x_{n-1}-p\|, 1\right\} \\
&\quad \times \left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\| \\
&\quad \times b\left(\left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\|\right) \\
&\leq \left(1-\frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1}-p\|^2 + 2\frac{\alpha_n\beta_n}{(1-\gamma_n)^2}\|u-p\|\,\|x_{n-1}-p\| \\
&\quad + 2\frac{\delta_n\beta_n}{(1-\gamma_n)^2}\|u_n-p\|\,\|x_{n-1}-p\| + \max\left\{\frac{\beta_n}{1-\gamma_n}\|x_{n-1}-p\|, 1\right\} \\
&\quad \times \left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\| \\
&\quad \times b\left(\left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\|\right).
\end{aligned} \tag{4.5}$$

Since $\delta_n = o(\alpha_n)$, there exists a sequence $\{t_n\} \subset [0,1]$ such that $t_n \to 0$ as $n \to \infty$ and $\delta_n = t_n\alpha_n$. Also

$$\begin{aligned}
\eta_n &= \left\|\frac{\alpha_n}{1-\gamma_n}(u-p) + \frac{\delta_n}{1-\gamma_n}(u_n-p)\right\| \\
&\leq \frac{\alpha_n}{1-\gamma_n}\|u-p\| + \frac{\delta_n}{1-\gamma_n}\|u_n-p\| \\
&\leq \frac{\alpha_n}{1-\gamma_n}\|u-p\| + N\frac{\delta_n}{1-\gamma_n} \\
&= \frac{\alpha_n}{1-\gamma_n}(\|u-p\| + Nt_n) \\
&\leq \frac{\alpha_n}{1-\gamma_n}(\|u-p\| + N) \\
&= \frac{\alpha_n}{1-\gamma_n}M'; \quad 0 < M' = \|u-p\| + N.
\end{aligned} \tag{4.6}$$

Substituting (4.6) in (4.5), we obtain

$$\begin{aligned} \|x_n - p\|^2 &\le \left(1 - \frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1} - p\|^2 + 2\frac{\alpha_n\beta_n}{(1-\gamma_n)^2}N\|u-p\| \\ &\quad + 2\frac{\delta_n\beta_n}{(1-\gamma_n)^2}N^2 + \max\{N,1\}M'^2\frac{\alpha_n}{1-\gamma_n}b\left(\frac{\alpha_n}{1-\gamma_n}\right) \\ &= \left(1 - \frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1} - p\|^2 + \frac{\alpha_n}{1-\gamma_n}\sigma_n; \end{aligned}$$

$$\sigma_n = 2N\frac{\beta_n}{1-\gamma_n}\|u-p\| + 2N^2\frac{t_n\beta_n}{1-\gamma_n} + \max\{N,1\}M'^2 b\left(\frac{\alpha_n}{1-\gamma_n}\right).$$

Now according to Lemma 7, we have $x_n \to p$.

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Arif Rafiq
COMSATS Institute of Information Technology
Islamabad
Pakistan
E-mail: aarafiq@gmail.com