F. MASSAMBA
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TOTALLY CONTACT UMBILICAL LIGHTLIKE HYPERSURFACES OF INDEFINITE SASAKIAN MANIFOLDS

Fortuné Massamba

Abstract

This paper investigates totally contact umbilical lightlike hypersurfaces which are tangent to the structure vector field. Theorems on Killing distributions, geodesibility of lightlike hypersurfaces are obtained. The geometrical configuration of such lightlike hypersurfaces and its screen distributions are established. We prove the non-existence of totally contact umbilical lightlike hypersurfaces and lightlike hypersurfaces with totally contact umbilical screen distributions in indefinite Sasakian space forms under some conditions. Some characterizations of totally contact geodesic lightlike hypersurfaces and screen distributions are also given.

1. Introduction

The totally contact umbilical concept was considered in [1], [6], [7], [12], [13] and others references therein where some classifications are given in submanifold of the Sasakian manifolds of codimension greater than 1. The present paper aims to investigate similar concept, namely, totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds.

As it is well known, contrary to timelike and spacelike hypersurfaces, the geometry of a lightlike hypersurface is different and rather difficult since the normal bundle and the tangent bundle have non-zero intersection. To overcome this difficulty, a theory on the differential geometry of lightlike hypersurfaces developed by Duggal and Bejancu [4] introduces a non-degenerate screen distribution and construct the corresponding lightlike transversal vector bundle. This enables to define an induced linear connection (depending on the screen distribution, and hence is not unique in general).

The paper is organized as follows. In Section 2, we recall some basic definitions and formulas for indefinite Sasakian manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In Section 3, for those lightlike hypersurfaces of indefinite Sasakian manifolds which are tangential to the structure

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vector field, the decomposition of almost contact metric is given. Theorems on Killing distributions, geodesibility of lightlike hypersurfaces of indefinite Sasakian manifolds are obtained. A characterization of lightlike hypersurfaces of indefinite Sasakian manifolds with parallel vector field is also given. In Section 4, we study totally contact umbilical lightlike hypersurfaces of an indefinite Sasakian manifold. By Theorem 4.3 and 4.10, we establish the geometrical configuration of such lightlike hypersurfaces, tangent to the structure vector field, and its screen distributions in Sasakian space forms. We prove the non-existence of totally contact umbilical lightlike hypersurfaces and lightlike hypersurfaces with totally contact umbilical screen distributions, tangent to the structure vector field, in indefinite Sasakian manifold under some conditions. Some characterizations of totally contact geodesic lightlike hypersurfaces and screen distributions are also given. Finally, we discuss, in section 5, the effect of the change of the screen distribution on different results found.

2. Preliminaries

A $(2n+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$ is said to be an *indefinite Sasakian manifold* if it admits an almost contact structure $(\bar{\phi}, \xi, \eta)$, i.e. $\bar{\phi}$ is a tensor field of type $(1,1)$ of rank $2n$, ξ is a vector field, and η is a 1-form, satisfying

$$
\begin{aligned}
(2.1)\qquad & \bar{\phi}^2 = -\mathbf{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \bar{\phi} = 0, \quad \bar{\phi}\xi = 0, \\
& \eta(\overline{X}) = \bar{g}(\xi, \overline{X}), \quad \bar{g}(\bar{\phi}\overline{X}, \bar{\phi}\overline{Y}) = \bar{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \quad \overline{\nabla}_{\overline{X}}\xi = -\bar{\phi}(\overline{X}), \\
& (\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = \bar{g}(\bar{\phi}\overline{X}, \overline{Y}), \quad (\overline{\nabla}_{\overline{X}}\bar{\phi})\overline{Y} = \bar{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X}, \quad \forall \overline{X}, \overline{Y} \in \Gamma(T\overline{M}),
\end{aligned}
$$

where $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\bar{g}$.

A plane section σ in $T_p\overline{M}$ is called a $\bar{\phi}$-section if it is spanned by $\overline{X}$ and $\bar{\phi}\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\bar{\phi}$-section σ is called a $\bar{\phi}$-sectional curvature. A Sasakian manifold $\overline{M}$ with constant $\bar{\phi}$-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by [13]

$$
\begin{aligned}
(2.2)\quad \overline{R}(\overline{X}, \overline{Y})\overline{Z} = {} & \frac{c+3}{4}\{\bar{g}(\overline{Y}, \overline{Z})\overline{X} - \bar{g}(\overline{X}, \overline{Z})\overline{Y}\} + \frac{c-1}{4}\{\eta(\overline{X})\eta(\overline{Z})\overline{Y} \\
& - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \bar{g}(\overline{X}, \overline{Z})\eta(\overline{Y})\xi - \bar{g}(\overline{Y}, \overline{Z})\eta(\overline{X})\xi + \bar{g}(\bar{\phi}\overline{Y}, \overline{Z})\bar{\phi}\overline{X} \\
& - \bar{g}(\bar{\phi}\overline{X}, \overline{Z})\bar{\phi}\overline{Y} - 2\bar{g}(\bar{\phi}\overline{X}, \overline{Y})\bar{\phi}\overline{Z}\}, \quad \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M}).
\end{aligned}
$$

Let $(\overline{M}, \bar{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \bar{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [4]. A complementary bundle of $TM^{\perp}$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen* *distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

THEOREM 2.1 [4]. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying $\bar{g}(N, E) = 1$ and $\bar{g}(N, N) = \bar{g}(N, W) = 0$, for any $W \in \Gamma(S(TM)|_{\mathcal{U}})$.*

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(\Xi)$ the smooth sections of the vector bundle Ξ. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decomposition

$$TM = S(TM) \perp TM^{\perp}, \tag{2.3}$$

$$T\overline{M} = TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \tag{2.4}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \bar{g})$, then by using the second decomposition of (2.3), we have Gauss and Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \quad \forall X \in \Gamma(TM),\ V \in \Gamma(N(TM)), \tag{2.6}$$

where $\nabla_X Y, A_V X \in \Gamma(TM)$ and $h(X, Y), \nabla_X^{\perp} V \in \Gamma(N(TM))$. ∇ is a symmetric linear connection on M called an induced linear connection, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$. h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.5) and (2.6) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \tag{2.7}$$

$$\text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.8}$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.7), we obtain

$$B(X, Y) = \bar{g}(\overline{\nabla}_X Y, E) \quad \text{and} \quad \tau(X) = \bar{g}(\nabla_X^{\perp} N, E) \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}).$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have

$$\nabla_X PY = \nabla_X^{*} PY + C(X, PY)E, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \tag{2.9}$$

$$\text{and} \quad \nabla_X E = -A_E^{*} X - \tau(X)E, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.10}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(2.11)\qquad (\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y), \quad \forall X, Y \in \Gamma(TM|_{\mathscr{U}}),$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \bar{g}(N,\cdot)$. Also, we have the following identities,

$$(2.12)\qquad \begin{gathered} g(A^*_E X, PY) = B(X,PY), \quad g(A^*_E X, N) = 0, \\ B(X,E) = 0, \quad \forall X, Y \in \Gamma(TM|_{\mathscr{U}}). \end{gathered}$$

Finally, using (2.7), $\bar{R}$ and R are the curvature tensor fields of $\overline{M}$ and M are related as

$$(2.13)\qquad \begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z)A_N X + \{(\nabla_X B)(Y,Z) \\ &- (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}N, \end{aligned}$$

$$(2.14)\qquad \text{where} \quad (\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

3. Lightlike hypersurfaces of indefinite Sasakian manifolds

Let $(\overline{M},\bar{\phi},\xi,\eta,\bar{g})$ be an indefinite Sasakian manifold and (M,g) be its lightlike hypersurface, *tangent to the structure vector field* ξ $(\xi \in TM)$[1]. If E is a local section of $TM^\perp$, then $\bar{g}(\bar{\phi}E,E) = 0$, and $\bar{\phi}E$ is tangent to M. Thus $\bar{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\bar{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\bar{\phi}(TM^\perp)$ as vector subbundle. Consider a local section N of $N(TM)$. Since $\bar{g}(\bar{\phi}N,E) = -\bar{g}(N,\bar{\phi}E) = 0$, we deduce that $\bar{\phi}N$ is also tangent to M. On the other hand, since $\bar{g}(\bar{\phi}N,N) = 0$, we see that the components of $\bar{\phi}N$ with respect to E vanishes. Thus $\bar{\phi}N \in \Gamma(S(TM))$. From (2.1), we have $\bar{g}(\bar{\phi}N,\bar{\phi}E) = 1$. Therefore, $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

It is known [3] that if M is tangent to the structure vector field ξ, then, ξ *belongs to* $S(TM)$. Using this, and since $\bar{g}(\bar{\phi}E,\xi) = \bar{g}(\bar{\phi}N,\xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n-4$ on M such that

$$(3.1)\qquad S(TM) = \{\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))\} \perp D_0 \perp \langle\xi\rangle,$$

where $\langle\xi\rangle$ is the distribution spanned by ξ, thats is, $\langle\xi\rangle = \mathrm{Span}\{\xi\}$.

[1] Many geometers use to consider ξ tangent to the manifold because in the theory of *CR* submanifolds the condition M normal to ξ leads to M anti-invariant submanifold (see [11]; Proposition 1.1, p. 43) and the condition ξ oblique gives very complicated embedding equations.

Proposition 3.1. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, the distribution D_0 is an invariant with respect to $\bar{\phi}$, that is, $\bar{\phi}(D_0) = D_0$.*

Proof. For any $X \in \Gamma(D_0)$ and $Y \in \Gamma(TM)$, we have $\bar{g}(\bar{\phi}X, Y) = -\bar{g}(X, \bar{\phi}Y)$. For $Y = \bar{\phi}E$, we obtain $\bar{g}(\bar{\phi}X, \bar{\phi}E) = \bar{g}(X, E) - \eta(X)\eta(E) = 0$. Thus $\bar{\phi}X \perp \bar{\phi}(TM^{\perp})$. On the other hand we have $\bar{g}(\bar{\phi}X, E) = -\bar{g}(X, \bar{\phi}E) = 0$, for any $E \in \Gamma(TM^{\perp})$. Hence $\bar{\phi}X \perp TM^{\perp}$. Also, we have $\bar{g}(\bar{\phi}X, \xi) = 0$ and $\bar{g}(\bar{\phi}X, \bar{\phi}N) = \bar{g}(X, N) - \eta(X)\eta(N) = 0$, for any $N \in \Gamma(N(TM))$. Thus

$$\bar{\phi}X \perp \{\{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(N(TM))\} \perp TM^{\perp} \perp \langle\xi\rangle\}.$$

Finally we derive $\bar{g}(\bar{\phi}X, N) = -\bar{g}(X, \bar{\phi}N) = 0$ and by summing up these results we deduce

$$\bar{\phi}X \perp \{\{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(N(TM))\} \perp TM^{\perp} \perp \langle\xi\rangle \oplus N(TM)\},$$

that is $\bar{\phi}(D_0) = D_0$ which proves our assertion. □

Example 3.2. Let $\mathbf{R}^7$ be the 7-dimensional real number space. We consider $\{x_i\}_{1\le i\le 7}$ as cartesian coordinates on $\mathbf{R}^7$ and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ a tensor field $\bar{\phi}$ of type $(1,1)$ by its matrix.

$$\begin{aligned}
\bar{\phi}\left(\frac{\partial}{\partial x_1}\right) &= -\frac{\partial}{\partial x_2}, \quad \bar{\phi}\left(\frac{\partial}{\partial x_2}\right) = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_7}, \quad \bar{\phi}\left(\frac{\partial}{\partial x_3}\right) = -\frac{\partial}{\partial x_4}, \\
\bar{\phi}\left(\frac{\partial}{\partial x_4}\right) &= \frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_7}, \quad \bar{\phi}\left(\frac{\partial}{\partial x_5}\right) = -\frac{\partial}{\partial x_6}, \\
\bar{\phi}\left(\frac{\partial}{\partial x_6}\right) &= \frac{\partial}{\partial x_5}, \quad \bar{\phi}\left(\frac{\partial}{\partial x_7}\right) = 0.
\end{aligned} \tag{3.2}$$

The differential 1-form η is defined by

$$\eta = dx_7 - x_4 dx_1 - x_6 dx_3. \tag{3.3}$$

The vector field ξ is defined by $\xi = \partial/\partial x_7$. It is easy to check (2.1) and thus $(\bar{\phi}, \xi, \eta)$ is an almost contact structure on $\mathbf{R}^7$. Finally we define metric $\bar{g}$ by

$$\begin{aligned}
\bar{g} = (x_4^2 - 1)dx_1^2 - dx_2^2 + (x_6^2 + 1)dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2 \\
- x_4 dx_1 \otimes dx_7 - x_4 dx_7 \otimes dx_1 + x_4 x_6 dx_1 \otimes dx_3 \\
+ x_4 x_6 dx_3 \otimes dx_1 - x_6 dx_3 \otimes dx_7 - x_6 dx_7 \otimes dx_3.
\end{aligned} \tag{3.4}$$

with respect to the natural field of frames. It is easy to check that $\bar{g}$ is a semi-Riemannian metric and $(\bar{\phi}, \xi, \eta, \bar{g})$ given by (3.2)–(3.4) is a Sasakian structure on $\mathbf{R}^7$.

We now define a hypersurface M of $(\mathbf{R}^7, \bar{\phi}, \xi, \eta, \bar{g})$ as $M = \{(x_1, \dots, x_7) \in \mathbf{R}^7 : x_5 = x_4\}$. Thus the tangent space TM is spanned by $\{U_i\}_{1 \le i \le 6}$ where

$$U_1 = \frac{\partial}{\partial x_1}, \quad U_2 = \frac{\partial}{\partial x_2}, \quad U_3 = \frac{\partial}{\partial x_3}, \tag{3.5}$$

$$U_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}, \quad U_5 = \frac{\partial}{\partial x_6}, \quad U_6 = \xi$$

and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E, where

$$E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}. \tag{3.6}$$

It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\mathbf{R}^7$. Also, the transversal bundle $N(TM)$ is spanned by

$$N = \frac{1}{2}\left(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}\right). \tag{3.7}$$

On the other hand, by using the almost contact structure of $\mathbf{R}^7$ and also by taking into account of the decomposition (3.1), the distribution D_0 is spanned by $\{F, \bar{\phi}F\}$, where $F = U_2$, $\bar{\phi}F = U_1 + x_4\xi$ and the distributions $\langle\xi\rangle$, $\bar{\phi}(TM^{\perp})$ and $\bar{\phi}(N(TM))$ are spanned, respectively, by

$$\xi, \bar{\phi}E = U_3 - U_5 + x_6\xi \quad \text{and} \quad \bar{\phi}N = \frac{1}{2}(U_3 + U_5 + x_6\xi). \tag{3.8}$$

Hence M is a lightlike hypersurface of $\mathbf{R}^7$.

Moreover, from (2.3) and (3.1) we obtain the decomposition

$$TM = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(N(TM))\} \perp D_0 \perp \langle\xi\rangle \perp TM^{\perp}, \tag{3.9}$$

$$\text{and} \quad T\overline{M} = \{\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(N(TM))\} \perp D_0 \perp \langle\xi\rangle \perp (TM^{\perp} \oplus N(TM)). \tag{3.10}$$

Now, we consider the distributions on M,

$$D := TM^{\perp} \perp \bar{\phi}(TM^{\perp}) \perp D_0, \quad D' := \bar{\phi}(N(TM)). \tag{3.11}$$

Then D is invariant under $\bar{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{3.12}$$

Let us consider the local lightlike vector fields

$$U := -\bar{\phi}N, \quad V := -\bar{\phi}E. \tag{3.13}$$

Then, from (3.12), any $X \in \Gamma(TM)$ is written as

$$X = RX + QX + \eta(X)\xi, \quad QX = u(X)U, \tag{3.14}$$

where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$.

Applying $\bar{\phi}$ to X and using (2.1) (note that $\bar{\phi}^2 N = -N$), we obtain $\bar{\phi}X = \phi X + u(X)N$, where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi X := \bar{\phi}RX$ and we also have

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \quad \forall X \in \Gamma(TM). \tag{3.15}$$

Now applying ϕ to $\phi^2 X$ and since $\phi U = 0$, we obtain $\phi^3 + \phi = 0$, which shows that ϕ is an f-structure [13] of constant rank. By using (2.1) we derive

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y), \tag{3.16}$$

where v is a 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \tag{3.17}$$

We have the following useful identities

$$\nabla_X \xi = -\phi X, \tag{3.18}$$

$$B(X, \xi) = -u(X), \tag{3.19}$$

$$C(X, \xi) = -v(X), \tag{3.20}$$

$$B(X, U) = C(X, V) \tag{3.21}$$

$$(\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X), \tag{3.22}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X - B(X, Y)U + u(Y)A_N X. \tag{3.23}$$

Proposition 3.3. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. The Lie derivative of g with respect to the vector field V is given by,*

$$(L_V g)(X, Y) = X.u(Y) + Y.u(X) + u([X, Y]) - 2u(\nabla_X Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.24}$$

Proof. From a straightforward calculation, we have, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}
u(\nabla_X Y) &= g(\nabla_X Y, V) = \bar{g}(\overline{\nabla}_X Y, V) = X.g(Y, V) - \bar{g}(Y, \overline{\nabla}_X V) \\
&= X.g(Y, V) - g(Y, [X, V]) - \bar{g}(Y, \overline{\nabla}_V X) \\
&= X.g(Y, V) - g(Y, [X, V]) - V.g(Y, X) + \bar{g}(\overline{\nabla}_V Y, X) \\
&= X.g(Y, V) + g(Y, [V, X]) - V.g(Y, X) + g([V, Y], X) + \bar{g}(\overline{\nabla}_Y V, X) \\
&= X.g(Y, V) - (L_V g)(X, Y) + Y.g(X, V) - \bar{g}(V, \overline{\nabla}_Y X) \\
&= X.u(Y) - (L_V g)(X, Y) + Y.u(X) + u([X, Y]) - u(\nabla_X Y).
\end{aligned}$$

(3.24) is proved. □

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Therefore, we investigate the integrability of the screen distributions. First, for any $X, Y \in \Gamma(D \perp \langle\xi\rangle)$,

$$u([X, Y]) = B(X, \phi Y) - B(\phi X, Y). \tag{3.25}$$

It is now easy to see that the distribution $D \perp \langle\xi\rangle$ is integrable if and only if $B(X, \phi Y) = B(\phi X, Y)$, $\forall X, Y \in \Gamma(D \perp \langle\xi\rangle)$. We have

Lemma 3.4. *Let (M, g) be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$. If M is D-totally geodesic, then $\bar{\phi}(TM^{\perp})$ is D-Killing distribution.*

Proof. Let M be a D-totally geodesic lightlike hypersurface. Then, for any $X, Y \in \Gamma(D)$, $B(X, Y) = 0$. So, $u(\nabla_X Y) = B(X, \phi Y) = 0$, since D is invariant under $\bar{\phi}$. We have $(L_V g)(X, Y) = u([X, Y])$, $\forall X, Y \in \Gamma(D)$ which implies $(L_V g)(X, Y) = -(L_V g)(Y, X)$. On the other hand, $(L_V g)(X, Y) - (L_V g)(Y, X) = 2(u([X, Y]) - B(X, \phi Y) + B(\phi X, Y)) = 0$. Therefore,

$$(L_V g)(X, Y) = (L_V g)(Y, X) = -(L_V g)(X, Y).$$

Thus, $(L_V g)(X, Y) = 0$ and $\bar{\phi}(TM^{\perp})$ is D-Killing distribution. □

We are now concerned with the structure equations of the immersions of a lightlike hypersurface in a semi-Riemannian manifold.

Let $\overline{M}(c)$ be an indefinite Sasakian space form and M be a lightlike hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathscr{U} \subset M$ (see Theorem 2.1) and by using (2.13), we obtain

$$\begin{aligned}(\nabla_X B)(Y, Z) &- (\nabla_Y B)(X, Z) \\ &= \tau(Y)B(X, Z) - \tau(X)B(Y, Z) \\ &\quad + \frac{c-1}{4}\{\bar{g}(\bar{\phi}Y, Z)u(X) - \bar{g}(\bar{\phi}X, Z)u(Y) - 2\bar{g}(\bar{\phi}X, Y)u(Z)\}.\end{aligned} \tag{3.26}$$

Theorem 3.5. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c, with $\xi \in TM$. Then, the Lie derivative of the second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X, Y) = -\tau(\xi)B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.27}$$

Moreover, ξ is a Killing vector field with respect to the second fundamental form B if and only if $\tau(\xi) = 0$ or M is totally geodesic.

Proof. Using (2.14) and (3.18), we obtain

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) + B(\phi X, Y) + B(X, \phi Y). \tag{3.28}$$

Likewise, Using (2.14), (3.18) and (3.19), we have

$$(\nabla_X B)(\xi, Y) = -X.u(Y) + B(\phi X, Y) + u(\nabla_X Y). \tag{3.29}$$

Subtracting (3.28) and (3.29), and using (3.22) we obtain

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = (L_\xi B)(X, Y) - u(Y)\tau(X). \tag{3.30}$$

From (3.26) and after calculations, the left hand side of (3.30) becomes

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = -u(Y)\tau(X) - \tau(\xi)B(X, Y). \tag{3.31}$$

The expressions (3.30) and (3.31) implies $(L_\xi B)(X, Y) = -\tau(\xi)B(X, Y)$. The last assertion is obvious by definitions of Killing distribution and totally geodesic submanifold. □

The second fundamental form h of M is said to be parallel if $(\nabla_X h)(Y, Z) = 0$, $\forall X, Y, Z \in \Gamma(TM)$. That is, $(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z)$. This means that, in general, the parallelism of h does not imply the parallelism of B and vice versa. We note that $(\nabla_X h)(Y, E) = (\nabla_X B)(Y, E)N$.

Lemma 3.6. *There exist no lightlike hypersufaces of indefinite Sasakian space forms $\overline{M}(c)$ $(c \neq 1)$ with $\xi \in TM$ and parallel second fundamental form.*

Proof. Suppose $c \neq 1$ and second fundamental form is parallel. Then, if we take $Y = E$ and $Z = U$ in (3.26), we obtain $((c-1)/4)u(X) = 0$. Taking $X = U$, we have $c = 1$, which is a contradiction. □

Lemma 3.7. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c, with $\xi \in TM$, such that its local second fundamental form B is parallel. If $\tau(E) \neq 0$, then $c = 1$ if and only if M is D'-totally geodesic.*

Proof. Suppose B is parallel. Then, taking $Y = E$ in (3.26), we obtain $3((c-1)/4)u(X)u(Z) = \tau(E)B(X, Z)$. Taking $X = Z = U$, we have $3((c-1)/4) = \tau(E)B(U, U)$ and if $\tau(E) \neq 0$, the equivalence follows. □

Theorem 3.8. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c with $\xi \in TM$. If the local second fundamental form B of M is parallel, then,*

$$(L_V g)(X, Y) = \tau(\xi)B(X, Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.32}$$

Moreover, if $\tau(\xi) \neq 0$, then M is totally geodesic if and only if $\bar{\phi}(TM^\perp)$ is a Killing distribution on M.

Proof. Using (2.14), (3.18), (3.19), (3.22) and (3.27), after calculations, we have, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} 0 &= (\nabla_\xi B)(X, Y) = L_\xi B(X, Y) + B(\phi X, Y) + B(X, \phi Y) \\ &= -\tau(\xi)B(X, Y) - (L_V g)(X, Y) - u(X)\tau(Y) - u(Y)\tau(X). \end{aligned} \tag{3.33}$$

Likewise, we obtain

$$\begin{aligned} 0 = (\nabla_Y B)(\xi, X) &= -Y.u(X) + B(X, \phi Y) + u(\nabla_Y X) \\ &= -(L_V g)(X, Y) - u(Y)\tau(X), \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} 0 = (\nabla_X B)(Y, \xi) &= X.B(\xi, Y) - B(\nabla_X Y, \xi) - B(Y, \nabla_X \xi) \\ &= -(L_V g)(X, Y) - u(X)\tau(Y). \end{aligned} \tag{3.35}$$

So substituting (3.34) and (3.35) in (3.33), we obtain (3.32). If $\tau(\xi) \neq 0$, the equivalence follows. □

THEOREM 3.9. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c, with $\xi \in TM$. If the second fundamental form h of M is parallel, then,*

(i) *$\bar{\phi}(TM^\perp)$ is a $D \perp \langle\xi\rangle$-Killing distribution.*
(ii) *For any, $X, Y \in \Gamma(TM)$, $B(A_E^* X, Y) = 0$.*
(iii) *For any $X, Y \in \Gamma(TM)$, $(L_E B)(X, Y) = -\tau(E)B(X, Y)$.*

Proof. Taking $Z = \xi$ in $(\nabla_Z h)(X, Y) = 0$, we have $(L_V g)(X, Y) = -u(X)\tau(Y) - u(Y)\tau(X)$ and for any $X, Y \in \Gamma(D \perp \langle\xi\rangle)$, $(L_V g)(X, Y) = 0$. This proves (i). (ii) is complete by using the following equation $0 = \bar{g}((\nabla_X h)(Y, E), E) = B(A_E^* X, Y)$. The last assertion (iii) is obtained as follows. Taking $Z = E$ in $(\nabla_Z h)(X, Y) = 0$, we have $(L_E B)(X, Y) = -\tau(E)B(X, Y) - 2B(A_E^* X, Y)$. So, by using the assertion (ii), we have $(L_E B)(X, Y) = -\tau(E)B(X, Y)$. □

A submanifold M is said to be totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies

$$B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{3.36}$$

where ρ is a smooth function on $\mathscr{U} \subset M$. The Gauss formula implies that $\bar{\phi}X = -\overline{\nabla}_X \xi = -\nabla_X \xi - B(X, \xi)N$. Since $\bar{\phi}\xi = 0$, we have $B(\xi, \xi) = 0$.

If we assume that M is totally umbilical lightlike hypersurface of the a semi-Riemannian manifold $\overline{M}$, then we have $B(X, Y) = \rho g(X, Y)$, for any $X, Y \in \Gamma(TM)$, which implies that $0 = B(\xi, \xi) = \rho$. Hence M is totally geodesic. Also, $\bar{\phi}X = \phi X - \rho\eta(X)N = \phi X$, that is M is invariant in $\overline{M}$. Therefore we have

Proposition 3.10. *Let (M,g) be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\bar{g})$ with $\xi \in TM$. If M is totally umbilical, then M is totally geodesic and invariant.*

It follows from the Proposition 3.10 that a Sasakian $\overline{M}(c)$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [1] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [6].

4. Totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds

In this section, we follow Bejancu [1] definition of totally contact umbilical submanifolds and state the following definition for totally contact umbilical lightlike hypersurfaces.

A submanifold M is said to be totally contact umbilical lightlike hypersurface of the a semi-Riemannian manifold $\overline{M}$ if the second fundemental form h of M satisfies:

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi) \tag{4.1}$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathscr{U} \subset M$). The notion of totally contact umbilical submanifolds of Sasakian manifolds corresponds to that of totally umbilical submanifolds of Kählerian manifolds (see [6] for more details). The totally contact umbilical condition (4.1) can be rewritten as,

$$h(X,Y) = B(X,Y)N = \{B_1(X,Y) + B_2(X,Y)\}N, \tag{4.2}$$

where $B_1(X,Y) = \lambda\{g(X,Y) - \eta(X)\eta(Y)\}$ and $B_2(X,Y) = -\eta(X)u(Y) - \eta(Y)u(X)$, since $h(X,\xi) = -u(X)N$. The covariant derivative of the local second fundamental form B of M is given by

$$(\nabla_X B)(Y,Z) = (\nabla_X B_1)(Y,Z) + (\nabla_X B_2)(Y,Z), \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.3}$$

If the $\lambda = 0$ (that is $B_1 = 0$), then the lightlike hypersurface M is said to be totally contact geodesic. The notion of totally contact geodesic submanifolds of Sasakian manifolds corresponds to that of totally geodesic submanifolds of Kaehlerian manifolds.

In the sequel, we need the following lemmas.

Lemma 4.1. *Let (M,g) be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M},\bar{g})$ with $\xi \in TM$. For any $X, Y \in \Gamma(TM)$*

$$g(\nabla_X V, Y) + u(A_E^* X)\theta(Y) = -B(X,\phi Y) - \tau(X)u(Y), \tag{4.4}$$

$$g(\nabla_X U, Y) + u(A_N X)\theta(Y) = -C(X,\phi Y) - \theta(X)\eta(Y) + \tau(X)v(Y). \tag{4.5}$$

Proof. By straightforward calculation and also by using (2.8) and (2.10)

$$\begin{aligned} g(\nabla_X V, Y) &= -g((\nabla_X \phi)E, Y) - g(\phi \nabla_X E, Y) \\ &= -g(\bar{\phi} \nabla_X E, Y) + u(\nabla_X E)\theta(Y) \\ &= -\bar{g}(A_E^* X, \bar{\phi} Y) - \tau(X) g(E, \bar{\phi} Y) - u(A_E^* X)\theta(Y) \\ &= -B(X, \phi Y) - \tau(X) u(Y) - u(A_E^* X)\theta(Y), \end{aligned}$$

and $g(\nabla_X U, Y) + u(A_N X)\theta(Y) = -\bar{g}(A_N X, \bar{\phi} Y) - \theta(X)\eta(Y) + \tau(X)v(Y)$

which completes the proof. □

LEMMA 4.2. *Let (M, g) be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$. Then, for any $X, Y, Z \in \Gamma(TM)$*

$$\begin{aligned} (\nabla_X B_1)(Y, Z) &= \lambda\{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\} \\ &\quad + \lambda\eta(Z)\{u(X)\theta(Y) + g(\phi X, Y)\} \\ &\quad + \lambda\eta(Y)\{u(X)\theta(Z) + g(\phi X, Z)\} \\ &\quad + \{g(Y, Z) - \eta(Y)\eta(Z)\}(X.\lambda), \end{aligned} \tag{4.6}$$

$$\begin{aligned} (\nabla_X B_2)(Y, Z) &= \{u(X)\theta(Y) + g(\phi X, Y)\}u(Z) \\ &\quad + \{u(X)\theta(Z) + g(\phi X, Z)\}u(Y) \\ &\quad + \{\tau(X)u(Y) + B(X, \phi Y)\}\eta(Z) \\ &\quad + \{\tau(X)u(Z) + B(X, \phi Z)\}\eta(Y). \end{aligned} \tag{4.7}$$

Proof. The proof follows from straightforward computing and by using the identities (2.11), (3.19) and (4.4). □

THEOREM 4.3. *Let $\overline{M}(c)$ be an indefinite Sasakian space form and M be a totally contact umbilical lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$. Then $c = -3$ ($\overline{M}(c)$ is of constant curvature -3) and λ satisfies the partial differential equations*

$$E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0, \tag{4.8}$$

$$\textit{and} \quad PX \cdot \lambda + \lambda\tau(PX) = 0, \quad \forall X \in \Gamma(TM). \tag{4.9}$$

Proof. Let M be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c. From (4.6) and (4.7), using (3.17) and the identity $B(X, \phi Y) = \lambda g(X, \phi Y)$, (3.26) becomes

$$\begin{aligned} &\lambda\{B(X, Z)\theta(Y) - B(Y, Z)\theta(X)\} + 2\lambda\{u(X)\theta(Y) + g(\phi X, Y)\}\eta(Z) \\ &\quad + \lambda\{\eta(Y)u(X) - \eta(X)u(Y)\}\theta(Z) + \lambda\{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\} \end{aligned} \tag{4.10}$$

$$\begin{aligned}
&+\{g(Y,Z)-\eta(Y)\eta(Z)\}X.\lambda-\{g(X,Z)-\eta(X)\eta(Z)\}Y.\lambda\\
&+2\{u(X)\theta(Y)+g(\phi X,Y)\}u(Z)+\{u(Y)g(\phi X,Z)-u(X)g(\phi Y,Z)\}\\
&+\{\tau(X)u(Y)+\lambda g(X,\phi Y)-\tau(Y)u(X)-\lambda g(Y,\phi X)\}\eta(Z)\\
&+\lambda\{\eta(Y)g(X,\phi Z)-\eta(X)g(Y,\phi Z)\}+\{\tau(X)\eta(Y)-\tau(Y)\eta(X)\}u(Z)\\
=&\frac{c-1}{4}\{\bar g(\bar\phi Y,Z)u(X)-\bar g(\bar\phi X,Z)u(Y)-2\bar g(\bar\phi X,Y)u(Z)\}\\
&+\tau(Y)B(X,Z)-\tau(X)B(Y,Z).
\end{aligned}$$

Putting $X=E$ in (4.10), we find

$$\begin{aligned}
(4.11)\qquad &-\lambda B(Y,Z)-2\lambda u(Y)\eta(Z)-\lambda\eta(Y)u(Z)+\{g(Y,Z)-\eta(Y)\eta(Z)\}(E.\lambda)\\
&\quad-3u(Y)u(Z)+\tau(E)\{u(Y)\eta(Z)+\eta(Y)u(Z)\}\\
&=\frac{3}{4}(c-1)u(Y)u(Z)-\tau(E)B(Y,Z).
\end{aligned}$$

Take $Y=Z=U$ in (4.11) we obtain $-3u(U)u(U)=\frac{3}{4}(c-1)u(U)u(U)$, that is, $c=-3$. On the other hand, by taking $Y=V$ and $Z=U$ in (4.11), we have $(B(V,U)=\lambda)$

$$E.\lambda+\lambda\tau(E)-\lambda^2=0. \tag{4.12}$$

Finally, substituting $X=PX$, $Y=PY$ and $Z=PZ$ into (4.10) with $c=-3$ and taking into account that $S(TM)$ is nondegenerate, we obtain

$$\{PX\cdot\lambda+\lambda\tau(PX)\}(PY-\eta(PY)\xi)=\{PY\cdot\lambda+\lambda\tau(PY)\}(PX-\eta(PX)\xi). \tag{4.13}$$

Putting $PX=\xi$ in (4.13), we have

$$\{\xi\cdot\lambda+\lambda\tau(\xi)\}(PY-\eta(PY)\xi)=0 \tag{4.14}$$

and by taking $Y=V$, we obtain

$$\xi.\lambda+\lambda\tau(\xi)=0. \tag{4.15}$$

Writing $PX\in\Gamma(S(TM))$ as $PX=PX'+\eta(PX)\xi$ $(PX'=\sum_i\alpha_iF_i+u(PX)U+v(PX)V$, $\{F_i\}_{1\le i\le 2n-4}$ an orthogonal basis of $D_0)$ and using (4.15), we have

$$\begin{aligned}
(4.16)\qquad PX\cdot\lambda+\lambda\tau(PX)&=(PX'+\eta(PX)\xi)\cdot\lambda+\lambda\tau(PX'+\eta(PX)\xi)\\
&=PX'\cdot\lambda+\lambda\tau(PX')+\eta(PX)(\xi\cdot\lambda+\lambda\tau(\xi))\\
&=PX'\cdot\lambda+\lambda\tau(PX')
\end{aligned}$$

which leads to get from (4.13)

$$\{PX'\cdot\lambda+\lambda\tau(PX')\}PY'=\{PY'\cdot\lambda+\lambda\tau(PY')\}PX'. \tag{4.17}$$

Now suppose that there exists a vector field X_0 on some neighborhood of M such that $PX_0'\cdot\lambda+\lambda\tau(PX_0')\neq 0$ at some point p in the neighborhood. Then, from (4.17) it follows that all vectors of the fibre $(S(TM)-\langle\xi\rangle)_p:=$

$(\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(N(TM)) \perp D_0)_p \subset S(TM)_p$ are collinear with $(PX_0')_p$. This contradicts $\dim(S(TM) - \langle\xi\rangle)_p > 1$. This implies (4.9). □

From Theorem 4.3 we obtain

Corollary 4.4. *There exist no totally contact umbilical lightlike real hypersurfaces of indefinite Sasakian space forms $\overline{M}(c)$ $(c \neq -3)$ with $\xi \in TM$.*

A part of the Theorem 4.3 is similar to that of the generic submanifold case given in [12]. Since the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M, that is, M is a hypersurface of indefinite Sasakian manifolds $\overline{M}$, M is also a generic submanifold. On the other hand, in the last part of the Theorem 4.3, namely, the equations (4.8) and (4.9), the geometry of the mean curvature vector H of M is discussed. These equations are similar to those of the indefinite Kählerian case (see [4] for details). However, there are non trivial differences arising in the details of the proof of our Theorem.

We also note that the partial differential equations (4.8) and the modified (4.9), $PX \cdot \lambda + \lambda\tau(PX) = 0$ with $PX \in \Gamma(S(TM) - \langle\xi\rangle)$ (that is, we exclude the partial differential equation in terms of ξ) arise when the submanifold M is a $D \oplus D'$-totally umbilical lightlike hypersurface, $B(X, Y) = \rho g(X, Y)$, $\forall X, Y \in \Gamma(D \oplus D')$. Because, in the direction of $D \oplus D'$, the function ρ is nowhere vanishing. In general, such a concept is called proper totally umbilical [4].

From (4.8) and (4.9), we have

$$\nabla_E^{\perp} H = \bar{g}(H, E)^2 N \quad \text{and} \quad \nabla_{PX}^{\perp} H = 0, \quad \forall X \in \Gamma(TM). \tag{4.18}$$

Lemma 4.5. *Let M be a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c = -3)$ with $\xi \in TM$. Then, the mean curvature vector H of M is $S(TM)$-parallel, that is,*

$$\nabla_{PX}^{\perp} H = 0, \quad \forall X \in \Gamma(TM). \tag{4.19}$$

Note that, if we choose, at each point $p \in M$, a connected open set G on M such that $T_pG = S(T_pM)$, then $\nabla_{PX}^{\perp} H = 0$ leads to H is a constant vector field in the direction of the screen distribution $S(TM)$.

A submanifold M is said to be an η-totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the second fundemental form h of M satisfies

$$h(X, Y) = \lambda\{g(X, Y) - \eta(X)\eta(Y)\}N, \quad \forall X, Y \in \Gamma(TM). \tag{4.20}$$

From this definition, we can deduce that the totally contact umbilical lightlike hypersurface M of $\overline{M}$ is also η-totally umbilical in the direction of $D \perp \langle\xi\rangle$, since the 1-form u vanishes in that direction.

If M is an η-totally umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$, we have

$$\bar{g}((\nabla_X h)(Y, Z), E) = (\nabla_X B_1)(Y, Z) + \lambda\tau(X)\{g(Y, Z) - \eta(Y)\eta(Z)\}. \tag{4.21}$$

Putting $Z = \xi$ in (4.21) and using (4.6), we obtain

$$(4.22)\qquad g((\nabla_X h)(Y,\xi),E) = (\nabla_X B_1)(Y,\xi) + \lambda\tau(X)\{g(Y,\xi) - \eta(Y)\eta(\xi)\}$$
$$= \lambda\bar{g}(\bar{\phi}X, Y).$$

If the second fundamental form h of the lightlike hypersurface M is parallel, then, we have

$$(4.23)\qquad 0 = g((\nabla_X h)(Y,\xi),E) = \lambda\bar{g}(\bar{\phi}X, Y)$$

which leads, by taking $X = E$ and $Y = U$, to $\lambda\bar{g}(\bar{\phi}E, U) = 0$, that is $\lambda = 0$. Hence, for any $X, Y \in \Gamma(TM)$, $B(X, Y) = 0$. Therefore we have

Theorem 4.6. *Let (M, g) be an η-totally umbilical lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$. If the second fundamental form h of M is parallel, then M is totally geodesic.*

This means that any η-totally umbilical parallel lightlike hypersurface M of an indefinite Sasakian manifold $\overline{M}$ admits a *metric connection*.

Theorem 4.7. *Let (M, g) be a totally contact geodesic lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$, with $\xi \in TM$. If the local second fundamental form B of M is parallel, then,*

(i) *The 1-form τ vanishes identically on M.*
(ii) *$\bar{\phi}(TM^{\perp})$ is a Killing distribution.*
(iii) *For any $X, Y \in \Gamma(TM)$, $B(A_E^* X, Y) = 0$.*
(iv) *ξ and E are Killing vector fields with respect to the local second fundamental form B of M.*

Proof. Using (4.7), we have, for any $X \in \Gamma(TM)$, $0 = (\nabla_X B)(\xi, U) = \tau(X)$. The others assertions follow from the latter and the Theorems 3.5, 3.8 and 3.9. □

Next, we deal with the geometry of the screen distribution of the lightlike hypersurfaces of indefinite Sasakian manifolds. From (2.2) and (2.13), a direct calculation shows that

$$(4.24)\qquad (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ)$$
$$= \frac{c+3}{4}\{\bar{g}(Y, PZ)\theta(X) - \bar{g}(X, PZ)\theta(Y)\}$$
$$+ \frac{c-1}{4}\{\eta(X)\eta(PZ)\theta(Y) - \eta(Y)\eta(PZ)\theta(X) + \bar{g}(\bar{\phi}Y, PZ)v(X)$$
$$- \bar{g}(\bar{\phi}X, PZ)v(Y) - 2\bar{g}(\bar{\phi}X, Y)v(PZ)\}.$$

From the differential geometry of lightlike hypersurfaces, we recall the following desirable property for lightlike geometry. The screen distribution $S(TM)$ of M is integrable if and only if the second fundamental form of $S(TM)$ is symmetric on $\Gamma(S(TM))$ (Theorem 2.3 in [4]).

PROPOSITION 4.8. *Let $(M, g, S(TM))$ be a lightlike hypersurface of indefinite Sasakian space form $\overline{M}(c)$, with $\xi \in TM$. If the screen distribution $S(TM)$ is integrable, then, for any $X, Y \in \Gamma(TM)$,*

$$(L_\xi C)(X, PY) = \tau(\xi) C(X, PY). \tag{4.25}$$

Moreover, ξ is a Killing vector field with respect to the second fundamental form C if and only if $\tau(\xi) = 0$ or the screen distribution $S(TM)$ is totally geodesic.

Proof. If the screen distribution $S(TM)$ of a lightlike hypersurface M is integrable, then, from (4.24) and using (3.20), we have, for any $X, Y \in \Gamma(TM)$,

$$(\nabla_\xi C)(X, PY) - (\nabla_X C)(\xi, PY) = -\eta(PY)\theta(X) + \tau(X)v(PY) + \tau(\xi)C(X, PY). \tag{4.26}$$

On the other hand, using (3.20) and (4.5), we have

$$\begin{aligned}(\nabla_\xi C)(X, PY) &= \xi.C(X, PY) - C(\nabla_\xi X, PY) - C(X, \nabla_\xi(PY)) \\ &= (L_\xi C)(X, PY) + C(\phi X, PY) + C(X, \phi PY)\end{aligned} \tag{4.27}$$

$$\begin{aligned}(\nabla_X C)(\xi, PY) &= -X.v(PY) + C(\phi X, PY) + v(\nabla_X PY) \\ &= -X.v(PY) + v(\nabla_X PY) + C(\phi X, PY) \\ &= C(X, \phi PY) + \theta(X)\eta(PY) - \tau(X)v(PY) + C(\phi X, PY).\end{aligned} \tag{4.28}$$

Putting (4.27) and (4.28) together in (4.26), we obtain, $(L_\xi C)(X, PY) = \tau(\xi)C(X, PY)$, for any $X, Y \in \Gamma(TM)$. The equivalence is obvious by definition. □

LEMMA 4.9. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$, with $\xi \in TM$. Then, the covariant derivative of v and the Lie derivative of g with respect to the vector field U are given, respectively, by*

$$(\nabla_X v)Y = -C(X, \phi Y) - \theta(X)\eta(Y) + \tau(X)v(Y), \tag{4.29}$$

$$(L_U g)(X, Y) = X.v(Y) + Y.v(X) + v([X, Y]) - 2v(\nabla_X Y), \tag{4.30}$$

for any $X, Y \in \Gamma(TM)$.

Proof. The proof of (4.29) follows from (4.5) and (4.30) follows from direct calculations. □

Now, we say that the screen distribution $S(TM)$ is totally contact umbilical if we have

$$\begin{aligned}C(X, Y) &= \alpha\{g(X, Y) - \eta(X)\eta(Y)\} + \eta(X)C(Y, \xi) + \eta(Y)C(X, \xi) \\ &= \alpha\{g(X, Y) - \eta(X)\eta(Y)\} - \eta(X)v(Y) - \eta(Y)v(X),\end{aligned} \tag{4.31}$$

where α is a smooth function on $\mathcal{U} \subset M$.

If we assume that the screen distribution of the lightlike hypersurface M of an indefinite Sasakian manifold, with $\xi \in TM$, is totally contact umbilical, then it follows that C is symmetric on $\Gamma(S(TM))$ and hence, according to what mentioned above, the distribution $S(TM)$ integrable.

THEOREM 4.10. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$, with $\xi \in TM$, such that $S(TM)$ is totally contact umbilical. Then $S(TM)$ is totally contact geodesic and $c = -3$.*

Proof. By a direct calculation of the right hand side in (4.24) and using (4.31), we get

$$\begin{aligned}(4.32)\quad &(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ)\\ &= \{g(Y, PZ) - \eta(Y)\eta(PZ)\}(X.\alpha) - \{g(X, Z) - \eta(X)\eta(PZ)\}(Y.\alpha)\\ &\quad + \alpha\{B(X, PZ)\theta(Y) - B(Y, PZ)\theta(X)\} + \alpha\{u(X)\theta(Y) + g(\phi X, Y)\\ &\quad - u(Y)\theta(X) - g(\phi Y, X)\}\eta(PZ)\\ &\quad + \alpha\{g(\phi X, PZ)\eta(Y) - g(\phi Y, PZ)\eta(X)\}\\ &\quad + \{u(X)\theta(Y) + g(\phi X, Y) - u(Y)\theta(X) - g(\phi Y, X)\}v(PZ)\\ &\quad + \{g(\nabla_Y U, PZ)\eta(X) - g(\nabla_X U, PZ)\eta(Y)\}\\ &\quad + \{g(\phi X, PZ)v(Y) - g(\phi Y, PZ)v(X)\}\\ &\quad + \{B(Y, U)\theta(X) + g(\nabla_Y U, X) - B(X, U)\theta(Y) - g(\nabla_X U, Y)\}\eta(PZ)\\ &\quad + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ).\end{aligned}$$

Putting $X = E$ in (4.32) and in the right hand side of (4.24), we obtain

$$\begin{aligned}(4.33)\quad &\{g(Y, PZ) - \eta(Y)\eta(PZ)\}(E.\alpha) - \alpha B(Y, PZ) - 2\alpha u(Y)\eta(PZ)\\ &- \alpha u(PZ)\eta(Y) - 2u(Y)v(PZ) - g(\nabla_E U, PZ)\eta(Y) - u(PZ)v(Y)\\ &+ \{B(Y, U) + g(\nabla_Y U, E) - g(\nabla_E U, Y)\}\eta(PZ) - \tau(E)C(Y, PZ)\\ &= \frac{c+3}{4}\bar{g}(Y, PZ) + \frac{c-1}{4}\{-\eta(Y)\eta(PZ) + u(PZ)v(Y) + 2u(Y)v(PZ)\}.\end{aligned}$$

Replacing $Y = PZ = U$ in (4.33), we have $\bar{g}(\overline{R}(E, U)U, N) = -\alpha B(U, U) = -\alpha C(U, V) = -\alpha^2 = 0$. The last assertion is obtained by taking $Y = V$ and $PZ = U$ in (4.33). □

COROLLARY 4.11. *There exist no lightlike hypersurfaces M of indefinite Sasakian space forms $\overline{M}(c)$ $(c \neq -3)$ with $\xi \in TM$ and totally contact umbilical screen distribution.*

It is easy to check that, when the screen distribution $S(TM)$ of a lightlike hypersurface M, with $\xi \in TM$, is η-totally umbilical, its second fundamental form C vanishes identically, that is, $S(TM)$ is totally geodesic.

THEOREM 4.12. *Let $(M, g, S(TM))$ be a lightlike hypersurface of Indefinite Sasakian space form $\overline{M}(c)$ with $\xi \in TM$, such that $S(TM)$ is totally contact geodesic. If $S(TM)$ is parallel. Then,*

(i) *The* 1*-form τ vanishes identically on M.*
(ii) *D' is a Killing distribution.*

Proof. If $S(TM)$ is parallel, then $C = 0$ [4]. (i) follows from $0 = (\nabla_X C)(\xi, V) = -\tau(X)$. (ii) is obvious. □

THEOREM 4.13. *Let $(M, g, S(TM))$ be a lightlike hypersurface of Indefinite Sasakian space form $\overline{M}(c)$, with $\xi \in TM$, such that $S(TM)$ is totally contact geodesic. If the local second fundamental form B is parallel, then, the following assertions are equivalent*

(i) *M is D-totally geodesic.*
(ii) *$A_E^* X = 0$, $\forall X \in \Gamma(D)$.*
(iii) *$TM^{\perp}$ is a D-parallel on M.*
(iv) *$\bar{\phi}(TM^{\perp})$ is a D-Killing distribution on M.*

Proof. The equivalence of (i) and (iv) follows from (3.32), since B is parallel. By using the second equation of (2.9), we obtain the equivalence of (ii) and (iii). Next, we prove the equivalence of (i) and (ii). Suppose M is D-totally geodesic. Then, for any $X, Y \in \Gamma(D)$, $B(X, Y) = \bar{g}(h(X, Y), E) = 0$. In particular, for any $X \in \Gamma(D)$ and $Y = V$, $B(X, V) = 0$. We have $u(A_E^* X) = g(A_E^* X, V) = 0$, i.e. $A_E^* X \in \Gamma(D \perp \langle \xi \rangle)$. Since $g(A_E^* X, N) = 0$ and $g(A_E^* X, \xi) = -u(X) = 0$, that is, $A_E^* X$ has no component in $\Gamma(TM^{\perp})$ and in $\langle \xi \rangle$, so $A_E^* X \in \Gamma(\bar{\phi}(TM^{\perp}) \perp D_0)$. If $A_E^* X = \beta V + Z$, $Z \in \Gamma(D_0)$, we have, $g(A_E^* X, Z) = \beta g(V, Z) + g(Z, Z) = g(Z, Z)$. On the other hand, we have

$$
\begin{aligned}
g(A_E^* X, Z) &= -\bar{g}(\overline{\nabla}_X E, Z) = -X.g(E, Z) + \bar{g}(E, \overline{\nabla}_X Z) \\
&= g(E, \nabla_X Z) + B(X, Z)\bar{g}(E, N) = 0.
\end{aligned}
$$

Thus, for any $Z \in \Gamma(D_0)$, $g(Z, Z) = 0$. Since D_0 is non-degenerate, then $Z = 0$. Finally $A_E^* X = \beta V \in \Gamma(\bar{\phi}(TM^{\perp}))$. Conversely, suppose that, for any $X \in \Gamma(D)$, $A_E^* X \in \Gamma(\bar{\phi}(TM^{\perp}))$. Let $\mathscr{B}_D = \{E, \bar{\phi}E, F_i, i = 1, 2, \ldots, 2n-4\}$ be a local orthonormal field of frames of D such that $D_0 = \mathrm{Span}\{F_i, i = 1, 2, \ldots, 2n-4\}$. Now, we want to show that $B(X, .)$ vanishes in each element of $\mathscr{B}_D$. For any $X \in \Gamma(D)$ $(X = RX)$, $u(A_E^* X) = 0$, i.e. $B(X, V) = 0$. $B(X, \xi) = \bar{g}(\overline{\nabla}_X \xi, E) = -\bar{g}(\bar{\phi}X, E) = -\bar{g}(\bar{\phi}RX, E) = 0$, since D is invariant under $\bar{\phi}$. $B(X, F_i) = -\bar{g}(\overline{\nabla}_X F_i, E) = g(F_i, \nabla_X E) = g(F_i, A_E^* X) = 0$, since $D_0 \perp \bar{\phi}(TM^{\perp})$. Let Y be an element of $\Gamma(D)$. Locally, we have

$$
Y = \theta(Y)E + v(Y)V + \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} F_i \in \Gamma(D),
$$

with $g(F_i, F_i) \neq 0$ because of the non-degeneracy of D_0. Consequently,

$$B(X, Y) = \theta(Y)B(X, E) + v(Y)B(X, V) + \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} B(X, F_i) = 0.$$

Hence, M is D-totally geodesic. It is easy to check that $A_E^* X = u(A_N X)V$. So, we have

$$A_E^* X = u(A_N X)V = C(X, V)V = -\eta(X)V = 0,$$

since $S(TM)$ is totally contact geodesic. □

5. Concluding remarks

It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.5)–(2.10) that in the case of lightlike hypersurfaces, there are interrelations between these geometric objects and those of its screen distributions. So, the geometry of lightlike hypersurfaces depends on the vector bundles $(S(TM), S(TM^{\perp})$ and $N(TM))$. However, it is important to investigate the relationship between some geometrical objects induced, studied above, with the change of the screen distributions. In this case, it is known that the local second fundamental form of M on $\mathscr{U}$ is independent of the choice of the above vector bundles. This means that all results of this paper which depend only on B are stable with respect to any change of those vector bundles.

Next, we study the effect of the change of the screen distribution on the results which also depend on other geometric objects. Recall the following four local transformation equations (see [4] page 87) of a change in $S(TM)$ to another screen $S(TM)'$:

$$W_i' = \sum_{j=1}^{2n-1} W_i^j (W_j - \varepsilon_j c_j E), \tag{5.1}$$

$$N' = N - \frac{1}{2}\left\{\sum_{i=1}^{2n-1} \varepsilon_i (c_i)^2\right\} E + \sum_{i=1}^{2n-1} c_i W_i, \tag{5.2}$$

$$\tau'(X) = \tau(X) + B(X, N' - N), \tag{5.3}$$

$$\nabla_X' Y = \nabla_X Y + B(X, Y)\left\{\frac{1}{2}\left(\sum_{i=1}^{2n-1} \varepsilon_i (c_i)^2\right) E - \sum_{i=1}^{2n-1} c_i W_i\right\}, \tag{5.4}$$

where $\{W_i\}$ and $\{W_i'\}$ are the local orthonormal basis of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E. Here c_i and W_i^j are smooth functions on $\mathscr{U}$ and $\{\varepsilon_1, \ldots, \varepsilon_{2n-1}\}$ is the signature of the base $\{W_1, \ldots, W_{2n-1}\}$.

Denote by ω the dual 1-form of $W = \sum_{i=1}^{2n-1} c_i W_i$ (characteristic vector field of the screen change) with respect to the induced metric g of M, that is $\omega(X) = g(X, W)$, $\forall X \in \Gamma(TM)$.

Let P and P' be projections of TM on $S(TM)$ and $S(TM)'$, respectively with respect to the orthogonal decomposition of TM. So, any vector field X on M can be written as $X = PX + \theta(X)E = P'X + \theta'(X)E$, where $\theta(X) = \bar{g}(X, N)$ and $\theta'(X) = \bar{g}(X, N')$. Then, using (5.2) we have

$$P'X = PX - \omega(X)E \quad \text{and} \quad C'(X, P'Y) = C'(X, PY), \quad \forall X, Y \in \Gamma(TM). \tag{5.5}$$

The relationship between the second fundamental forms C and C' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by (using (5.2) and (5.4))

$$\begin{aligned} C'(X, PY) &= C(X, PY) - \frac{1}{2} g(W, W)B(X, Y) + g(\nabla_X PY, W) \\ &= C(X, PY) - \frac{1}{2} g(\nabla_X PY + B(X, Y)W, W) \\ &= C(X, PY) - \frac{1}{2} \omega(\nabla_X PY + B(X, Y)W), \end{aligned} \tag{5.6}$$

Note that if the lightlike hypersurface M is totally geodesic, by (5.4), the linear connection ∇ is unique.

Proposition 5.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any $X, Y, Z \in \Gamma(TM)$,*

$$\bar{g}((\nabla'_X h')(Y, Z), E) = \bar{g}((\nabla_X h)(Y, Z), E) + \mathscr{L}(X, Y, Z), \tag{5.7}$$

where $\mathscr{L}$ is given by $\mathscr{L}(X, Y, Z) = B(X, Y)B(Z, W) + B(X, Z)B(Y, W) + B(Y, Z)B(X, W)$.

We note that $\mathscr{L}(X, Y, Z)$ is symmetric with respect to X, Y and Z. Moreover $\mathscr{L}(\cdot, \cdot, E) = 0$ and $\mathscr{L}_\xi(X, Y) = \mathscr{L}(X, Y, \xi) = -u(W)B(X, Y) - u(X)B(Y, W) - u(Y)B(X, W)$. Also, it is easy to check that the parallelism of h is independent of the screen distribution $S(TM)$ $(\nabla' h' \equiv \nabla h)$ if and only the second fundamental form B of M vanishes identically on M.

Also, we have the following lemmas.

Lemma 5.2. *The forms v and v' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: $v'(X) = v(X) - \frac{1}{2}\omega(-2\phi X + u(X)W)$.*

Lemma 5.3. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \bar{g})$ with $\xi \in TM$ such that its screen distribution is totally contact umbilical. Then, the second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: $C'(X, Y)$*

$= C(X, Y) + \frac{1}{2}K(X, Y)$, *where K is a symmetric bilinear form defined by* $K(X, Y) = \eta(X)\omega(\phi Y + u(Y)W) + \eta(Y)\omega(\phi X + u(X)W)$.

Therefore, the results expressed in terms of C and these two last Lemmas are independent of the screen distribution $S(TM)$ if and only if $\omega(\nabla_X PY + B(X, PY)W) = 0$, $\forall X, Y \in \Gamma(TM)$.

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Fortuné Massamba
Department of Mathematics
University of Botswana
Private Bag 0022 Gaborone
Botswana
E-mail: massfort@yahoo.fr
massambaf@mopipi.ub.bw