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HYPERPLANE SECTION PRINCIPLE OF LEFSCHETZ ON CONIC-BUNDLE AND BLOWING-DOWN

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1. Introduction

We work over the complex number field.

(#) Let N_1 be an $n(>3)$-dimensional projective variety which is a locally complete intersection and A a smooth ample Cartier divisor. Moreover let A be a blowing-up of a smooth projective variety B along a smooth subvariety C.

Under the condition (#) we consider the following

PROBLEM 1.0. Under which condition there exists an n-dimensional complete Moishezon space N_2 containing B as a divisor where N_2 is a locally complete intersection, N_1 the blowing-up of N_2 along the subvariety C and where A a strict transform of B on N_1.

Let $r = \dim B - \dim C > 1$ and E in A the exceptional locus via the blowing-up of B along the subvariety C. It is already known that there exists N_2 in Problem 1.0 under each condition:

1) (Fu80) $r > 2$ and N_1 is smooth.
2) (Fa84, Fa86) $r = 2$, N_1 is a smooth 4-fold with $\kappa(N_1) \geq 0$ and $E \neq \mathbf{P}^1 \times \mathbf{P}^1$.

Thus we can pose a

CONJECTURE 1.1. *Let us maintain the notations and the condition* (#) *as above. Assume* $\kappa(N_1) \geq 0$ *and* $n \geq 5$. *Then there exists* N_2 *in Problem* 1.0 *and* $n = \dim N_1 \leq 5$, *if* E *is not isomorphic to* $\mathbf{P}^{n-3} \times \mathbf{P}^1$. *When* E *is isomorphic to* $\mathbf{P}^{n-3} \times \mathbf{P}^1$, *there is a birational morphism* $g : N_1 \to N_3$ *whose exceptional locus* D *is contracted to* $\mathbf{P}^1$ *via* g.

In this paper we show

THEOREM 1.2. *Under the condition* (#) *we assume that* $r = 2$ *and* $\kappa(N_1) \geq 0$. *Then the conclusion of above conjecture* 1.1 *holds if* $n = 5$ *and* E *is not isomorphic to* $\mathbf{P}^{n-3} \times \mathbf{P}^1$.

See an example in Remark 4.4 for Thereom 1.2.

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Remark 1.3. (1) We need a condition: $\kappa(N_1) \geq 0$. If otherwise we have a counterexample: In fact for a smooth $(n-1)$-fold $B(n > 3)$, we have only to consider a $\mathbf{P}^1$-bundle N_1 over B in the Zariski topology and a smooth ample Cartier divisor A in N_1 which yields a tautological line bundle. (see introduction [Fa86])

(2) As a matter of fact even if we replace $\kappa(N_1) \geq 0$ by the condition: a fiber $\mathbf{P}^1$ in the exceptional locus E via the blowing-up $h : A \to B$ does not deform to fill up N_1, we have the conclusion of Theorem 1.2.

In order to show Theorem 1.2, we take the locus D which is shown to be a divisor in N_1 where $D \cap A = E$. Next we show that D has a $\mathbf{P}^2$-bundle structure over C which is the extension of $\mathbf{P}^1$-bundle $E \to C$ and finally that D in N_1 collapses to C in N_2. Thus we pose more general problem on the extension of a morphism.

In the following, setting $r = 2$, we change the notation by the following way: D to M, E to A, C to S, and $n-1$ to n.

PROBLEM 1.4. Let M be a projective $n(\geq 4)$-fold which is a locally complete intersection, A a smooth ample Cartier divisor and $\pi : A \to S$ a conic bundle over a smooth projective $(n-2)$-fold S. (see below as for the definition of conic bundle) Assume that $\rho(A) = \rho(S) + 1$ and that A is not isomorphic to $\mathbf{P}^{n-2} \times \mathbf{P}^1$. Then π is extended to a morphism $\phi : M \to S$. Particularly assume that $\pi : A \to S$ is a $\mathbf{P}^1$-bundle over a smooth projective $(n-2)$-fold S. Then the morphism $\phi : M \to S$ is a $\mathbf{P}^2$-bundle with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)$ and $\dim S \leq 2$.

DEFINITION 1.4.1. A non-singular projective variety X is called a quadric bundle over a smooth projective variety Y if there exists a surjective morphism $f : X \to Y$ such that every fiber X_y is isomorphic to a possibly singular hyperquadric of the same dimension m. When $m = 1$, it is called a conic bundle. See Proposition 3.4 due to Mori and Mukai [MoMu85] and Proposition 3.5 in [Mi83] about the property of conic bundles with $\rho(X) = \rho(Y) + 1$.

(1.4.2) Remark in case of $\dim X = 2$ that a conic bundle $f : X \to Y$ with $\rho(X) = \rho(Y) + 1$ is a geomerically ruled surface over Y and therefore Problem 1.4 for $n = 3$ holds due to Badescu [Ba80] (see Theorem 5.5.3 [BS95] also). The latter part in Problem 1.4 is called **Sommese's conjecture**.

In the section 2 and 3 we show

THEOREM 1.5. *Let M be a projective 4-fold which is a locally complete intersection, A a smooth ample Cartier divisor and $\pi : A \to S$ a conic bundle over a smooth projective surface S. Assume that $\kappa(S) \geq 0$ and $\rho(A) = \rho(S) + 1$. Then π is extended to a morphism $\phi : M \to S$ which is one of the following*:

i) *$\mathbf{P}^2$-bundle over S with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^2}(1)$ (Case 3 (b.2)).*

ii) *$\mathbf{P}^2$-bundle over S with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^2}(2)$ (Case 4 (b.2)).*

iii) *Quadric-bundle over S (Case 4 (b.2)), namely which is contained in a $\mathbf{P}^3$-bundle $g : V \to S$ over S with $\mathcal{O}_V(M)|_{g^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^3}(2)$.*

Thus combining Main theorem in [FaSaSo87] and [SaSp86], we get

COROLLARY 1.6 (Sommese's conjecture in case of $n = 4$). *Let M be a projective 4-fold which is a locally complete intersection, A an ample Cartier divisor and $\pi : A \to S$ a $\mathbf{P}^1$-bundle over a smooth projective surface S. If A is not isomorphic to $\mathbf{P}^2 \times \mathbf{P}^1$, then π is extended to a morphism $\phi : M \to S$ which is a $\mathbf{P}^2$-bundle with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^2}(1)$.*

Remark 1.7. 1) In [SaZh00] Corollary 1.6 is shown under the assumption of smoothness of M. Thus in this paper the investigations in the singular case are made carefully.

2) Under the assumption: $H^0(S, K_S) \neq 0$ Sommese conjecture holds. It is obtained by modifying some part in the proof in Theorem III (4.29) [Sa87] slightly. Thus under the same one Theorem 1.2 holds if $\dim N_1 \geq 5$.

Remark 1.8. For a variety B $N_1(B)$ denotes $\Sigma\mathbf{Z}\{1\text{-cycle of } B/\text{numerical equivalence}\} \otimes \mathbf{R}$. Assume $\dim M \geq 4$. Then an embedding $i : A \hookrightarrow M$ induces an isomorphism $i_* : N_1(A) \cong N_1(M)$. Thus Theorem 1.5 and Corollary 1.6 say that an extremal ray R in $N_1(A)$ induced by π goes to the one $i_*(R)$ in $N_1(M)$ induced by ϕ.

This paper is organized as follows. In section 2 and section 3 we study basic facts and show Proposition 2.1. Using this Proposition we prove Theorem 1.2 and Theorem 1.5 in section 4.

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2. Preliminary

In this section we state Proposition 2.1 which is necessary to get Theorem 1.5. We give the setup for the proof, divide the proof into cases (from case 1) to 4)) and state the proof in Case 1), 2) and 3). The main case Case 4) will be dealt in the next section.

The method of the proof is basically the same as the one in [SaZh00]. Especially we need to check three cases: (b2) in Case 3, (a1), (b2) and (b3) in Case 4 [SaZh00] carefully.

Hereafter we assume that

(2.0) the Kodaira dimension of a smooth projective surface S is non-negative.

We begin with a well-known fact related with the above assumption (2.0) which is used hereafter.

(**Kod**) Let $f : T \to S$ be a morphism from a projective ruled surface T to a smooth projective surface S. Then if the morphism f is surjective, then S is

ruled. Equivalently if the Kodaira dimension of S is non-negative, then the image $f(T)$ is a point or a curve.

PROPOSITION 2.1. *Let M be a projective 4-fold which is a locally complete intersection, A a smooth ample Cartier divisor and $\pi : A \to S$ a conic bundle over a smooth surface S. Assume $\kappa(S) \geq 0$ and $\rho(A) = \rho(S) + 1$. Then we have the following two possibilities*:

1) *$\pi : A \to S$ is extended to a morphism $\phi : M \to S$ which is one of the following*:

i) *$\mathbf{P}^2$-bundle over S with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^2}(1)$ (Case 3 (b.2))*.

ii) *$\mathbf{P}^2$-bundle over S with $\mathcal{O}_M(A)|_{\phi^{-1}(s)} \cong \mathcal{O}_{\mathbf{P}^2}(2)$ (Case 4 (b.2))*.

iii) *Quadric-bundle over S (Case 4 (b.2))*.

2) *There is a birational morphism $f : M \to W$ onto a projective variety W where W is a locally complete intersection and M is the blowing-up of W along a smooth subvariety F contained in the smooth part of W.*

Moreover $f(A)$ has the following properties:

1. *$f(A)$ is a smooth ample Cartier divisor contained in the smooth part of W and $f|_A : A \to f(A)$ is the blowing-up of $f(A)$ along the subvariety $F(\subset f(A))$.*

2. *$f(A)$ has a conic-bundle structure $\pi' : f(A) \to S'$ over a smooth projective surface S'.*

3. *There is a birational morphism $f' : S \to S'$ with the commutativity $(f|_A)\pi' = \pi f'$. Here the Picard number $\rho(S)$ of S is equal to $\rho(S') + 1$. (Case 4 (a.1))*

Remark 2.1.1. 1) The case 2) in Proposition 2.1 does not occur, which is shown in the proof in Theorem 1.5.

2) In 1) of Proposition 2.1 the conic bundle $\pi : A \to S$ of the subcase i) turns out to be $\mathbf{P}^1$-bundle in the Zariski topology and the one of ii) with singular fibers.

We first consider a sufficient condition for a variety to have rational singularities.

Remark 2.2. (1) Grothendieck [Gro68] showed the following:

Let (R, m) be a regular local ring, p a prime ideal of R and $A = R/p$. Assume that p is generated by R-sequence. If A_q is UFD for each prime ideal q in Spec A with $htq \leq 3$, then A is UFD. This says that

(2) Let X be an $n(\geq 4)$-dimensional variety which is a locally complete intersection with at most isolated singularities. Then X is locally-factorial. In particular M in Problem 1.4 is locally-factorial. In fact since A is a smooth ample Cartier divisor in M, M has at most isolated singularities.

From now on we state a property which makes it possible to use the contraction theorem by extremal ray.

PROPOSITION 2.3. *Let X be a locally-factorial and Gorenstein projective variety and Y an irreducible divisor of X. Assume that there is a surjective morphism $b : Y \to Z$ to projective variety Z so that a general fiber l of b is $\mathbf{P}^1$. Moreover assume $(Y.l)_X \geq 0$. Then $H^0(X, K_X) = 0$.*

Proof. From the first assumption we see that $(K_Y.l)_Y = -2$ for a smooth general fiber l of b. Assume $H^0(X, K_X) \neq 0$. If K_X is not isomorphic to $\mathcal{O}_X$, we have an effective divisor D with $\mathcal{O}_X(D) \cong K_X$. For such a general fiber l of b we have $(l.D) \leq -2$ by $(K_X + Y.l)_X = (K_Y.l)_Y = -2$. Thus noting X is locally-factorial, we can take an irreducible component D' of D with $(l.D') \leq -1$ which implies $l \subset D'$. Hence D' contains a smooth general fiber l of b and therefore D' coincides Y, contradicting the assumption $(Y.l)_X \geq 0$. When $K_X \cong \mathcal{O}_X$, we get $(K_X + Y.l)_X = (Y.l)_X = -2$, a contradiction. q.e.d.

COROLLARY 2.4. *Let M be a projective $n(\geq 4)$-fold which is a locally complete intersection, A an ample Cartier divisor. Assume that A is smooth and that there is a surjective morphism $\pi : A \to S$ to a projective $(n-2)$-fold S where a general fiber l of π is $\mathbf{P}^1$. Then M has at most isolated rational Gorenstein singularities.*

Proof. Since A is a smooth ample divisor in M, M has at most isolated singularities which is Gorenstein by Remark 2.2 (2). On the other hand when M has irrational singular points $\mathrm{Irr}(M)$ which is finite, it is shown that $h^0(K_M) + h^0(K_A) \geq h^0(K_M + A) \geq \#(\mathrm{Irr}\ M)$ by virtue of Corollary 0.2.2 [So86] and by the following exact sequence: $0 \to K_M \to K_M + A \to K_A \to 0$. From the structure of A we see $(l.K_A) = -2$ where l is a fiber of π. Thus we get $h^0(K_A) = 0$. Moreover we have $H^0(M, K_M) = 0$ by Proposition 2.3 which yields that M has no irrational singular points. Thus we complete the proof. q.e.d.

From now on we begin with the proof of Proposition 2.1.

Since A is a conic bundle over S, K_A is not nef and therefore $K_M + A$ is not nef. Let us set $L = \mathcal{O}_M(A)$ and $K = K_M$. Since M has rational Gorenstein singularities by Corollary 2.4, we apply our (M, L) to [Fu87]. Since L is ample, there is a positive integer j with $2 \leq j \leq 5$ so that $K + jL$ is nef and $K + (j-1)L$ is not nef. Thus we study the following four cases separately:

Case k) $K + (6-k)L$ is nef and $K + (5-k)L$ is not nef where k runs over 1, 2, 3, 4.

First we begin with

(2.6) **Case 1** and **Case 2**

The arguments of case 1 and case 2 stated in [SaZh00] work well in these cases. In fact we infer that S is ruled, a contradiction to the assumption $\kappa(S) \geq 0$. As a consequence these two cases do not happen.

Next we study

(2.7) **Case 3** ($K + 3L$ is nef and $K + 2L$ is not nef)

We begin with a reformed version of Theorem 3′ [Fu87] under the weaker following assumption. The proof is given with few changes of Fujita's argument.

PROPOSITION 2.8. *Let U be an $n(>3)$-dimensional projective normal variety with only rational Gorenstein isolated singularities. Let L be an ample line bundle on U. Assume that $K_U + (n-1)L$ is nef and that $K_U + (n-2)L$ is not nef.*

Moreover assume that U is a locally complete intersection. If Picard number of U, $\rho(U) \geq 2$, then we have

(a) *There is a birational morphism $f : U \to W$ and the exceptional locus E is $\mathbf{P}^{n-1}$ with $(E, L_E) \cong (\mathbf{P}^{n-1}, \mathcal{O}(1))$.*

(b) *There is a surjective morphism $f : U \to W$ to a smooth projective variety W. Let F be a general fiber of f.*

(b.1) $\dim W = 1$ *and* (F, L_F) *is* $((n-1)$*-dimensional hyperquadric*, $\mathcal{O}(1))$.

(b.2) $\dim W = 2$, *f makes (U, L) a scroll over a smooth surface W and* $(F, L_F) \cong (\mathbf{P}^{n-2}, \mathcal{O}(1))$.

Proof. First since $n \geq 4$ and $\rho(U) \geq 2$, (b.1) follows immediately from [Fu87, Theorem 3 b1]. Next remark that every Weil divisor in U is Cartier by Remark 2.2 (2). As for (a), taking general $(n-2)$ hyperplane sections $H_1, \ldots, H_{n-2}$ in U, we see that $H_1 \cap, \ldots, \cap H_{n-2}$ is a smooth surface, noting that U has at most isolated singularities. Since the exceptional locus E of U is Cartier, the proof of the case (a) in [Fu87] works well.

(b.2) follows from the following (2.9). Fujita actually showed (2.9) in (2.12) [Fu87], although he assumed U is smooth.

(2.9) Let $f : U \to S$ be a surjective morphism between normal projective varieties U, S and L an ample line bundle on U. Suppose that U is a locally complete intersection and that $\dim U - \dim S > \dim \operatorname{Sing} U$. Moreover assume that $\dim Z = r$ for every fiber Z of f and that $(F, L_F) \cong (\mathbf{P}^r, \mathcal{O}(1))$ for every general fiber F of f. Then S is smooth and f makes (U, L) a scroll over S.

As for (2.9) see Proposition 3.2.1 in [BS95]. q.e.d.

To finish the case 3) we return to the observation of case 3 in (2.7). Applying two subcases (a), (b.1) of Proposition 2.8, we infer that S is ruled (**Kod** in (2.0)), contradicting the assumption of non-negative $\kappa(S)$ as shown in [SaZha00]. Thus (a), (b.1) do not occur and only the subcase (b.2) does. In this case $A \cap f^{-1}(w)$ is $\mathbf{P}^1$ for a general point w in W. First by $\kappa(S) \geq 0$, $\pi(A \cap f^{-1}(w))$ is one point. Let $h = (\pi, f \mid A) : A \to S \times W$. Since $\rho(A) > \rho(h(A))$, we get $\rho(S) = \rho(h(A)) = \rho(W)$. Hence an induced morphism $h(A) \to S$ is a finite birational morphism and therefore an isomorphism. Similarly an induced morphism $h(A) \to W$ is an isomorphism, which is a case 1) i) of Proposition 2.1.

3. Proof of Proposition 2.1 in Case 4)

In this section we investigate Case 4 to complete the proof of Proposition 2.1.

First we begin with

(3.1) **Case 4.** ($K + 2L$ is nef and $K + L$ is not nef)

First we begin with

PROPOSITION 3.2. *Let U be an $n(>3)$-dimensional projective normal variety with only rational Gorenstein isolated singularities and L an ample line bundle on*

U. Assume that $K_U + (n-2)L$ is nef and $K_U + (n-3)L$ is not nef. Moreover assume that $|L|$ contains a smooth divisor A. Then if $f : U \to W$ is a birational elememtary contraction induced by a curve R with $(R.K_U + (n-3)L) < 0$, it is divisorial.

Proof. Let $X = \{w \in W \mid \dim f^{-1}(w) > 0\}$ and $E = f^{-1}(X)$. Then by virtue of (2.5) in [Fu87], $\dim f^{-1}(w) \geq n-2 \geq 2 (w \in W)$ and therefore $\dim(f^{-1}(w) \cap A) \geq 1$. Thus we can take an irreducible rational curve C on A so that $(K_A + (n-4)A|_A.C) = (K_U + (n-3)A.C) < 0$. Note $K_A + (n-3)A|_A$ is nef on A. Hence letting $g : A \to A'$ be an elementary contraction induced by the curve C on A and noting A is smooth, we infer that g is a birational contraction and that the exceptional locus $E(g)$ of g is a divisor in A namely, $\dim E(g) = n-2$, by virtue of Theorem (0.4) Ionescu [Io86]. Thus we see that $E(g) \subset E$ and that $E(g)$ is contained in the smooth locus of U. Now if f would be small, $E(g)$ would be an irreducible component of E. Therefore since the irreducible component $E(g)$ is contained in the smooth locus of U, the argument in Theorem (1.1) due to Wisniewski [Wi91] works and the following inequality of the conclusion holds:

$\dim F + \dim(\text{locus of } R) \geq \dim X + l(R) - 1$. (Here F is an ireducible of a non-trivial fiber f.) The curve C above applies to R. However this contradicts the assumption $(R.K_U + (n-3)L) < 0$. Thus we complete the proof. q.e.d.

Thus by virtue of Proposition 3.2 and Remark 2.2 (2) we have the consequence of theorem 4 [Fu87] under the weaker condition as the one. Note that the case of small contraction does not occur.

PROPOSITION 3.3. *Let U be an $n(>3)$-dimensional projective normal variety with only rational Gorenstein isolated singularities and L an ample line bundle on U. Assume that $|L|$ contains a smooth divisor A and that U is a locally complete intersection. If $K + (n-2)L$ is nef and $K + (n-3)L$ is not nef, then we have the following cases under the condition $\rho(U) > 1$:*

(a) *There is a birational morphism $f : U \to W$ onto a normal projective variety W with $\rho(W) = \rho(U) - 1$. Let $X = \{x \in W \mid \dim f^{-1}(x) > 0\}$. Then $E = f^{-1}(X)$ is an irreducible divisor and we have two cases*:

(a.1) $\dim X = 1$ *and for any smooth point x in X $(E_x, L_x) \cong (\mathbf{P}^{n-2}, \mathcal{O}(1))$ where E_x is the fiber of $f : E \to X$ over x and $L_x = L|_{E_x}$. In this case the restriction $\mathcal{O}_U(E)$ to E_x is $\mathcal{O}(-1)$.*

(a.234) *X is a point. $(E, L|_E)$ is $(\mathbf{P}^3, \mathcal{O}(2))$ or $(\mathbf{P}^{n-1}, \mathcal{O}(1))$ or $(Q, \mathcal{O}_Q(1))$. Here Q denotes a hyperquadric.*

(b) *There is a surjecive morphism $f : U \to W$ onto a normal projective variety W and* $\dim W < 4$. *Let F be a general fiber of f.*

(b.1) $\dim W = 1$ *and* $(F, L_F) \cong (\mathbf{P}^3, \mathcal{O}(j))$ *with* $j = 2, 3$, $(\mathbf{P}^4, \mathcal{O}(2))$, $(Q(\subset \mathbf{P}^4), \mathcal{O}_Q(2))$ *or* (*del Pezzo manifold*, $\mathcal{O}(1)$).

(b.2) $\dim W = 2$ *and* $(F, L_F) \cong (\mathbf{P}^2, \mathcal{O}(2))$ *or* $(Q(\subset \mathbf{P}^{n-1}), \mathcal{O}_Q(1))$.

(b.3) $\dim W = 3$ *and* $(F, L_F) \cong (\mathbf{P}^{n-3}, \mathcal{O}(1))$.

Before studying the case 4, we state two facts on conic bundle.

PROPOSITION 3.4 (Proposition 4.8 in [MoMu85]). *Let $f : X \to Y$ be a conic bundle from a projective smooth 3-fold X to a smooth projective surface Y. Then the following conditions are equivalent to each other.*

(1) *A general fiber X_y of f is an extremal rational curve.*

(2) $\rho(X) = \rho(Y) + 1$

(3) *For every irreducible curve C on Y, $f^{-1}(C)$ is irreducible.*

PROPOSITION 3.5 (Lemma 4.7 in [Mi83]). *Let $f : X \to Y$ be a conic bundle from a projective smooth 3-fold X to a smooth projective surface Y with $\rho(X) = \rho(Y) + 1$. Then the following conditions are equivalemt to each other.*

(1) *$f : X \to Y$ is not a standard conic bundle,*

Here a conic bundle $f : X \to Y$ is said to be standard if $\operatorname{Pic} X \cong f^* \operatorname{Pic} Y \oplus \mathbf{Z}K_X$.

(2) *$f : X \to Y$ is a $\mathbf{P}^1$-bundle in the Zariski topology.*

Remark 3.5.1. We denote Δ_π as a closed set $\{s \in S \mid \pi^{-1}(s)$ is singular$\}$ (possibly empty) and remark that Δ_π has only normal crossing as singularities where Δ_π is of 1-dimension with $\dim \operatorname{Sing} \Delta_\pi \leq 0$ and that $\pi^{-1}(s)$ is a smooth conic, a reducible conic or a double line according as $s \notin \Delta_\pi$, $s \in \Delta_\pi$ or, $s \in \operatorname{Sing} \Delta_\pi$ by [Be77].

Hereafter till the end of this section a conic bundle $\pi : A \to S$ means the one from a projective smooth 3-fold A to a smooth projective surface S with $\rho(A) = \rho(S) + 1$.

Next we state a property about a birational contraction of a conic bundle which is necessary to study subcases (a.1) and (b.3).

PROPOSITION 3.6. *Let $\pi : A \to S$ be a conic bundle* (3.5.1) *and $g : A \to A'$ an elementary contraction by an extremal rational curve R. Assume that g is birational and the Kodaira dimension of S is non-negative. Then the exceptional divisor E via g is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ where two morphisms $g|_E : E \to g(E)$, $\pi|_E : E \to \pi(E)$ correspond to one of two projections: $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$ respectively and there is an open set U containing $\pi(E)$ in S where $\pi : A \to S$ is $\mathbf{P}^1$-bundle over U, namely $\Delta_\pi \cap U = \emptyset$. Moreover there is a birational morphism $g' : S \to S'$ which is blowing down $\pi(E)$ of S to a smooth surface S' where $\pi' : A' \to S'$ is a conic bundle over S' wuth the commutativity $g'\pi = \pi' g$.*

Proof. Mori Theory says that the exceptional locus E is one of $\mathbf{P}^1$-bundle over a smooth curve C, $\mathbf{P}^2$, $\mathbf{P}^1 \times \mathbf{P}^1$ and an irreducible singular quadric surface where E in only the first case goes to a curve C via g. Hence from **Kod** in (2.0) $\pi(E)$ is an irreducible curve by non-negative Kodaira dimension of S, $\pi^{-1}(\pi(E))$ is irreducible by virtue of (3) in Proposition 3.4 and therefore $\pi^{-1}(\pi(E)) = E$.

Particularly the third case that $g(\mathbf{P}^1 \times \mathbf{P}^1)$ is a point is ruled out in view of the property of g and π. Consequently we have only to study the first case. Thus $g|_E$ is a $\mathbf{P}^1$-bundle over $g(E) \cong \mathbf{P}^1$ and $\pi(E)$ is a rational curve. Noting that every fiber of π is connected, we see that E is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ where the first projection is $g|_E$ and where $\pi|_E$ is the composite of the second projection and the normalization $h : \mathbf{P}^1 \to \pi(E)$. Particularly a general fiber of $\pi|_E : E \to \pi(E)$ is a smooth rational curve. Hence each fiber of $\pi|_E$ is a smooth rational curve or a double line. Thus we conclude that if $\pi(E)$ intersects with Δ_π then $\pi(E) \cap \Delta_\pi$ is contained in the singular part of Δ_π. Now we have a

Sublemma: $\pi(E)$ does not intersects with Δ_π. Therefore $\pi(E)$ is smooth and $\pi : A \to S$ is $\mathbf{P}^1$-bundle over an open set around $\pi^{-1}(E)$.

In fact if otherwise, we take a point s in $\pi(E) \cap \Delta_\pi$ and let l be the reduced part ($\cong \mathbf{P}^1$) of $\pi^{-1}(s)$. For subvarieties $l \subset E \subset A$ we have an exact sequence:

$$0 \to N_{l/E} \to N_{l/A} \to N_{E/A|l} \to 0.$$

Since l is a fiber of the second projection of $\mathbf{P}^1 \times \mathbf{P}^1$, $N_{l/E}$ is a trivial line bundle on $\mathbf{P}^1$. Moreover $N_{E/A|l}$ is trivial since $(E.l) = 0$ by $\pi^{-1}(\pi(E)) = E$. Thus we infer that $N_{l/A} = \mathcal{O} \oplus \mathcal{O}$. On the other hand we have a

Claim. For a smooth conic bundle $h : B \to T$ over a smooth surface T let l be a reduced part of non-reduced fiber. Then $\det N_{l/B} = -1$.

In fact for a smooth fiber C of the conic bundle we have $(K_B.C) = -2$. Since C is numerically equivalent to $2l$. Thus we get $(K_B.l) = -1$ and consequently $\det N_{l/B} = -1$. We get a claim.

Thus we have a contradiction and that $\pi(E) \cap \Delta_\pi$ is empty. Thus we get sublemma.

Noting $(\mathcal{O}_E(E).\bar{l}) = -1$ for a fiber $\bar{l}$ of a $\mathbf{P}^1$-bundle $g|_E : E \to g(E)$, we get $\pi(E)^2 = -1$, which yields the remainder. Thus we complete the proof. q.e.d.

(3.7) Let us return the proof of Proposition 2.1 and first show that the subcase (a.234) does not occur.

(3.7.1) Subcase (a.234) in case 4)

Let E be the exceptional locus of f with a point $P := f(E)$. Then $E \cap A$ is one of $\mathbf{P}^2$, a singular quadric surface and $\mathbf{P}^1 \times \mathbf{P}^1$. By $\kappa(S) \geq 0$ and **Kod** we infer that the first two cases are ruled out and $\pi(E \cap A)$ is a curve. Moreover we get $\pi^{-1}(\pi(E \cap A)) = E \cap A$ by (3) in Proposition 3.4. Since f collapses a fiber of π to the point P, we see $\dim f(A) < 3$, which contradicts to the ampleness of A (see (a1) in p. 317 [SaZa00]). Thus this subcase does not occur.

Subcase (a.1) in case 4).

(1) Let $f : M \to W$ be a birational morphism induced by an extremal rational curve C in A where $(K + 2A, C) = 0$, and $f(C)$ is a point. Let $X = \{x \in W \mid \dim f^{-1}(x) > 0\}$ and $E = f^{-1}(X)$ an irreducible divisor. Thus since $K_M + 2A|_A$ is nef and not ample, we see that the morphism $f|_A : A \to f(A)$ factors $h : A \to A' = h(A)$ and $h' : A' \to f(A)$ where h is a contraction by an extremal rational curve in A. On the other hand since f is birational and A is

ample, h is birational and the exceptional locus $\mathrm{Exc}(h)$ via h is contained in $A \cap E$.

We show

(2) $\mathrm{Exc}(h) = A \cap E$ and h can be identified as the morphism g in Proposition 3.6.

For a point x in X let us set $E_x := f^{-1}(x)$ and $l_x := E_x \cap A$. Then recalling that E is irreducible and $\dim E_x = 2$, we infer by the same argument as in (a.234) that for each point x in X E_x is not contained in A, $\dim l_x = 1$ and $f(A \cap E)$ is a curve X, since E_x is P^2 for each smooth point x in X and therefore each component of E_x is a ruled surface for each point x in X [Ma68]. Noting that $A \cap E = \bigcup_{x \in X} l_x$ and that $l_x \cong P^1$ for each point x of an open set X_0 in X, we see that $A \cap E$ is irreducible. In fact if othewise, we could find an irreducible component D of $A \cap E$ which is contained in $\bigcup_{x \in X - X_0} l_x$ and get $\dim D = 1$. It is absurd. Hence since $\mathrm{Exc}(h)$ is in $A \cap E$ these two coincide. Thus we get (2).

(3) By Proposition 3.6, we see $A \cap E \cong \mathbf{P}^1 \times \mathbf{P}^1$. Since M is a locally complete intersection, E is Cohen-Macauley. Hence E is normal. Thus letting $p : X' \cong \mathbf{P}^1 \to X$ be the normalization of X, we have a morphism $f' : E \to X'$ induced by $f|_E : E \to f(E)$ with $f|_E = pf'$. Then f' is $\mathbf{P}^2$-bundle over X' and the restriction of two morphisms π, f' to $A \cap E$ corresponds to two canonical projections of $\mathbf{P}^1 \times \mathbf{P}^1$ respectively. Thus $\mathrm{Sing}\, M \cap E$ is empty and $\mathcal{O}_M(E)|_F = \mathcal{O}_{\mathbf{P}^2}(-1)$ with a fiber F of $f' : E \to X'$. Note that both $\mathrm{Sing}\, M \cap A$ and $\pi^{-1}(\Delta_\pi) \cap E$ are empty.

Thus we get

(4) M can be blown down along the direction $f' : E \to X'$ to an algebraic space N [Na71][Ar70]. Set the morphism as $\bar{f} : M \to N$. Then $\bar{f}|_A$ can be identified as $h : A \to h(A)$ by Proposition 3.6. $\bar{f}(A)$ is ample in N and N is a projective variety by Step 5 [SaZh00]. Moreover it is a locally complete intersection, $B \subset \mathrm{Reg}\, N$ and B is a conic bundle over a smooth surface S' where S' is a blowing-down of S along an exceptional curve $\pi(A \cap E)$.

(5) $f : M \to W$ can be identified as $\bar{f} : M \to N$. Hence since $p : X' \cong \mathbf{P}^1 \to X$ is an isomorphism, $f|_E : E \to X$ is $\mathbf{P}^2$-bundle over $\mathbf{P}^1$.

In fact we have $\rho(M) = \rho(N) + 1$ and $\rho(M) = \rho(W) + 1$. Define a morphism $h = (\phi, f) : M \to W \times N$ and set the image of h as $\overline{M}$. Since $\rho(M) \geq \rho(\overline{M}) + 1$ and therefore $\rho(\overline{M}) = \rho(W) = \rho(N)$, two natural projections $\overline{M} \to W$ and $\overline{M} \to N$ are isomophisms by Zariski Main Theorem. Thus we get (5).

We finish the subcase (a.1).

We consider subcase (b).

Subcase (b.1). This case does not occur by the proof in [SaZh00].

Subcase (b.2). We show that the case occurs that only $\pi : A \to S$ has a standard conic bundle structure. First $\pi(A \cap F)$ is a point. If otherwise, S is ruled, a contradicton since $\pi(A \cap F)$ is a rational curve. Thus the morphism $f|_A : A \to W$ factors $A \xrightarrow{\pi} S \xrightarrow{j} W$ where $j : S \to W$ is a surjective morphism. Note that a general fiber $A \cap F$ of $f|_A : A \to W$ is irreducible, since $A \cap F$ is an ample divisor in F. Consequently we see that the morphism j is birational. Moreover by $\rho(M) = \rho(W) + 1$ and $\rho(A) = \rho(S) + 1$, we get $\rho(S) = \rho(W)$.

Hence j is an isomorphism. Since A is ample in M, $f : M \to S$ is of equi-dimension and therefore flat.

First consider the case that for a general fiber F of f, $(F, L_F) \cong (Q(\subset \mathbf{P}^3), \mathcal{O}_Q(1))$. Then we see easily that M is a quadric-bundle over S.

In case of $F \cong \mathbf{P}^2$ taking a line bundle $L' = -K_M - A$ on M, we see that $L'|_F \cong \mathcal{O}_{\mathbf{P}^2}(1)$. Note that L' is relatively ample with respect to f. Hence M is $\mathbf{P}^2$-bundle over S by (2.9). Consequently the case does not happen that $A \to S$ is a $\mathbf{P}^1$-bundle over S, as stated in [SaZh00], [Ba80] and (1.4.2) but the one does that $\pi : A \to S$ is a standard conic bundle over S with "singular fibers".

Next to get Corollary 1.6 shown after, we state

Remark 3.7.1.1. Under the conditions and assumptions in Theorem 1.5, we assume, moreover, that a conic bundle $\pi : A \to S$ is a $\mathbf{P}^1$-bundle over a smooth projective surface S as in Corollary 1.6. Then the subcase (b.2) does not occur.

It is proved just before that the case does not occur that $M \to S$ is a $\mathbf{P}^2$-bundle over S. Hence let F be a fiber of the quadric bundle $f : M \to S$. Since $A \cap F$ is a smooth conic in F by assumption, F is a smooth quadric surface or a possibly singular quadric surface with one vertex. Thus we remark that each line l on F is not contained in A where $l \cap A$ is scheme-theoretically one point. Now take a line l on a smooth quadric surface F and consider the Hilbert scheme R' of l in M. Then we have the following exact sequence:

$$0 \to N_{l/F} \to N_{l/M} \to N_{F|M}|_l \to 0$$

Since $N_{l/M} = \mathcal{O}_{\mathbf{P}^1}^{\oplus 3}$, and therefore $H^1(l, N_{l/M}) = 0$, we can take a 3-dimensional irreducible component R of R' containing the line l. Let Q be the universal scheme of R and $p : Q \to M$, $q : Q \to R$ two canonical morphisms. Note that the degree of p is 2. From $(l.A) = 1$ the Cartier divisor $p^{-1}(A)$ yields a section of $q : Q \to R$ and Q is a proj of rank-2 vector bundle E on 3-fold R with the following exact sequence on R:

$$0 \to \mathcal{O} \to E \to G \to 0.$$

Here G is a line bundle on R where $P(G)$ corresponds to a section $p^{-1}(A)$ in $Q = P(E)$. Now $\mathrm{Sing}(M/S)_f$ denotes a closed set $\{m \in M \mid f^{-1}(f(m))$ is a singular quadric surface with the vertex $m\}$. Then we have

Claim: $\mathrm{Sing}(M/S)_f$ consists of at most finite subset in M.

In fact A is off $\mathrm{Sing}(M/S)_f$ since A is a smooth ample Cartier divisor.

Now take a very ample smooth curve C in S where C does not intersect with $f(\mathrm{Sing}(M/S)_f)$. Let us set $M_C = f^{-1}(C)$. Moreover let $p_C : p^{-1}(M_C) \to M_C$ be a canonical morphism between 3-folds obtained by taking the base change of $p : Q \to M$ over M_C. Hence $p_C : p^{-1}(M_C) \to M_C$ is a double covering. Since A is an ample divisor in M, so is $p_C^{-1}(A \cap M_C)$ in $p^{-1}(M_C)$. Here remark there is a surface R_C in R with $p^{-1}(M_C) = q^{-1}(R_C)$ from the constraction of Q, R. Consequently both $E|_{R_C}$ and $G|_{R_C}$ are ample vector bundles over a surface R_C.

If necessary, taking the base change of above exact sequence by the normalzation $h: R'_C \to R_C$ of R_C, we see that the above sequence splits over R_C by the vanishing theorem $H^1(R_C, -G|_{R_C}) = 0$ due to Mumford [Mu67]. This contradicts to the ampleness of $E|_{R_C}$.

Subcase (b.3). We show this case does not occur.

First $f|_A : A \to W$ is a birational morphism. Since $\rho(M) = \rho(W) + 1$, $f|_A$ is not finite. Take a curve C in A with a point $f(C)$. By assumption we see $(K_M + A.C) < 0$ and therefore $(K_A.C) < 0$. Hence we can find an extremal rational curve C_1 in A such that $f(C_1)$ is a point. By $\rho(A) = \rho(W) + 1$ we infer that $f|_A : A \to W$ is an elementary contraction induced by a smooth rational curve C_1 on A. Moreover we see by Proposition 3.6 that $\pi^{-1}(\pi(C_1)) = \mathbf{P}^1 \times \mathbf{P}^1$ and that W is a smooth 3-fold. Here note that the morphism $f|_A : A \to W$ corresponds to the one $g : A \to A'$ in Proposition 3.6. Let $\mathrm{Exc}(f|A)$ be the image of the exceptional locus $\{w \in W \mid \dim(f|_A)^{-1}(w) = 1\}$ of $f|_A : A \to W$. Then it coincides with $f(\pi^{-1}(\pi(C_1)))(= \mathbf{P}^1)$ in W. Moreover since A is ample in M, $\dim f^{-1}(w) = 1$ for each point w in $W - \mathrm{Exc}(f|A)$. Now remark that $f^{-1}(w)$ is a smooth rational curve for a general point w in $W - \mathrm{Exc}(f|A)$. $f : M \to W$ is $\mathbf{P}^1$-bundle over a smooth open subscheme $W - \mathrm{Exc}(f|A)$ in W since f is flat over there. Hence we can take a smooth curve C_2 in S which does not intersect with the exceptional curve $\pi(C_1)$ on S via the blowing-down $g' : S \to S'$. Then $f^{-1}(f(\pi^{-1}(C_2))) \to f(\pi^{-1}(C_2))$ is a $\mathbf{P}^1$-bundle over $f(\pi^{-1}(C_2))(= F \subset W)$ with a section $f^{-1}(F) \cap A$. In case of $\mathbf{P}^1$-bundle there are a rank-2 vector bundle E and a quotient line bundle G over F enjoying an exact sequence on F: $0 \to \mathcal{O} \to E \to G \to 0$ with $f^{-1}(F) \cong \mathbf{P}(E)$ and $f^{-1}(F) \cap A \cong \mathbf{P}(G)$. Since A is an ample divisor, E and G are ample vector bundles. On the other hand the exact sequence splits from $H^1(F, -G) = 0$ by Kodaira vanishing Theorem, a contradiction to the ampleness of E. Thus the case (b.3) does not happen. q.e.d.

Thus we have finished the proof of Proposition 2.1.

4. Proof of Theorem 1.2 and Theorem 1.5

(4.1) **Proof of Thereom 1.5.**

We use induction on $\rho(S)$. When $\rho(S) = 1$, (M, A) is in case 1) of Proposition 2.1. Next assume $\rho(S) > 1$.

Hereafter we assume that (M, A) is in case 2) of Proposition 2.1 and study $(W, f(A))$. Since $\rho(S) = \rho(S') + 1$, by induction assumption we have a morphism $h' : W \to S'$ in one of the case 1). By the argument below we will get a contradiction and show that case 2) of Proposition 2.1 does not occur. Consequently we complete the proof of Theorem 1.5.

Let $s'_0 := h'(F)$ be a point in S' and $W_0 = h'^{-1}(s'_0)$ a fiber in W. Note that in case i) and case ii) W_0 is isomorphic to P^2 and in case iii) it a quadric surface with at most one singularity. Similarly F is a line in W_0 in case i) and a smooth conic in the other cases.

Thus in each case take a line $C'(\neq F)$ in the fiber W_0 where in case ii) $C' \cap F$ consists of two points. Moreover let us take an irreducible and reduced curve C in M with $f(C) = C'$. Then the intersection number $(C.E)$ in M is 1, 2 and 1 in each case respectively. On the other hand we have an equality $(C'.F)_{W_0} = (C'.A') = (C.f^*A') = (C.A + E) = (C.A) + (C.E)$. But the direct calculation yields the inequality since $(C.A)$ is positive by the ampleness of A, a contradiction. Consequently the case 2) of Proposition 2.1 does not occur.

Thus we finish the proof of Theorem 1.5. q.e.d.

(4.2) **Proof of Corollary 1.6.**

First let us consider the case of $\kappa(S) \geq 0$. Then from Remark 3.7.1.1 only the case i) happens and two cases ii) and iii) in Theorem 1.5 are ruled out.

Next let us consider the case of $\kappa(S) = -\infty$. In [FaSaSo87] it is shown in Thoerem 2.0 when $S = \mathbf{P}^2$ and in Theorem when there is a surjective holomorphic map from S to a curve except the special case. The remainder is proved in [SaSp86]. q.e.d.

(4.3) **Proof of Theorem 1.2**

Let $\phi : A \to B$ be the blow-up of B along C and E the exceptional locus via the blow-up. Then E is a $\mathbf{P}^1$-bundle over C and let E_c be a fiber $\mathbf{P}^1$ of a point $c \in C$ via the $\phi : A \to B$. Then we have the following exact sequence:

$$0 \to N_{E_c/E} \to N_{E_c/A} \to N_{E/A|E_c} \to 0$$

Since $N_{E_c/E} \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus 2}$ and $N_{E/A|E_c} \cong \mathcal{O}_{\mathbf{P}^1}(-1)$, we have

$$N_{E_c/A} \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}(-1).$$

Moreover by the assumption $\kappa(N_1) \geq 0$ and the following exact sequence:

$$0 \to N_{E_c/A} \to N_{E_c/N_1} \to N_{A/N_1|E_c} \to 0$$

we get the following with $b = E_c.A (> 0)$

$$N_{E_c/N_1} \cong \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(a_2) \oplus \mathcal{O}_{\mathbf{P}^1}(a_3) \oplus \mathcal{O}_{\mathbf{P}^1}(-1),$$

with $a_1 \geq a_2 \geq a_3 \geq 0$ and $\Sigma a_i = b$.

Thus the deformation of E_c in N_1 provides us with an irreducible divisor D in N_1. We get the following property:

1. $D \cap A = E$,

(for the proof see the proof of (1.2) Theorem [So81] for example)

2. A is an ample divisor in N_1, so is E in D. Therefore D has at most isolated singularities.

By virtue of Corollary 1.6 D is a $\mathbf{P}^2$-bundle over a smooth surface C and

E is a tautological line bundle. Thus we get $1 = (E_c.E)_D = (E_c.A)_{N_1} = b$ and therefore $(D.E_c) = -1$. Thus Theorem 1.2 follows. q.e.d.

Remark 4.4. We state an example on Theorem 1.2.

Let N_2' be a hypersurface of degree $d+1$ defined by $F = X_0X_3^d + X_1X_4^d + X_2X_5^d + X_6^{d+1} = 0$ and V_1 the one of degree $e+1$ by $G = X_0X_4^e + X_1X_5^e + X_2X_3^e + X_6X_3^e + X_6^{e+1} = 0$ in $\mathbf{P}^6$ with $d, e > 0$. Let C be a plane in $\mathbf{P}^6$ defined by $X_0 = X_1 = X_2 = X_6 = 0$.

We have the following properties:

1) N_2', V_1 and $N_2' \cap V_1$ are smooth around C.

2) A hypersurface of degree b defined by $(X_3 + X_0)^b - (X_3 - X_0)^b + (X_4 + X_1)^b - (X_4 - X_1)^b + (X_5 + X_2)^b - (X_5 - X_2)^b + (X_6)^b = 0$ is a smooth variety containing the C.

Let us take N_2 as a hypersurface of degree $d+1$ defined by $X_0A_0 + X_1A_1 + X_2A_2 + X_6A_3 = 0$ where A_i $(0 \le i \le 3)$ are homegenous polynomials of degree d with generic coefficients and V as the one of degree $e+1$ similarly. Noting that the property of smoothness is an open condition in the set of subschemes containing the C, we see easily that N_2, V and $N_2 \cap V$ are smooth varieties in $\mathbf{P}^6$ containing C.

Let $\bar{f} : \bar{\mathbf{P}} \to \mathbf{P}^6$ be the blowing-up of $\mathbf{P}^6$ with the center of C. Moreover let us set N_1 the proper transform of N_2 and A of $B := N_2 \cap V$ respectively. We see that $\bar{P}$ has a $\mathbf{P}^3$-bundle structure: $\bar{g} : \bar{P} \to \mathbf{P}^3$. Letting $\bar{D}$ be the exceptional divisor of $\bar{P}$ via $\bar{f}$, we see that $\bar{D}$ is isomorphic to $\mathbf{P}^2 \times \mathbf{P}^3$ and $\bar{f}|_{\bar{D}}$ is the first projection and $\bar{g}|_{\bar{D}}$ the second one. Let us set two morphisms $f := \bar{f}|_{N_1} : N_1 \to N_2$ and $g := \bar{g}|_{N_1} : N_1 \to \mathbf{P}^3$. Then f is a birational morphism whose exceptional locus $D(:= \bar{D} \cap N_1)$ has a $\mathbf{P}^2$-bundle structure over $C = f(D) = \mathbf{P}^2$. A fiber of the morphism g is a surface of degree d. Let C_1 be a line in a fiber of $f|_D$ and C_2 a curve in a fiber of g. Thus we get

PROPOSITION 4.5. 1) $\overline{NE}(N_1) = \mathbf{R}_{>0}C_1 + \mathbf{R}_{>0}C_2$

2) *$f|_A : A \to B$ is a birational morphism whose exceptional locus $E(:= D \cap A)$ is a $\mathbf{P}^1$-bundle structure over C.*

3) $(A.C_1)_{N_1} = 1$ *and* $(A.C_2)_{N_1} = (\deg C_2)e$. *Here the degree* $(= \deg C_2)$ *of C_2 denotes the one in a fiber* $(= \mathbf{P}^3)$ *of $\bar{g}$.*

Thus if $e > 0$, A is a smooth, ample divisor in N_1. Consequently these yield a desired example.

Note $f^*B = A + D$ in N_1 and each fiber of $\bar{g}$ is embedded in $\mathbf{P}^6$ via $\bar{f}$ as a linear space. Thus deg C_2 is equal to the degree of $f(C_2)$ in $\mathbf{P}^6$. We show only the latter part of 3). In N_1 take a curve C_2 in a fiber of g, not contained in D. Then $f(C_2)$ is a space curve in $\mathbf{P}^6$, not contained in the plane C. Thus we get $(D.C_2)_{N_1} = \deg C_2$ and $(B.f(C_2))_{N_2} = (\mathcal{O}_{N_2}(e+1), f(C_2)) = (e+1) \deg C_2$.

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