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A QUOTIENT GROUP OF THE GROUP OF SELF HOMOTOPY EQUIVALENCES OF SO(4)

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Abstract

The author studies the quotient group $\mathscr{E}(\mathrm{SO}(4))/\mathscr{E}_{\#}(\mathrm{SO}(4))$, where $\mathscr{E}(\mathrm{SO}(4))$ is the group of homotopy classes of self homotopy equivalences of the rotation group SO(4) and $\mathscr{E}_{\#}(\mathrm{SO}(4))$ is the subgroup of it consisting of elements that induce the identity on homotopy groups.

1. Introduction

For a space X with a base point, let $\mathscr{E}(X)$ denote the group of homotopy classes of based self homotopy equivalences of X and let $\mathscr{E}_{\#}(X)$ be the normal subgroup of $\mathscr{E}(X)$ consisting of elements that induce the identity on homotopy groups. These groups have been studied by many people [5]. But the group structures are still unknown except for a few special cases. In particular, while $\mathscr{E}_{\#}(\mathrm{SO}(4))$ is known [4], $\mathscr{E}(\mathrm{SO}(4))$ is unknown. The purpose of this paper is to study the quotient group $\mathscr{E}(\mathrm{SO}(4))/\mathscr{E}_{\#}(\mathrm{SO}(4))$. The following basic theorem is due to Sieradski [6] and Yamaguchi [7].

Theorem 1.1. $\mathscr{E}(\mathrm{SO}(4))/\mathscr{E}_{\#}(\mathrm{SO}(4)) \cong \mathrm{Inv}(M_2(\sqrt{2}))$.

Here $M_2(\sqrt{2})$ is the ring of 2×2-matrices

$$\begin{bmatrix} a_{11} & \sqrt{2}a_{12} \\ \sqrt{2}a_{21} & a_{22} \end{bmatrix} \quad (a_{ij} \in \mathbf{Z})$$

and $\mathrm{Inv}(M_2(\sqrt{2}))$ is the group of invertible elements of $M_2(\sqrt{2})$. Our main results are stated as follows.

Theorem 1.2. *Let* $A \in \mathrm{Inv}(M_2(\sqrt{2}))$.

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(1) *The order of* A *is finite if and only if* $A = \pm E$ *or* $\mathrm{tr}(A) = 0$.
(2) *If* $\mathrm{tr}(A) = 0$, *then the order of* A *is* $3 + \det(A)$.
(3) *If* A *is of order* 4, *then* $A^2 = -E$.
Here E *denotes the unit matrix, and* $\mathrm{tr}, \det : \mathrm{Inv}(M_2(\sqrt{2})) \to \mathbf{Z}$ *denote the trace and the determinant, respectively.*

THEOREM 1.3. *The group* $\mathrm{Inv}(M_2(\sqrt{2}))$ *is not nilpotent and generated by*

$$A = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with relations:

$$C^2 = (AB^{-1})^4 = E, \quad (AB^{-1})^2 = (B^{-1}A)^2, \quad CA = A^{-1}C, \quad CB = B^{-1}C. \tag{1.1}$$

COROLLARY 1.4. *The order of any element of* $\mathscr{E}(\mathrm{SO}(4))/\mathscr{E}_\#(\mathrm{SO}(4))$ *is* 1, 2, 4 *or* ∞.

In Section 2, for completeness, we prove Theorem 1.1 by our methods. We prove Theorem 1.2 and Theorem 1.3 in Section 3 and Section 4, respectively.

2. A proof of Theorem 1.1

In this paper spaces are assumed to be based, maps and homotopies preserve base points, and the base point of a topological group is the unit. The group $\mathscr{E}(X \times Y)/\mathscr{E}_\#(X \times Y)$ with X, Y group-like spaces was studied by Sieradski [6], and his method was applied to the case $X = \mathrm{S}^3$ and $Y = \mathrm{SO}(3)$ by Yamaguchi [7]. Recall that there is a homeomorphism $\mathrm{SO}(4) \approx \mathrm{S}^3 \times \mathrm{SO}(3)$, where $\mathrm{SO}(3) = \mathbf{P}^3$, the real projective space of dimension 3, and that it induces the isomorphisms $\mathscr{E}(\mathrm{SO}(4)) \cong \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$, $\mathscr{E}_\#(\mathrm{SO}(4)) \cong \mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3)$ and $\mathscr{E}(\mathrm{SO}(4))/\mathscr{E}_\#(\mathrm{SO}(4)) \cong \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)/\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3)$. Hence Theorem 1.1 can be stated as follows.

THEOREM 2.1 ([6, 7]). $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)/\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3) \cong \mathrm{Inv}(M_2(\sqrt{2}))$.

We shall prove Theorem 2.1. For convenience we use the same notations for a map and its homotopy class and we do not distinguish them. Given a topological group G and a space X, let $[X, G]$ denote the set of homotopy classes of maps from X into G. It inherits a group structure from G; its multiplication is denoted by $+$. In the special case $X = G$, we denote $[X, G]$ by $\mathscr{H}(G)$, because the notation $[G, G]$ may be confused with the commutator subgroup of G. If $\alpha : X \to Y$ and $\beta : Y \to Z$ are maps (or homotopy classes of them), then their composition is denoted by $\beta \circ \alpha$. The following result is well known.

LEMMA 2.2. *For any maps* $\alpha, \beta : Y \to G$ *and* $\gamma : X \to Y$, *we have* $(\alpha + \beta) \circ \gamma = \alpha \circ \gamma + \beta \circ \gamma$.

We use the following notations as in [3]: $\mathbf{P}^n$ the real projective space of dimension n; $q : S^3 \times \mathbf{P}^3 \to S^3 \wedge \mathbf{P}^3$ and $q_3 : \mathbf{P}^3 \to \mathbf{P}^3/\mathbf{P}^2 = S^3$ the quotient maps; $i : S^3 \vee \mathbf{P}^3 \to S^3 \times \mathbf{P}^3$, $i_1' : S^3 \to S^3 \vee \mathbf{P}^3$ and $i_2' : \mathbf{P}^3 \to S^3 \vee \mathbf{P}^3$ the inclusion maps; $i_k = i \circ i_k'$ $(k = 1, 2)$; $p : S^3 \to \mathbf{P}^3$ the canonical double covering map.

We have the following exact sequence of groups.

$$1 \to [S^3 \wedge \mathbf{P}^3, S^3 \times \mathbf{P}^3] \xrightarrow{q^*} \mathscr{H}(S^3 \times \mathbf{P}^3) \xrightarrow{i^*} [S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3] \to 1$$

We define a binary operation $\bullet$ of $[S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3]$ as follows:

$$\alpha \bullet \beta = i^*(i^{*-1}(\alpha) \circ i^{*-1}(\beta)) \quad (\alpha, \beta \in [S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3]). \tag{2.1}$$

The operation $\bullet$ is well-defined. For, if $\tilde{\alpha}, \tilde{\alpha}' \in i^{*-1}(\alpha)$ and $\tilde{\beta}, \tilde{\beta}' \in i^{*-1}(\beta)$, then $\tilde{\alpha}' = \tilde{\alpha} + q^*(a)$ for some $a \in [S^3 \wedge \mathbf{P}^3, S^3 \times \mathbf{P}^3]$ and

$$i^*(\tilde{\alpha}' \circ \tilde{\beta}') = \tilde{\alpha}' \circ \beta = (\tilde{\alpha} + q^*(a)) \circ \beta = \tilde{\alpha} \circ \beta + a \circ q \circ \beta = \tilde{\alpha} \circ \beta = i^*(\tilde{\alpha} \circ \tilde{\beta}),$$

since $q \circ \beta$ is null-homotopic.

Lemma 2.3. *The triple* $([S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3], +, \bullet)$ *is a unitary ring such that* i *is the unit and* $i^* : \mathscr{H}(S^3 \times \mathbf{P}^3) \to [S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3]$ *is additive and multiplicative, that is,* $i^*(x + y) = i^*(x) + i^*(y)$ *and* $i^*(x \circ y) = i^*(x) \bullet i^*(y)$.

Proof. By definitions, i^* is additive and multiplicative. Thus it suffices to prove the following equalities:

$$i \bullet \alpha = \alpha = \alpha \bullet i, \tag{2.2}$$

$$(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma), \tag{2.3}$$

$$(\alpha + \beta) \bullet \gamma = \alpha \bullet \gamma + \beta \bullet \gamma, \tag{2.4}$$

$$\alpha \bullet (\beta + \gamma) = \alpha \bullet \beta + \alpha \bullet \gamma, \tag{2.5}$$

where $\alpha, \beta, \gamma \in [S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3]$.

Since $i^*(1) = i$, (2.2) is obvious. Hence i is the unit. We have (2.3) and (2.4) from (2.1) and Lemma 2.2. To prove (2.5), consider the homomorphism

$$\begin{aligned} \Theta : [S^3 \vee \mathbf{P}^3, S^3 \times \mathbf{P}^3] &\xrightarrow{(1 \vee p)^*} [S^3 \vee S^3, S^3 \times \mathbf{P}^3] \\ &\xrightarrow{\cong} \pi_3(S^3 \times \mathbf{P}^3) \oplus \pi_3(S^3 \times \mathbf{P}^3) \end{aligned} \tag{2.6}$$

which is defined by $\Theta(\alpha) = i_1'^*(\alpha) \oplus p^* i_2'^*(\alpha)$. Since Θ is injective, it suffices for (2.5) to prove the following two equalities:

$$i_1'^*(\alpha \bullet (\beta + \gamma)) = i_1'^*(\alpha \bullet \beta + \alpha \bullet \gamma),$$
$$p^* i_2'^*(\alpha \bullet (\beta + \gamma)) = p^* i_2'^*(\alpha \bullet \beta + \alpha \bullet \gamma).$$

Let $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ satisfy $i^*(\tilde{\alpha}) = \alpha$, $i^*(\tilde{\beta}) = \beta$, $i^*(\tilde{\gamma}) = \gamma$. Then we have $i^*(\tilde{\beta} + \tilde{\gamma}) = \beta + \gamma$ and

$$\begin{aligned} i_1'^*(\alpha \bullet (\beta + \gamma)) &= \tilde{\alpha} \circ (\tilde{\beta} + \tilde{\gamma}) \circ i_1 = \tilde{\alpha} \circ (\tilde{\beta} \circ i_1 + \tilde{\gamma} \circ i_1) = \tilde{\alpha}_*(\tilde{\beta} \circ i_1 + \tilde{\gamma} \circ i_1) \\ &= \tilde{\alpha}_*(\tilde{\beta} \circ i_1) + \tilde{\alpha}_*(\tilde{\gamma} \circ i_1) \\ &\qquad (\text{since } \tilde{\alpha}_* : \pi_3(\mathrm{S}^3 \times \mathbf{P}^3) \to \pi_3(\mathrm{S}^3 \times \mathbf{P}^3) \text{ is a homomorphism}) \\ &= (\tilde{\alpha} \circ \tilde{\beta} + \tilde{\alpha} \circ \tilde{\gamma}) \circ i_1 = i_1'^*(\alpha \bullet \beta + \alpha \bullet \gamma) \end{aligned}$$

and

$$\begin{aligned} p^* i_2'^*(\alpha \bullet (\beta + \gamma)) &= \tilde{\alpha} \circ (\tilde{\beta} + \tilde{\gamma}) \circ i_2 \circ p = \tilde{\alpha}_*(\tilde{\beta} \circ i_2 \circ p + \tilde{\gamma} \circ i_2 \circ p) \\ &= \tilde{\alpha}_*(\tilde{\beta} \circ i_2 \circ p) + \tilde{\alpha}_*(\tilde{\gamma} \circ i_2 \circ p) = p^* i_2'^*(\alpha \bullet \beta + \alpha \bullet \gamma). \end{aligned}$$

Hence we obtain (2.5). This completes the proof of Lemma 2.3. □

By Lemma 2.3, the set of invertible elements

$$\mathrm{Inv} := \{\alpha \in [\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3] \mid \exists \beta \in [\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3]; \alpha \bullet \beta = i = \beta \bullet \alpha\}$$

becomes a group.

Lemma 2.4. (1) $\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3) = i^{*-1}(i)$ *and* $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3) = i^{*-1}(\mathrm{Inv})$.
(2) $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)/\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3) \cong \mathrm{Inv}$.

Proof. (1). Let Θ be the monomorphism in (2.6). If $f \in \mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3)$, then $\Theta(i^*(f)) = \Theta(i)$ so that $i^*(f) = i$. Hence $\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3) \subset i^{*-1}(i)$. Conversely let $g \in i^{*-1}(i)$. Since $i_* : \pi_*(\mathrm{S}^3 \vee \mathbf{P}^3) \to \pi_*(\mathrm{S}^3 \times \mathbf{P}^3)$ is surjective, the equality $i^*(g) = i$ implies $g \in \mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3)$. Thus $\mathscr{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3) = i^{*-1}(i)$.

Let $f \in \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$. Take $g \in \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$ such that $f \circ g = 1 = g \circ f$. Then $i^*(f) \bullet i^*(g) = i^*(f \circ g) = i = i^*(g \circ f) = i^*(g) \bullet i^*(f)$. Hence $i^*(f) \in \mathrm{Inv}$ and so $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3) \subset i^{*-1}(\mathrm{Inv})$.

Conversely let $f \in i^{*-1}(\mathrm{Inv})$. Then there exists $g \in \mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ such that $i^*(f) \bullet i^*(g) = i = i^*(g) \bullet i^*(f)$. Hence $i^*(f \circ g) = i^*(1) = i^*(g \circ f)$, and so $f \circ g - 1$ and $g \circ f - 1$ belong to the image of q^*. Since any element of the image of q^* induces the trivial homomorphism on homotopy groups, it follows that $f \circ g$ and $g \circ f$ induce the identity homomorphism on homotopy groups so that f is a homotopy equivalence, that is, $f \in \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$, and so $i^{*-1}(\mathrm{Inv}) \subset \mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$. Therefore $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3) = i^{*-1}(\mathrm{Inv})$.

(2). By (1) and Lemma 2.3, the assertion follows. □

We define $f_{kl} \in \mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3)$ and $f'_{kl} \in [\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3]$ by

$$f_{11} = i_1 \circ \mathrm{pr}_1, \quad f_{21} = i_2 \circ p \circ \mathrm{pr}_1, \quad f_{12} = i_1 \circ q_3 \circ \mathrm{pr}_2, \quad f_{22} = i_2 \circ \mathrm{pr}_2, \quad f'_{kl} = f_{kl} \circ i,$$

where $\mathrm{pr}_1 : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathrm{S}^3$ and $\mathrm{pr}_2 : \mathrm{S}^3 \times \mathbf{P}^3 \to \mathbf{P}^3$ are the projections. Then, as is easily shown, we have

$$[\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3] = \bigoplus_{1 \le k, l \le 2} \mathbf{Z}\{f'_{kl}\}.$$

Given a 2×2-matrix (a_{ij}) with $a_{ij} \in \mathbf{Z}$, let

$$(a_{ij})' = \begin{bmatrix} a_{11} & \sqrt{2}a_{12} \\ \sqrt{2}a_{21} & a_{22} \end{bmatrix} \in M_2(\sqrt{2}). \tag{2.7}$$

LEMMA 2.5. *The function* $\varphi : [\mathrm{S}^3 \vee \mathbf{P}^3, \mathrm{S}^3 \times \mathbf{P}^3] \to M_2(\sqrt{2})$ *defined by* $\varphi(\sum a_{kl} f'_{kl}) = (a_{kl})'$ *is an isomorphism of rings.*

Proof. Obviously φ is an additive isomorphism. By direct calculation, we have

$$f_{kl} \circ f_{mn} = \begin{cases} \varepsilon(k,l,n) f_{kn} & l = m \\ 0 & l \neq m \end{cases} \quad \text{and so} \quad f'_{kl} \bullet f'_{mn} = \begin{cases} \varepsilon(k,l,n) f'_{kn} & l = m \\ 0 & l \neq m \end{cases},$$

where

$$\varepsilon(k,l,n) = \begin{cases} 2 & (k,l,n) = (1,2,1),(2,1,2) \\ 1 & \text{otherwise.} \end{cases}$$

Hence

$$\left(\sum_{k,l} a_{kl} f'_{kl}\right) \bullet \left(\sum_{m,n} b_{mn} f'_{mn}\right) = \sum_{k,n} c_{kn} f'_{kn},$$

where $c_{kn} = \sum_l a_{kl} b_{ln} \varepsilon(k,l,n)$. The last equality implies $(c_{kn})' = (a_{kn})'(b_{kn})'$, that is, $\varphi((\sum a_{kl} f'_{kl}) \bullet (\sum b_{kl} f'_{kl})) = \varphi(\sum a_{kl} f'_{kl}) \varphi(\sum b_{kl} f'_{kl})$. Therefore φ is multiplicative. This completes the proof. $\square$

Proof of Theorem 2.1. It follows from Lemma 2.4 and Lemma 2.5 that the surjection $\varphi \circ i^* : \mathscr{H}(\mathrm{S}^3 \times \mathbf{P}^3) \to M_2(\sqrt{2})$ induces a multiplicative surjection $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3) \to \mathrm{Inv}(M_2(\sqrt{2}))$ with $\mathscr{E}_{\#}(\mathrm{S}^3 \times \mathbf{P}^3)$ the kernel. Hence we obtain Theorem 2.1. $\square$

3. Proof of Theorem 1.2

We have $\mathrm{Inv}(M_2(\sqrt{2})) = \det^{-1}\{1,-1\}$ and we write

$$\mathrm{Inv}_+(M_2(\sqrt{2})) = \det^{-1}(1), \quad \mathrm{Inv}_-(M_2(\sqrt{2})) = \det^{-1}(-1).$$

Then $\mathrm{Inv}_+(M_2(\sqrt{2}))$ is a subgroup of $\mathrm{Inv}(M_2(\sqrt{2}))$ of index 2.

To prove Theorem 1.2 we need three lemmas. Given an integer δ, we define a sequence of integers $\beta_n = \beta_n(\delta)$ $(n \geq 1)$ by

$$\beta_1 = 1, \quad \beta_2 = \delta, \quad \beta_{n+1} = \delta\beta_n - \beta_{n-1} \quad (n \geq 2).$$

The following two lemmas are easily proved by the induction.

LEMMA 3.1. *If* $A \in \mathrm{Inv}_+(M_2(\sqrt{2}))$ *and* $\delta = \mathrm{tr}(A)$, *then*

$$A^n = -\beta_{n-1} E + \beta_n A \quad (n \geq 2).$$

LEMMA 3.2. *We have*

$$\beta_{2n-1} = \sum_{i=0}^{n-1}(-1)^{n+i-1}\binom{n+i-1}{2i}\delta^{2i}, \quad \beta_{2n} = \sum_{i=0}^{n-1}(-1)^{n+i-1}\binom{n+i}{2i+1}\delta^{2i+1}.$$

The third lemma we need is

LEMMA 3.3. *If δ is even and $\beta_{2n-1} = -1$, then n is even and $\delta = 0$.*

Proof. Suppose that δ is even and $\beta_{2n-1} = -1$. Since $\beta_{2n-1} \equiv (-1)^{n-1} \pmod{\delta^2}$ by Lemma 3.2, it follows that n is even. Set $n = 2m$ and define

$$g_{2m} := \sum_{i=1}^{2m-1}(-1)^{i-1}\binom{2m+i-1}{2i}\delta^{2i-2} = \sum_{j=0}^{2m-2}(-1)^{j}\binom{2m+j}{2j+2}\delta^{2j}.$$

Then $\delta^2 g_{2m} = \beta_{2n-1} + 1 = 0$. Thus it suffices to prove that $g_{2m} \neq 0$. Since $g_2 = 1$, we assume $m \geq 2$. Note that

$$g_{2m} = (2m-1)m + \sum_{j=1}^{2m-2}(-1)^{j}\frac{(2m-j-1)(2m-j)\cdots(2m+j)}{(2j+2)!}\delta^{2j}.$$

We prove that if $1 \leq j \leq 2m-2$ and δ is a non-zero even integer, then

$$\Phi(j) := v_2\left(\frac{(2m-j-1)(2m-j)\cdots(2m+j)}{(2j+2)!}\delta^{2j}\right) \geq v_2(m) + 1. \tag{3.1}$$

Here $v_2(k)$ is the exponent of 2 in the integer k, that is, $k = 2^{v_2(k)}l$ such that $v_2(k)$ is a non-negative integer and l is an odd integer. If (3.1) holds, then $g_{2m} \equiv 2^{v_2(m)} \pmod{2^{v_2(m)+1}}$ and so $g_{2m} \neq 0$. Now we prove (3.1). Let $\varepsilon(k)$ denote the sum of all coefficients in the 2-adic expansion of the positive integer k. As is well known, $v_2(k!) = k - \varepsilon(k)$. We have

$$\begin{aligned}\Phi(j) &= \varepsilon(2j+2) - (2j+2) + 2jv_2(\delta) + \sum_{i=0}^{2j+1} v_2(2m-j-1+i) \\ &\geq \varepsilon(2j+2) - 2 + \sum_{i=0}^{2j+1} v_2(2m-j-1+i) =: \Psi(j).\end{aligned}$$

It suffices for (3.1) to prove

$$\Psi(j) \geq v_2(m) + 1 \quad \text{if } 1 \leq j \leq 2m-2. \tag{3.2}$$

If $l \geq 0$ and $2l+2 \leq 2m-2$, then

$$\begin{aligned}\Psi(2l+1) &= \varepsilon(4l+4) - 2 + v_2(2m-2l-2) + \cdots + v_2(2m) + \cdots + v_2(2m+2l) \\ &\geq 1 - 2 + 1 + v_2(2m) = v_2(m) + 1\end{aligned}$$

and

$$\begin{aligned}\Psi(2l+2) &= \varepsilon(4l+6) - 2 + \nu_2(2m-2l-2) + \cdots + \nu_2(2m) + \cdots + \nu_2(2m+2l+2)\\ &\geq 2 - 2 + \nu_2(2m-2l-2) + \nu_2(2m) + \nu_2(2m+2l+2)\\ &\geq \nu_2(m) + 3 > \nu_2(m) + 1.\end{aligned}$$

This proves (3.2) and completes the proof of Lemma 3.3. □

Proof of Theorem 1.2. Let $A = (a_{ij})' \in \mathrm{Inv}(M_2(\sqrt{2}))$ and write $\delta = \mathrm{tr}(A)$, where $(a_{ij})'$ is the matrix defined in (2.7). Since $\det(A) = a_{11}a_{22} - 2a_{12}a_{21} = \pm 1$, it follows that a_{11} and a_{22} are odd so that δ is even.

(i) Suppose $\det(A) = 1$ and $A \neq \pm E$. We prove that the following conditions are equivalent: (1) the order of A is 4; (2) the order of A is finite; (3) $\delta = 0$; (4) $A^2 = -E$. Note that the order of A is $3 + \det(A)$ if (1) holds. It is obvious that (1) and (4) imply (2) and (1), respectively. It follows from Lemma 3.1 that (3) implies (4). By Lemma 3.1, for $n \geq 2$, the equality $A^n = E$ holds if and only if

$$-\beta_{n-1} + \beta_n a_{11} = -\beta_{n-1} + \beta_n a_{22} = 1, \quad \beta_n a_{12} = \beta_n a_{21} = 0.$$

Assume (2), that is, $A^n = E$ for $n \geq 2$. Then $\beta_n = 0$ and $\beta_{n-1} = -1$ by the assumption $A \neq \pm E$. Hence n is even by Lemma 3.2 so that $n \equiv 0 \pmod 4$ and $\delta = 0$ by Lemma 3.3. Thus (3) holds.

(ii) Suppose $\det(A) = -1$. We prove that the following conditions are equivalent: (1) the order of A is 2; (2) the order of A is finite; (3) $\delta = 0$. Note that the order of A is $3 + \det(A)$ if (1) holds. It is obvious that (1) implies (2). Assume (2), that is, $A^n = E$ for $n \geq 2$. Then $(A^2)^n = E$ with $A^2 \in \mathrm{Inv}_+(M_2(\sqrt{2}))$. Hence $A^2 = \pm E$ or $\mathrm{tr}(A^2) = 0$ by (i). Since

$$A^2 = \begin{bmatrix} 1 + a_{11}\delta & \sqrt{2}a_{12}\delta \\ \sqrt{2}a_{21}\delta & 1 + a_{22}\delta \end{bmatrix}, \tag{3.3}$$

it follows that $\mathrm{tr}(A^2) = 2 + \delta^2 \geq 2$ so that $A^2 = \pm E$. Then the assumption $\det(A) = -1$ and (3.3) imply that $\delta = 0$ and $A^2 = E$, that is, (1) and (3) follows. This completes the proof of Theorem 1.2. □

For a group G, let $\mathrm{Tor}\, G$ denote the subset of G consisting of elements with finite order.

Corollary 3.4. (1) $\mathrm{Tor}\,\mathrm{Inv}(M_2(\sqrt{2}))$ *is not a subgroup of* $\mathrm{Inv}(M_2(\sqrt{2}))$.
(2) $\mathrm{Tor}\,\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$ *is not a subgroup of* $\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$.

Proof. Let

$$P = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & \sqrt{2} \\ 0 & -1 \end{bmatrix}.$$

Then $P^2 = Q^2 = E$ and so $P, Q \in \mathrm{Tor}\,\mathrm{Inv}(M_2(\sqrt{2}))$, while their product

$$PQ = \begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}$$

has an infinite order by Theorem 1.2. This implies (1). Since $\mathscr{E}_{\#}(\mathrm{S}^3 \times \mathbf{P}^3) \subset \mathrm{Tor}\,\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$ by [4], $\mathrm{Tor}\,\mathscr{E}(\mathrm{S}^3 \times \mathbf{P}^3)$ is mapped onto $\mathrm{Tor}\,\mathrm{Inv}(M_2(\sqrt{2}))$ under the epimorphism that induces the isomorphism of Theorem 2.1. This implies (2). $\square$

4. Proof of Theorem 1.3

Let A, B, C be the matrices defined in Theorem 1.3. For each $n \in \mathbf{Z}$, we have

$$A^n = \begin{bmatrix} 1 & 0 \\ \sqrt{2}n & 1 \end{bmatrix}, \quad B^n = \begin{bmatrix} 1 & \sqrt{2}n \\ 0 & 1 \end{bmatrix}. \tag{4.1}$$

The following lemma is a part of Theorem 1.3.

Lemma 4.1. *The group* $\mathrm{Inv}_+(M_2(\sqrt{2}))$ *is not nilpotent and generated by* A *and* B *with relations*:

$$(AB^{-1})^4 = E, \quad (AB^{-1})^2 = (B^{-1}A)^2.$$

Proof. Direct calculation implies that $\mathrm{Inv}(M_2(\sqrt{2}))$ and $\mathrm{Inv}_+(M_2(\sqrt{2}))$ have the same center $\mathbf{Z}_2\{-E\}$ and that the following equalities hold

$$(AB^{-1})^2 = (A^{-1}B)^2 = (B^{-1}A)^2 = (BA^{-1})^2 = -E. \tag{4.2}$$

Hence $(AB^{-1})^4 = E$. Define $C_n \in \mathrm{Inv}_+(M_2(\sqrt{2}))$ inductively by

$$C_1 = ABA^{-1}B^{-1}, \quad C_n = C_{n-1}BC_{n-1}^{-1}B^{-1} \quad (n \geq 2).$$

By the induction on n, we can easily prove that the $(2,1)$-component of C_n is $-\sqrt{2}2^{2^n-1}$ and so $C_n \neq E$ for all $n \geq 1$. Hence $\mathrm{Inv}_+(M_2(\sqrt{2}))$ is not nilpotent.

In the rest of the proof we prove that $\mathrm{Inv}_+(M_2(\sqrt{2}))$ is generated by A and B. That is, we will show that if $X = (x_{ij})' \in \mathrm{Inv}_+(M_2(\sqrt{2}))$, then $X \in \langle A, B \rangle$, where $(x_{ij})'$ is the notation of (2.7), and $\langle A, B \rangle$ is the subgroup generated by A and B. By the definition, we have

$$x_{11}x_{22} - 2x_{12}x_{21} = 1. \tag{4.3}$$

Hence x_{11} is odd and so $x_{11} \neq 0$. By (4.2), $X \in \langle A, B \rangle$ if and only if $-X = X(-E) \in \langle A, B \rangle$. So we can assume $x_{11} > 0$ without loss of generality. By the induction on $l \geq 1$, we prove that if $X = (x_{ij})' \in \mathrm{Inv}_+(M_2(\sqrt{2}))$ with $x_{11} = 2l - 1$, then

$$X \in \langle A, B \rangle. \tag{4.4}$$

If $l = 1$, then $x_{11} = 1$ and, by (4.1) and (4.3), we have $X = A^{x_{21}}B^{x_{12}}$ and so (4.4) holds in this case. Assume that (4.4) is true if $1 \le x_{11} \le 2l - 1$ with $l \ge 1$. Suppose $x_{11} = 2l + 1$. By (4.3), x_{11} and x_{21} are prime each other and so we can write $x_{21} = kx_{11} + i$ with $1 \le i < x_{11}$. We have $X = A^k BD$, where

$$D = \begin{bmatrix} x_{11} - 2i & \sqrt{2}\{(2k+1)x_{12} - x_{22}\} \\ \sqrt{2}i & -2kx_{12} + x_{22} \end{bmatrix}.$$

Note that $x_{11} - 2i$ is odd and $|x_{11} - 2i| \le x_{11} - 2$. By the inductive hypothesis, D or $D(-E)$ is an element of $\langle A, B\rangle$ according to whether $x_{11} - 2i$ is positive or negative. Hence, anyway, $D \in \langle A, B\rangle$ and so $X \in \langle A, B\rangle$. This completes the induction. $\square$

By the algorithm in the above proof, we have

Proposition 4.2. *Let $X = (x_{ij})' \in \mathrm{Inv}_+(M_2(\sqrt{2}))$. Then $|x_{11}| = 2n - 1$ with $n \ge 1$ and*

$$X = (A^{k_1}B)(A^{k_2}B)\cdots(A^{k_{m-1}}B)(A^{k_m}B^k)(-E)^{\varepsilon}. \tag{4.5}$$

for some integers $k_1, \dots, k_m$, k, ε such that $1 \le m \le n$, $\varepsilon = 0, 1$, and that if $m \ge 2$, then $k_m \ne 0$.

The decomposition (4.5) is unique for $n \le 2$, while it is not unique for $n \ge 3$ because (4.2) implies

$$\begin{bmatrix} 2n-1 & \sqrt{2} \\ (n-1)\sqrt{2} & 1 \end{bmatrix} = BA^{n-1} = BA^{n-2}BA^{-1}B(-E).$$

Proof of Theorem 1.3. By Lemma 4.1, $\mathrm{Inv}(M_2(\sqrt{2}))$ is not nilpotent. Since the map

$$\mathrm{Inv}_+(M_2(\sqrt{2})) \to \mathrm{Inv}_-(M_2(\sqrt{2})), \quad X \mapsto CX$$

is a bijection, it follows from Lemma 4.1 that $\mathrm{Inv}(M_2(\sqrt{2}))$ is generated by A, B and C. Direct calculation implies the following equalities:

$$C^2 = E, \quad CA = A^{-1}C, \quad CB = B^{-1}C.$$

This and Lemma 4.1 complete the proof of Theorem 1.3. $\square$

Problem 4.3. *Are* (1.1) *the defining relations of* $\mathrm{Inv}(M_2(\sqrt{2}))$?

While $\mathcal{E}_\#(\mathrm{S}^3 \times \mathbf{P}^3)$ is nilpotent by [1] (or [4]), Theorem 2.1 and Theorem 1.3 imply

Corollary 4.4. *The group $\mathcal{E}(\mathrm{S}^3 \times \mathbf{P}^3)$ is not nilpotent.*

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