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NON-HYPERELLIPTIC RIEMANN SURFACES OF GENUS FIVE ALL OF WHOSE WEIERSTRASS POINTS HAVE MAXIMAL WEIGHT[1]

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Abstract

It is proved that any non-hyperelliptic Riemann surface of genus five all of whose Weierstrass points have maximal weight has only Weierstrass points with gap sequence $(1, 2, 3, 5, 9)$, and a defining equation of such a surface is given.

§1. Introduction

In their papers ([6], [7]), Kuribayashi and al. proved that if a non-hyperelliptic compact Riemann surface of genus three has exactly twelve Weierstrass points, then its defining equation is given by one of the followings:

$$\begin{gathered} x^4 + y^4 + 1 = 0 \\ x^4 + y^4 + 3(x^2y^2 + x^2 + y^2) + 1 = 0 \end{gathered} \tag{1}$$

As to the case of genus five, in his paper ([2]) Centina showed the existence of elliptic-hyperellitic curves of genus five having exactly twenty four Weierstrass points which constitute the set of fixed points of three distinct elliptic-hyperellitic involutions on them, and proved that all such curves are double covers of Fermat's quartic.

The purpose of this paper is to prove that any non-hyperellitic Riemann surfaces of genus five all whose Weierstrass points have maximal weight has only Weierstrass points with gap sequence $(1, 2, 3, 5, 9)$ (Proposition 2), and to give an explicit defining equation of such a surface.

The referee showed us that the example given by Centina and ours are equivalent. The author thanks the referee for granting generous permission to write it in Theorem 4, which makes our small contribution fruitful.

In this paper the term *linear series* means a linear series of positive dimension.

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§2. Gap sequences

It is known ([3]) that the maximal weight of Weierstrass points on compact non-hyperelliptic Riemann surfaces of genus five is five, and the total Weierstrass weight is $120 = 4 \cdot 5 \cdot 6$. We shall be concerned with the existence of non-hyperelliptic surfaces all whose Weierstrass points have maximal weight, in other words, having $24 = 120/5$ Weierstrass points, and determining defining equations of such surfaces if there exist.

There are two types of gap sequence with maximal weight five, i.e.,

$$(1,2,4,5,8), \quad (1,2,3,5,9). \tag{2}$$

Let $W_3^{\max}$ be the set of Weierstrass points with the gap sequence $(1,2,4,5,8)$, and $W_4^{\max}$ the set of Weierstrass points with the gap sequence $(1,2,3,5,9)$ of a Riemann surface.

We shall prove that non-hyperelliptic Riemann surfaces of genus five all whose Weierstrass points have maximal weight have only Weierstrass points in $W_4^{\max}$.

Lemma 1. *There is at most one linear series of degree three on surfaces of genus $g \geq 5$ ([5], p. 553).*

Proposition 1. *There is no surface of genus five with all Weierstrass points in $W_3^{\max}$.*

Proof. If there were a surface of genus five with all Weierstrass points in $W_3^{\max}$, then by Lemma 1 and the Riemann-Hurwitz relation for the three-sheeted covering of $\mathbf{P}^1$ assured by the non-gap 3 of a Weierstrass point in $W_3^{\max}$, we would see that the surface has at most seven points in $W_3^{\max}$.

Lemma 2. *For each point $P \in W_4^{\max}$, we consider the linear series $|4P|$ of degree four without fixed points. Then the number of points of $W_4^{\max}$ that determine the same linear series is at most five.*

Proof. The Riemann-Hurwitz relation for the four-sheeted covering of $\mathbf{P}^1$ assured by the non-gap 4 of a point in $W_4^{\max}$ is

$$2 \cdot 5 - 2 = 4(2 \cdot 0 - 2) + V, \quad V = 16, \tag{3}$$

where V is the total ramification number. We see from $[16/3] = 5$ that at most five points among $W_4^{\max}$ determine the same linear series of degree four without fixed points.

Proposition 2. *Any non-hyperellitic Riemann surfaces of genus five all whose Weierstrass points have maximal weight has no Weierstrass point in $W_3^{\max}$.*

Proof. In the proof of Corollary 5, Section 4, Part III of his lecture notes ([1]), Accola showed that a Riemann surface of genus five admitting a linear series of degree three without fixed points can admit at most one half-canonical linear series of degree four.

If the surface has a Weierstrass points in $W_3^{\max}$, then by the proof of Proposition 1 it must have at least 17 Weierstrass points in $W_4^{\max}$ so that by Lemma 2 it must have at least four distinct linear series. This contradicts the fact shown by Accola.

Thus we only have to consider the Riemann surfaces with all Weierstrass points in $W_4^{\max}$.

§3. A definig equation

LEMMA 3. *Any Riemann surface of genus five all whose Weierstrass points have maximal weight has at least five distinct half-canonical linear series of degree four.*

Proof. We note that n is a gap at a point P if and only if there is a holomorphic differential with zero of order $n-1$ at P. Thus for a point $P \in W_4^{\max}$, $|8P|$ is canonical, and then by Lemma 2 we have the result.

The next theorem seems to suggest that a Riemann surface with all Weierstrass points in $W_4^{\max}$ would be elliptic-hyperelliptic.

THEOREM 1 (Accola [2]). *If a non-hyperelliptic Riemann surface of genus five has four even half-integer theta-characteristics $[\eta_i]$, $i = 1, 2, 3, 4$ such that $\eta_1 + \eta_2 + \eta_3 = \eta_4$, and $\theta[\eta_i](u)$, $i = 1, 2, 3, 4$ vanishes to order two at $u = 0$, then the Riemann surface has an elliptic-hyperelliptic involution.*

So we have looked for Riemann surfaces with all Weierstrass points in $W_4^{\max}$ among the elliptic-hyperelliptic Riemann surfaces, and found the following.

THEOREM 2. *All Weierstrass points of the Riemann surface defined by the equation*

$$y^4 = x(x^2 - 1)(x^2 + 1)^2 \tag{4}$$

are in $W_4^{\max}$.

Before giving the proof of this Theorem, we need the next proposition.

PROPOSITION 3 ([3], p. 84–86). *Let $\{\phi_1, \dots, \phi_n\}$ be a basis for a finite-dimensional space A of holomorphic functions on a complex domain D. For $z \in D$ one can find a basis $\{\psi_1, \dots, \psi_n\}$ with*

$$\mathrm{ord}_z\, \psi_1 < \mathrm{ord}_z\, \psi_2 < \cdots < \mathrm{ord}_z\, \psi_n \tag{5}$$

by linearly transforming $\{\phi_1, \dots, \phi_n\}$. Let $\mu_j = \mathrm{ord}_z\, \psi(z)$, and

$$\tau(z) = \sum_{j=1}^{n} (\mu_j - j + 1) \tag{6}$$

be the weight of z with respect to A. Consider the Wronskian

$$\Phi(z) = \det \begin{bmatrix} \psi_1(z) & \cdots & \psi_n(z) \\ \psi_1'(z) & \cdots & \psi_n'(z) \\ \vdots & & \vdots \\ \psi_1^{(n-1)}(z) & \cdots & \psi_n^{(n-1)}(z) \end{bmatrix}. \tag{7}$$

Then

$$\mathrm{ord}_z\, \Phi = \tau(z). \tag{8}$$

Proof of Theorem 2. The surface defined by (4) is obviously elliptic-hyperelliptic, that is, a two-sheeted covering of the elliptic curve defined by

$$Y^2 = x(x^2 - 1) \quad (y^2 = Y(x^2 + 1)). \tag{9}$$

The projection $(x, y) \to x$ defines a four-sheeted covering of the Riemann sphere P_x with four ramification points $(x, y) = (0, 0), (\pm 1, 0), (\infty, \infty)$ of multiplicity four and four ramification points $(x, y) = (\pm i, 0)$ of multiplicity two. A basis of the holomorphic differentials on the surface is given by

$$\frac{1}{y}\,dx, \quad \frac{(x^2+1)}{y^2}\,dx, \quad \frac{(x^2+1)}{y^3}\,dx, \quad \frac{x(x^2+1)}{y^3}\,dx, \quad \frac{x^2(x^2+1)}{y^3}\,dx. \tag{10}$$

It is obvious that four ramification points of multiplicity four including the infinite point $(x, y) = (\infty, \infty)$ belong to $W_4^{\max}$. Except the point $(x, y) = (\infty, \infty)$ we use x as a local parameter, since the orders of zeros of the Wronskian do not depend on local parameters ([4], p. 148–149). Direct calculation of its Wronskian and Proposition 3 proves that the 24 points on it : the four point with $x = 0, \pm 1, \infty$, the four points with $x = \pm i$ and the sixteen points with $x = -1 \pm \sqrt{2}$ and $x = 1 \pm \sqrt{2}$ are all the Weierstrass points in $W_4^{\max}$. But it is cumbersome to calculate by hand, and one can use Maple as follows:

```
> restart :
> D(x) := 1 :
> f[0] := y∧4 − x∗(x∧2 − 1)∗(x∧2 + 1)∧2 :
> D(f[0]) :
> f[1] := D(y) = factor(solve(%, D(y))) :
> for i from 2 to 4 do   g[i] := D(f[i − 1]) : f[i] := factor(subs(f[1], g[i])) : od :
> with(linalg) :
> A1 := matrix([[1/y, (x∧2 + 1)/y∧2, (x∧2 + 1)/y∧3, x∗(x∧2 + 1)/y∧3,
                  x∧2∗(x^2 + 1)/y∧3]]) :
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> for i from 2 to 5 do  A||i := map(D, A||(i − 1)) : od :
> B := stackmatrix(A1, A2, A3, A4, A5) :
> det(B) :
> factor(subs(f[1], f[2], f[3], f[4], %)) :
> factor(subs(y = (x * (x ∧ 2 − 1) * (x ∧ 2 + 1) ∧ 2) ∧ (1/4), %));
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$$\frac{63(x^2-2x-1)^5(x^2+2x-1)^5}{1024(x^2+1)^5(x-1)^{10}(x+1)^{10}x^{10}} \tag{11}$$

Remark. One might think that the elliptic-hyperelliptic curves would have all its Weierstrass points in $W_4^{\max}$, but this is not the case. Some Weierstrass points of the curves

$$y^4 = x(x^2-a)(x^2+1)^2 \quad \text{or} \quad y^4 = x(x^2-1)(x^2+a)^2 \quad (a \neq 1) \tag{12}$$

have not maximal weight 5. This fact suggests that the Riemann surfaces having only Weierstrass points in $W_4^{\max}$ need to have some automorphisms besides the elliptic-hyperelliptic involution.

Here we give a proof of the equivalence of an example by Centina and the one in Theorem 2, which is announced in Introduction.

Theorem 4. *Centina's example* ([2]) *defined by*

$$\begin{cases} X_1^2 + X_4^2 + X_5^2 = 0 \\ X_2^2 + X_4^2 - X_5^2 = 0 \\ X_3^2 + X_4X_5 = 0 \end{cases} \tag{13}$$

is birationally equivalent to

$$y^4 = x(x^2-1)(x^2+1)^2. \tag{14}$$

Proof. Set $x_i = \dfrac{X_i}{X_5}$ $(i = 1, \dots, 4)$, and we have

$$\begin{cases} x_1^2 + x_4^2 + 1 = 0 \\ x_2^2 + x_4^2 - 1 = 0 \\ x_3^2 + x_4 = 0 \end{cases}. \tag{15}$$

Next we set $u = x_1 - x_2$, $v = x_3$, then obviously we have $C(x_1, x_2, x_3, x_4) \supset C(u, v)$. Subtracting the second equation from the first of (15), we have $x_1^2 - x_2^2 = -2$ and then $x_1 + x_2 = -\dfrac{2}{u}$. Thus we have $x_1 = \dfrac{u}{2} - \dfrac{1}{u}$, $x_2 = -\left(\dfrac{u}{2} + \dfrac{1}{u}\right)$. In addition, we have $x_4 = -v^2$ so that $C(x_1, x_2, x_3, x_4) \subset C(u, v)$. Consequently we have

$$C(x_1, x_2, x_3, x_4) = C(u, v) \tag{16}$$

with the relation:

$$\frac{u^6}{4} + u^2 + (uv)^4 = 0. \tag{17}$$

Set $s = cu$, $t = duv$ and choose the constants c, d properly, we have $C(u, v) = C(s, t)$ with $t^4 = s^6 + s^2$. Furthermore, if we set

$$\begin{cases} x = \dfrac{i(-is + \zeta)}{(is + \zeta)} \\ y = \dfrac{(x + i)^2 t}{\sqrt[4]{8}} \end{cases}, \quad \text{where } \zeta = (-1)^{1/4} \tag{18}$$

then we ultimately have $y^4 = x(x^2 - 1)(x^2 + 1)^2$.

The problem to find all defining equations of non-hyperelliptic Riemann surfaces all whose Weierstrass points in $W_4^{\max}$ still remains.

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