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A COEFFICIENT INEQUALITY FOR CERTAIN MEROMORPHIC UNIVALENT FUNCTIONS

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1. Let Σ_0 denote the class of functions $g(z)$ univalent in $|z|>1$, regular apart from a simple pole at the point at infinity and having the expansion at that point

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}. \tag{1}$$

Garabedian and Schiffer [1] obtained the sharp estimate $|b_3|\leqq(1+2e^{-6})/2$ and at the same time they remarked that if all the coefficients b_n of $g(z)$ are real then $b_3\leqq 1/2$. Further Jenkins [5] proved that if $b_j=0$ for $j<n$ then $|b_{2n+1}|\leqq (n+1)^{-1}[1+2\exp\{-(2n+4)/n\}]$ and that if $b_j=0$ for $j\leqq(n-1)/2$ then $|b_n|\leqq 2/(n+1)$.

In this paper we shall be concerned with the coefficient b_5 and we shall prove

THEOREM. *If all the coefficients b_n of $g(z)$ are real, then*

$$b_5\leqq\frac{1}{3}+\frac{4}{507}$$

with equality holding only for the function $\tilde{g}(z)$ which satisfies the algebraic equation

$$\left(w^2+\frac{12}{13}\right)^3=\left(z^3+\frac{6}{13}z+\frac{6}{13}z^{-1}+z^{-3}\right)^2, \qquad w=\tilde{g}(z).$$

The expansion of $\tilde{g}(z)$ at the point at infinity begins

$$z-\frac{4}{13}z^{-1}+\frac{16}{169}z^{-3}+\left(\frac{1}{3}+\frac{4}{507}\right)z^{-5}+\cdots.$$

Our proof is due to Jenkins' General Coefficient Theorem.

2. Firstly we give several lemmas which will be used later on.

LEMMA A. *Let $Q(w)dw^2=\alpha(w^4+\beta_1 w^3+\beta_2 w^2+\beta_3 w+\beta_4)dw^2$ be a quadratic differential on the w-sphere and let*

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$$g^*(z)=z+\sum_{n=1}^{\infty} b_n^* z^{-n}$$

be a function of class Σ_0 *which maps* $|z|>1$ *onto a domain* D *admissible with respect to* $Q(w)dw^2$. *Let* $g(z)$ *be a function of class* Σ_0 *having the expansion at the point at infinity*

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

where $b_1=b_1^*$. *Then*

$$\Re\alpha\{b_5-b_5^*+\beta_1(b_4-b_4^*)+(\beta_2+3b_1^*)(b_3-b_3^*)$$

$$+(\beta_3+2\beta_1 b_1^*+2b_2^*)(b_2-b_2^*)+(b_2-b_2^*)^2\}\leqq 0\,.$$

In the case $b_j=b_j^*$, $j=1, 2$ *equality occurs only for* $g(z)\equiv g^*(z)$.

Proof. Let $\phi(w)$ be the inverse of $g^*(z)$ defined in D. Then we apply the General Coefficient Theorem in its extended form [6] with $\Re$ the w-sphere, $Q(z)dz^2$ being $\alpha(w^4+\beta_1 w^3+\beta_2 w^2+\beta_3 w+\beta_4)dw^2$, the admissible domain D and the admissible function $g(\phi(w))$. The function $g(\phi(w))$ has the expansion at the point at infinity

$$w+\sum_{n=2}^{\infty} a_n w^{-n}$$

where

$$a_2=b_2-b_2^*\,,$$

$$a_3=b_3-b_3^*\,,$$

$$a_4=b_4-b_4^*+2b_1^*(b_2-b_2^*)\,,$$

$$a_5=b_5-b_5^*+3b_1^*(b_3-b_3^*)+2b_2^*(b_2-b_2^*)\,.$$

Hence we have the desired inequality. The equality statement follows at once from the general equality conditions in the General Coefficient Theorem.

LEMMA B. *Let* $g(z)$ *be a function of class* Σ_0 *having the expansion* (1) *at the point at infinity. Then*

$$\left|b_5+b_1b_3+b_2^2+\frac{1}{3}b_1^3\right|\leqq\frac{1}{3}\,.$$

Proof. Let $G_\mu(w)$ be the μ-th Faber polynomial which is defined by

$$G_\mu(g(z))=z^\mu+\sum_{\nu=1}^{\infty} b_{\mu\nu}z^{-\nu}\,.$$

Then Grunsky's inequality [2] has the form

$$\left|\sum_{\mu,\nu=1}^{m} \nu b_{\mu\nu}x_\mu x_\nu\right|\leqq\sum_{\nu=1}^{m}\nu|x_\nu|^2\,.$$

Putting $m=3$, $x_1=x_2=0$ and $x_3=1$ we have the desired inequality.

The following lemma is a simple consequence of the area theorem.

LEMMA C. *Let $g(z)$ be a function of class Σ_0 having the expansion* (1) *at the point at infinity. Then*

$$|b_1|^2+3|b_3|^2+5|b_5|^2\leqq 1.$$

3. Next we give certain functions which play the role of extremal functions.

LEMMA 1. *Let $Q^*(w;X)dw^2$ be the quadratic differential $(w^4-2Xw^2+X^2)dw^2$, $(0\leqq X\leqq 4)$. Let Y be a real number satisfying the condition*

$$2Y-X+2\geqq 0, \qquad 6Y-X^{3/2}+2\leqq 0. \tag{2}$$

Then there is a function $g^*(z;X,Y)\in\Sigma_0$ which satisfies the algebraic equation

$$w^3-3Xw=z^3-(6Y+3)z-(6Y+3)z^{-1}+z^{-3} \tag{3}$$

and which maps $|z|>1$ onto a domain admissible with respect to $Q^*(w;X)dw^2$. The expansion of $g^*(z;X,Y)$ at the point at infinity begins

$$z+(X-2Y-1)z^{-1}+(2XY-4Y^2+X-6Y-2)z^{-3}+\Big(\frac{1}{3}+\Phi(X,Y)\Big)z^{-5}+\cdots$$

where

$$\Phi(X,Y)=-\frac{1}{3}X^3+8XY^2-\frac{40}{3}Y^3+10XY-28Y^2+3X-18Y-\frac{11}{3}.$$

Proof. There are three end domains $\mathcal{E}_1^*, \mathcal{E}_2^*, \mathcal{E}_3^*$ in the trajectory structure of $Q^*(w;X)dw^2$ on the upper half w-plane. For a suitable choice of determination the function

$$\zeta=\int(w^2-X)dw$$

maps $\mathcal{E}_1^*$, $\mathcal{E}_2^*$, $\mathcal{E}_3^*$ respectively onto an upper half-plane, a lower half-plane and an upper half-plane, the points $X^{1/2}$, $-X^{1/2}$ corresponding to the points $-2X^{3/2}/3$, $2X^{3/2}/3$ respectively.

On the other hand there are three end domains $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ in the trajectory structure of the quadratic differential

$$z^{-8}(z-1)^2(z+1)^2(z-e^{i\theta})^2(z+e^{i\theta})^2(z-e^{-i\theta})^2(z+e^{-i\theta})^2dz^2, \quad 0\leqq\theta\leqq\frac{\pi}{2}$$

on the domain $|z|>1$, $\Im z>0$. For a suitable choice of determination the function

$$\zeta=\int z^{-4}(z-1)(z+1)(z-e^{i\theta})(z+e^{i\theta})(z-e^{-i\theta})(z+e^{-i\theta})dz \tag{4}$$

maps $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ respectively onto an upper half-plane, a lower half-plane and an upper half-plane, the points 1, $e^{i\theta}$, $-e^{-i\theta}$, -1 corresponding to the points $-(12\cos 2\theta+4)/3$, $-16\cos^3\theta/3$, $16\cos^3\theta/3$, $(12\cos 2\theta+4)/3$ respectively.

If X and θ satisfy the condition

$$4\cos 2\theta+\frac{4}{3}\leqq\frac{2}{3}X^{3/2}\leqq\frac{16}{3}\cos^3\theta, \tag{5}$$

then we can combine the above two functions to obtain a function which maps the domain $|z|>1$, $\Im z>0$ into the upper half w-plane. We put $Y=\cos 2\theta$. Then the condition (5) is equivalent to the condition (2). By reflection this function extends to a function $g^*(z; X, Y)$ which maps $|z|>1$ onto a domain admissible with respect to $Q^*(w; X)dw^2$. The function $g^*(z; X, Y)$ satisfies the algebraic equation (3). Inserting

$$w=z+b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+b_5z^{-5}+\cdots$$

in (3) we have

$$\begin{aligned} &b_0=0, \\ &b_1=X-2Y-1, \\ &b_2=0, \\ &b_3=2XY-4Y^2+X-6Y-2, \\ &b_4=0, \\ &b_5=\frac{1}{3}+\Phi(X, Y). \end{aligned} \tag{6}$$

This completes the proof of Lemma 1.

Let $\mathfrak{D}_1$ denote the domain in the XY-plane defined by $X>0$, $2Y-X+2>0$ and $32Y-4X+11<0$. We can verify that if $(X, Y)\in\bar{\mathfrak{D}}_1$ then X and Y satisfy the condition (2) and that $\bar{\mathfrak{D}}_1$ is mapped by (6) onto the closed domain in the b_1b_3-plane defined by $b_3+b_1^2-b_1\leqq 0$ and $12b_3-4b_1^2-b_1+5\geqq 0$.

LEMMA 2. *On* $\bar{\mathfrak{D}}_1$

$$\Phi(X, Y)\leqq\frac{1}{147}.$$

Equality occurs only for $X=3/7$, $Y=-3/7$.

Proof. The points which satisfy $\Phi_X=\Phi_Y=0$ are the following four points

$$\left(\frac{3}{7}, -\frac{3}{7}\right),\ \left(0, -\frac{1}{2}\right),\ \left(\frac{3+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{4}\right),\ \left(\frac{3-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{4}\right).$$

These points are not contained in $\mathfrak{D}_1$ except the point $(3/7, -3/7)$. At the point $(3/7, -3/7)$ we have $\Phi=1/147$, $\Phi_{XX}<0$, $\Phi_{XY}^2-\Phi_{XX}\Phi_{YY}<0$. On the other hand we have on the boundary of $\mathfrak{D}_1$

$$\Phi(0,\, Y)=-\frac{40}{3}Y^3-28Y^2-18Y-\frac{11}{3}\leqq 0\,, \quad -1\leqq Y\leqq -\frac{1}{3}\,,$$

$$\Phi\Big(X,\frac{1}{2}X-1\Big)=-\frac{1}{3}, \qquad 0\leqq X\leqq \frac{7}{4}$$

and

$$\Phi\Big(X,\frac{1}{8}X-\frac{11}{32}\Big)=-\frac{15}{64}X^3+\frac{87}{256}X^2+\frac{75}{1024}X$$

$$-\frac{3025}{12288}<0\,, \quad 0\leqq X\leqq \frac{7}{4}\,.$$

Hence we have the desired result.

LEMMA 3. *Let $\tilde{Q}(w\,;X)dw^2$ be the quadratic differential $(w^4+Xw^2)dw^2$, $(0\leqq X\leqq 4)$. Let Y be a real number satisfying the condition*

$$(7) \qquad Y+1\geqq 0\,, \qquad 12Y+X^{3/2}+4\leqq 0\,.$$

Then there is a function $\tilde{g}(z\,;X,\,Y)\in\Sigma_0$ which satisfies the algebraic equation

$$(8) \qquad (w^2+X)^3=\{z^3-(6Y+3)z-(6Y+3)z^{-1}+z^{-3}\}^2$$

and which maps $|z|>1$ onto a domain admissible with respect to $\tilde{Q}(w\,;X)dw^2$. The expansion of $\tilde{g}(z\,;X,\,Y)$ at the point at infinity begins

$$z-\Big(\frac{1}{2}X+2Y+1\Big)z^{-1}-\Big(\frac{1}{8}X^2+XY+4Y^2+\frac{1}{2}X+6Y+2\Big)z^{-3}$$

$$+\Big(\frac{1}{3}+\Psi(X,\,Y)\Big)z^{-5}+\cdots$$

where

$$\Psi(X,\,Y)=-\frac{1}{16}X^3-\frac{3}{4}X^2Y-4XY^2-\frac{40}{3}Y^3-\frac{3}{8}X^2-5XY-28Y^2$$

$$-\frac{3}{2}X-18Y-\frac{11}{3}.$$

Proof. There are three end domains $\tilde{\mathscr{E}}_1$, $\tilde{\mathscr{E}}_2$, $\tilde{\mathscr{E}}_3$ in the trajectory structure of $\tilde{Q}(w\,;X)dw^2$ on the upper half w-plane. For a suitable choice of determination the function

$$\zeta=\int w(w^2+X)^{1/2}dw$$

maps $\tilde{\mathscr{E}}_1$, $\tilde{\mathscr{E}}_2$, $\tilde{\mathscr{E}}_3$ respectively onto an upper half-plane, a lower half-plane and an upper half-plane, the positive real axis corresponding to the half-infinite segment $\Im\zeta=0$, $X^{3/2}/3<\Re\zeta<\infty$. If X and θ satisfy the condition

$$(9) \qquad 4\cos 2\theta+\frac{4}{3}\leqq -\frac{1}{3}X^{3/2}\,,$$

then we can combine this function with (4) to obtain a function which maps the domain $|z|>1$, $\Im z>0$ into the upper half w-plane. We put $Y=\cos 2\theta$. Then the condition (9) is equivalent to the condition (7). By reflection this function extends to a function $\tilde{g}(z; X, Y)$ which maps $|z|>1$ onto a domain admissible with respect to $\tilde{Q}(w; X)dw^2$. The function $\tilde{g}(z; X, Y)$ satisfies the algebraic equation (8). Inserting

$$w=z+b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+b_4z^{-4}+b_5z^{-5}+\cdots$$

in (8) we have

$$\begin{aligned}
&b_0=0\,,\\
&b_1=-\frac{1}{2}X-2Y-1\,,\\
&b_2=0\,,\\
&b_3=-\frac{1}{8}X^2-XY-4Y^2-\frac{1}{2}X-6Y-2\,,\\
&b_4=0\,,\\
&b_5=\frac{1}{3}+\Psi(X, Y)\,.
\end{aligned}\tag{10}$$

This completes the proof of Lemma 3.

Let $\mathfrak{D}_2$ denote the domain in the XY-plane defined by $X>0$, $Y+1>0$, $12Y+X+4<0$ and $36Y+7X+8<0$. We can verify that if $(X, Y)\in\bar{\mathfrak{D}}_2$ then X and Y satisfy the condition (7) and that $\bar{\mathfrak{D}}_2$ is mapped by (10) onto the closed domain in the b_1b_3-plane defined by $b_3+b_1^2-b_1\geqq 0$, $2b_3+b_1^2-1\leqq 0$, $8b_3+53b_1^2+98b_1+45\geqq 0$ and $8b_3+5b_1^2+6b_1+5\geqq 0$.

LEMMA 4. *On* $\bar{\mathfrak{D}}_2$

$$\Psi(X, Y)\leqq\frac{4}{507}.$$

Equality occurs only for $X=12/13$, $Y=-15/26$.

Proof. The points which satisfy $\Psi_X=\Psi_Y=0$ are the following four points

$$\left(\frac{12}{13}, -\frac{15}{26}\right),\ \left(0, -\frac{1}{2}\right),\ \left(2+\frac{2\sqrt{3}}{3}, -1\right),\ \left(2-\frac{2\sqrt{3}}{3}, -1\right).$$

These points are not contained in $\mathfrak{D}_2$ except the point $(12/13, -15/26)$. At the point $(12/13, -15/26)$ we have $\Psi=4/507$, $\Psi_{XX}<0$, $\Psi_{XY}^2-\Psi_{XX}\Psi_{YY}<0$. On the other hand we have on the boundary of $\mathfrak{D}_2$

$$\Psi(0, Y)=-\frac{40}{3}Y^3-28Y^2-18Y-\frac{11}{3}\leqq 0,\quad -1\leqq Y\leqq -\frac{1}{3},$$

$$\Psi(X, -1)=-\frac{1}{16}X^3+\frac{3}{8}X^2-\frac{1}{2}X-\frac{1}{3}<0,\quad 0\leqq X\leqq 4,$$

$$\Psi\left(X, -\frac{1}{12}X-\frac{1}{3}\right)=-\frac{13}{648}X^3-\frac{7}{216}X^2+\frac{1}{27}X$$
$$-\frac{23}{81}<0, \quad 0\leqq X\leqq 1$$

and

$$\Psi\left(X, -\frac{7}{36}X-\frac{2}{9}\right)=\frac{527}{17496}X^3-\frac{1775}{5832}X^2+\frac{640}{729}X$$
$$-\frac{1975}{2187}<0, \quad 1\leqq X\leqq 4.$$

Hence we have the desired result.

4. Now we prove the following theorem which includes as a special case the theorem stated in § 1.

THEOREM. *Let $g(z)$ be a function of class Σ_0 having the expansion at the point at infinity*

$$g(z)=z+\sum_{n=1}^{\infty} b_n z^{-n}$$

where b_1 and b_2 are real.
If $b_1\geqq 0$, then

$$\Re b_5\leqq\frac{1}{3}+\frac{1}{147}$$

with equality holding only for the function $g^*\left(z;\frac{3}{7},-\frac{3}{7}\right)$. The expansion of this function at the point at infinity begins

$$z+\frac{2}{7}z^{-1}-\frac{5}{49}z^{-3}+\left(\frac{1}{3}+\frac{1}{147}\right)z^{-5}+\cdots.$$

If $b_1\leqq 0$, then

$$\Re b_5\leqq\frac{1}{3}+\frac{4}{507}$$

with equality holding only for the function $\tilde{g}\left(z;\frac{12}{13},-\frac{15}{26}\right)$. The expansion of this function at the point at infinity begins

$$z-\frac{4}{13}z^{-1}+\frac{16}{169}z^{-3}+\left(\frac{1}{3}+\frac{4}{507}\right)z^{-5}+\cdots.$$

Proof. Firstly we consider the case $b_1\geqq 0$. We divide this case into several subcases.

Case 1. $\Re b_3\geqq 0$. By Lemma B we have

$$\Re b_5\leqq\Re\left\{b_5+b_1b_3+b_2^2+\frac{1}{3}b_1^3\right\}\leqq\frac{1}{3}.$$

Case 2. $(4b_1^2+b_1-5)/12\leqq\Re b_3\leqq 0$. In this case there is a point (X_0, Y_0) in $\bar{\mathfrak{D}}_1$

such that

$$b_1=X_0-2Y_0-1\,,$$

$$\Re b_3=2X_0Y_0-4Y_0^2+X_0-6Y_0-2\,.$$

We apply Lemma A with

$$Q(w)dw^2=(w^4-2X_0w^2+X_0^2)dw^2\,,$$

$$g^*(z)=g^*(z\,;\,X_0,\,Y_0)\,.$$

Then we have

$$\Re\{b_5+i(X_0-6Y_0-3)\Im b_3+b_2^3\}\leqq\frac{1}{3}+\Phi(X_0,\,Y_0)\,.$$

Hence by using Lemma 2 we obtain

$$\Re b_5\leqq\frac{1}{3}+\frac{1}{147}\,.$$

Case 3. $\Re b_3\leqq(4b_1^2+b_1-5)/12$. By Lemma C we have

$$|b_5|^2\leqq\frac{1}{5}-\frac{3}{5}\left|\frac{1}{3}b_1^2+\frac{1}{12}b_1-\frac{5}{12}\right|^2-\frac{1}{5}|b_1|^2$$

$$=-\frac{1}{720}(48b_1^4+24b_1^3+27b_1^2-30b_1+11)+\frac{1}{9}.$$

Put $P(x)=48x^4+24x^3+27x^2-30x+11$. It is very easy to prove that $P'(x)$ is monotone increasing for $0\leqq x\leqq 1$ and $P'(0)<0$, $P'(1/3)>0$. Let λ be the root of $P'(x)=0$, $0<\lambda<1/3$. Construct $N(x)=4P(x)-xP'(x)$. Then $N(x)$ is monotone decreasing for $0\leqq x\leqq 1/3$ and $N(1/3)>0$. Hence $N(x)>0$ for $0\leqq x\leqq 1/3$. Especially $N(\lambda)>0$ which implies that $P(\lambda)>0$. Therefore $P(x)>0$ for $0\leqq x\leqq 1$. Hence by the above inequality we have $\Re b_5<1/3$.

Thus we obtain that if $b_1\geqq 0$ then

$$\Re b_5\leqq\frac{1}{3}+\frac{1}{147}\,.$$

If equality occurs, then $b_2=0$ and, by Lemma 2, $b_1=2/7$. Hence by Lemma A we have $g(z)\equiv g^*\left(z\,;\frac{3}{7},\,-\frac{3}{7}\right)$.

Next we consider the case $b_1\leqq 0$. We also divide this case into several subcases.

Case 1. $\Re b_3\geqq-(b_1^2-1)/2$. By Lemma C we have

$$|b_5|^2\leqq\frac{1}{5}-\frac{3}{5}\left|-\frac{1}{2}b_1^2+\frac{1}{2}\right|^2-\frac{1}{5}|b_1|^2$$

$$=-\frac{3}{20}\left(b_1^2-\frac{1}{3}\right)^2+\frac{1}{15}<\frac{1}{9}\,.$$

This implies that $\Re b_5<1/3$.

Case 2. $\max\{-b_1^2+b_1, -(5b_1^2+6b_1+5)/8, -(53b_1^2+98b_1+45)/8\}\leqq\Re b_3\leqq-(b_1^2-1)/2$. In this case there is a point (X_0, Y_0) in $\overline{\mathfrak{D}}_2$ such that

$$b_1=-\frac{1}{2}X_0-2Y_0-1,$$

$$\Re b_3=-\frac{1}{8}X_0^2-X_0Y_0-4Y_0^2-\frac{1}{2}X_0-6Y_0-2.$$

We apply Lemma A with

$$Q(w)dw^2=(w^4+X_0w^2)dw^2,$$

$$g^*(z)=\tilde{g}(z; X_0, Y_0).$$

Then we have

$$\Re\left\{b_5-i\left(\frac{1}{2}X_0+6Y_0+3\right)\Im b_3+b_2^2\right\}\leqq\frac{1}{3}+\Psi(X_0, Y_0).$$

Hence by using Lemma 4 we have

$$\Re b_5\leqq\frac{1}{3}+\frac{4}{507}.$$

Case 3. $(4b_1^2+b_1-5)/12\leqq\Re b_3\leqq-b_1^2+b_1$. In this case there is a point (X_0, Y_0) in $\overline{\mathfrak{D}}_1$ such that

$$b_1=X_0-2Y_0-1,$$

$$\Re b_3=2X_0Y_0-4Y_0^2+X_0-6Y_0-2.$$

Hence we have

$$\Re b_5\leqq\frac{1}{3}+\frac{1}{147}.$$

Case 4. $-5/16\leqq b_1\leqq 0$ and $\Re b_3\leqq(4b_1^2+b_1-5)/12$. By Lemma C we have

$$|b_5|^2\leqq\frac{1}{5}-\frac{3}{5}\left|\frac{1}{3}b_1^2+\frac{1}{12}b_1-\frac{5}{12}\right|^2-\frac{1}{5}|b_1|^2$$

$$=-\frac{1}{720}(48b_1^4+24b_1^3+27b_1^2-30b_1+11)+\frac{1}{9}<\frac{1}{9}.$$

This implies that $\Re b_5<1/3$.

Case 5. $-2/3\leqq b_1\leqq-5/16$ and $\Re b_3\leqq\max\{-b_1^2+b_1, -(5b_1^2+6b_1+5)/8\}$. In this case $b_1\leqq-5/16$, $\Re b_3\leqq-2/5$. Hence by Lemma C we have

$$|b_5|^2\leqq\frac{1}{5}-\frac{3}{5}\cdot\frac{4}{25}-\frac{1}{5}\cdot\frac{25}{256}<\frac{1}{9}.$$

This implies that $\Re b_5<1/3$.

Case 6. $b_1\leqq-2/3$ and $\Re b_3\leqq-(53b_1^2+98b_1+45)/8$. By Lemma C we have

$$|b_5|^2\leqq\frac{1}{5}-\frac{1}{5}\cdot\frac{4}{9}=\frac{1}{9}.$$

This implies that $\Re b_5 \leqq 1/3$.

Thus we have that if $b_1 \leqq 0$ then

$$\Re b_5 \leqq \frac{1}{3} + \frac{4}{507}.$$

If equality occurs, then $b_2=0$ and, by Lemma 4, $b_1=-4/13$. Hence by Lemma A we have $g(z) \equiv \tilde{g}\left(z; \frac{12}{13}, -\frac{15}{26}\right)$.

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