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METRICS AND CONNECTIONS IN THE TANGENT BUNDLE

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1. Introduction. One of the present authors has been concerned, in collaboration with others [8, 9, 10][1], with the relations between a manifold M and its tangent bundle TM. Three mappings from M to TM have been considered, called the vertical, the complete, and the horizontal lifts, the first two not depending on any connection which may be defined on M. In this note we want to consider further the metrics and the connections which can be defined on TM. In particular, we want to take into account the fact that the tangent bundle is an almost product space with one of the distributions (the fibres) being integrable. In the greater part of the paper, we shall take the case in which the manifold M is a Riemannian space, while at the end we shall consider the case in which it is a Finsler space.

Let π be the projection $TM \to M$. If p is a point of the open set $U \subset M$ in which the local coordinates are x^h $(h, i, j, k, \cdots = 1, 2, \cdots, n)$ the local coordinates in $\pi^{-1}(U) \subset TM$ are ξ^A $(A, B, C, \cdots = 1, 2, \cdots, 2n)$ where $\xi^h = x^h$ and $\xi^{n+h} = \xi^{h^*} = y^h$, where y^h is a system of coordinates in each tangent space $T_p(M)$ referred to $\partial_i = \partial/\partial x^i$ at p. The coordinates (x^h, y^h) are called *induced coordinates* in $\pi^{-1}(U)$. The local expressions for the various lifts of a vector field X of components X^h are respectively

$$
(1.1) \qquad
\begin{aligned}
&X^V: \begin{pmatrix} 0 \\ X^h \end{pmatrix} && \text{for the vertical lift,} \\
&X^C: \begin{pmatrix} X^h \\ \partial X^h \end{pmatrix} && \text{for the complete lift, and} \\
&X^H: \begin{pmatrix} X^h \\ -\Gamma^h_i X^i \end{pmatrix} && \text{for the horizontal lift,}
\end{aligned}
$$

where $\partial = y^j \partial_j$ and $\Gamma^h_i = y^j \Gamma^h_{ji}$ with Γ^h_{ji} representing the connection coefficients in the base manifold M.

Mappings of tensors from M to TM have been given in the papers cited, and, since the metric in M is given locally by the components of a $(0, 2)$ tensor G written in a coordinate neighbourhood (U, x^h) as g_{ji}, the vertical lift G^V and the complete lift G^C of this tensor are respectively written as

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1) These numbers refer to the papers listed at the end.

$$G^V: \begin{pmatrix} g_{ji} & 0 \\ 0 & 0 \end{pmatrix}, \qquad G^C: \begin{pmatrix} \partial g_{ji} & g_{ji} \\ g_{ji} & 0 \end{pmatrix}. \tag{1.2}$$

The definition given of the horizontal lift of tensors [9] would make the horizontal lift of the metric tensor coincide with that of the complete lift. We shall suggest another definition of G^H in this paper, demanding that the horizontal lift of a vector is equal in length to the original vector.

Using (1.1) and (1.2) we immediately have

$$\begin{aligned} &G^V(X^V, Y^V)=0, && G^V(X^V, Y^H)=0, \\ &G^V(X^H, Y^H)=g(X, Y)\circ\pi, && G^V(X^C, Y^C)=g(X, Y)\circ\pi, \end{aligned} \tag{1.3}$$

where $g(X, Y)$ represents the inner product in M itself. Hence we state

PROPOSITION 1.1. *With respect to the vertical metric in TM,* (i) *the vertical vectors are null vectors,* (ii) *the horizontal vectors have the same length as the vectors to which they project,* (iii) *the vertical and horizontal vectors are orthogonal.*

The corresponding formulae for the complete lift are

$$\begin{aligned} &G^C(X^V, Y^V)=0, && G^C(X^V, Y^H)=g(X, Y)\circ\pi, \\ &G^C(X^H, Y^H)=0, && G^C(X^C, Y^C)=(g(X, Y))^C. \end{aligned} \tag{1.4}$$

We correspondingly state

PROPOSITION 1.2. *With respect to the complete metric in TM, both the vertical and the horizontal vectors are null. The vertical lift of a vector X and the horizontal lift of a vetor Y will only be orthogonal if they are orthogonal with respect to the metric g in M.*

2. The tangen bundle as an almost product space. Let us take any C^∞ manifold $\bar{M}$ of dimension N on which are defined globally two distributions of dimension n and $N-n$ respectively. Let H and V represent the distributions which are assumed to form a complete system so that their sum is the distribution of the tangent planes to $\bar{M}$. Associated with the distributions there are two (1, 1) tensors H and V of rank n and $N-n$ respectively which characterize the distributions and satisfy

$$H^2=H, \quad V^2=V, \quad HV=VH=0, \quad H+V=I. \tag{2.1}$$

The torsion tensor of the almost product structure is defined for any two vector fields X and Y by ([4], p. 37)

$$\begin{aligned} S_{H,V}(X, Y)&=S_{V,H}(X, Y)=-S_{H,V}(Y, X) \\ &=[HX, VY]-H[X, VY]-V[HX, Y]+HV[X, Y] \\ &\quad+[VX, HY]-V[X, HY]-H[VX, Y]+VH[X, Y], \end{aligned}$$

which reduces, for the case in which H and V satisfy (2.1) to

(2. 2) $$S_{H,V}(X, Y)=-2H[VX, VY]-2V[HX, HY].$$

In this form the torsion tensor of the almost product structure immediately gives information about the integrability of the distributions. The case that we shall be concerned with is that in which the bracket $[VX, VY]$ lies in V, and hence the expression for the torsion of the almost product structure reduces to

(2. 3) $$S_{H,V}(X, Y)=-2V[HX, HY].$$

If any metric $\bar{G}$ has been defined on the manifold M, a new metric G can be defined with respect to which the distribution H and V are orthogonal by putting

(2. 4) $$G(X, Y)=\bar{G}(HX, HY)+\bar{G}(VX, VY)$$

so that

$$G(HX, VY)=0.$$

Connections which are closely related to the distributions have been studied by Walker [7] whose results have been applied to the particular case of a tangent bundle [2]. If we are given any symmetric connection, such as the unique torsion-free connection $\overset{G}{\nabla}$ associated with a metric, another symmetric connection $\mathring{\nabla}$ can be constructed such that, with respect to $\mathring{\nabla}$ the $V(H)$ distribution is parallel for a displacement in the $H(V)$ distribution, which is expressed by

(2. 5) $$V\mathring{\nabla}_{VX}(HY)=0 \quad \text{and} \quad H\mathring{\nabla}_{HX}(VY)=0.$$

This connection will also have the property that a path (auto-parallel) of the connection whose tangent at a point is in $H(V)$ will have its tangent at every point in $H(V)$. This is expressed by the conditions

(2. 6)
$$\text{(a)} \qquad V\{\mathring{\nabla}_{HX}(HY)+\mathring{\nabla}_{HY}(HX)\}=0,$$
$$\text{(b)} \qquad H\{\mathring{\nabla}_{VX}(VY)+\mathring{\nabla}_{VY}(VX)\}=0.$$

The $\mathring{\nabla}$ is determined from $\overset{G}{\nabla}$ by the formula

(2. 7) $$\mathring{\nabla}_X Y=\overset{G}{\nabla}_X Y+B(X, Y)(\overset{G}{\nabla}),$$

where

$$2B(X, Y)(\overset{G}{\nabla})=-2\{A(X, Y)+A(Y, X)\}+\{A(HX, HY)+A(HY, HX)\}$$
$$+\{A(VX, VY)+A(VY, VX)\}$$

and where

(2. 8) $$A(X, Y)=H\overset{G}{\nabla}_X(VY)+V\overset{G}{\nabla}_X(HY).$$

This symmetric connection does not satisfy the condition $\mathring{\nabla}G=0$, since the unique connection for which G is constant is the coonection determined by G

itself, which we denote by $\overset{G}{\nabla}$. One of Walker's main results however is the following:

THEOREM I. *Given a metric G with respect to which two complementary distributions are orthogonal, it is possible to construct a global connection* ∇ *with torsion with respect to which the two distributions are parallel and such that G is constant, i.e.* $\nabla G=0$.

We note that the two conditions of relative parallelism and of path parallelism satisfied by the $\overset{\circ}{\nabla}$ are weaker than the condition of parallelism, which would demand both

$$H\overset{\circ}{\nabla}_X(VY)=0 \quad \text{and} \quad V\overset{\circ}{\nabla}_X(HY)=0. \tag{2. 9}$$

Such a connection is obtained from any symmetric conection $\overset{S}{\nabla}$ by forming

$$\nabla_X Y=\overset{S}{\nabla}_X Y-A(X, Y)(\overset{S}{\nabla}). \tag{1. 10}$$

For the paraticular case of the tangent bundle the HX and VX are the horizontal and vertical lifts X^H and X^V of the vector tangent to a manifold M, where the vertical distribution V is given by the fibres and the horizontal distribution H is the complementary distribution which determines the connection in M. For this case the following relations hold for the brackets [3]

$$\begin{aligned}
&[X^V, Y^V]=0,\\
&[X^V, Y^H]=[X, Y]^V-(\overset{g}{\nabla}_X Y)^V,\\
&H[X^H, Y^H]=[X, Y]^H,\\
&V[X^H, Y^H]=-\gamma R(X, Y)(\overset{g}{\nabla}),
\end{aligned} \tag{2. 11}$$

γ applied to a tensor field $R_i{}^h$ of type (1, 1) of M being a vector field $\begin{pmatrix} 0 \\ R_i{}^h y^i \end{pmatrix}$ of TM, which, on comparing with the expression (2. 2) for the torsion $S_{V,H}(X, Y)$ of the almost product structure immediately gives a relation between that torsion and the curvature tensor of the base manifold M.

The horizontal distribution H is spanned by the n vectors

$$e_i=\frac{\partial}{\partial x^i}-\Gamma_i^h\frac{\partial}{\partial y^h}, \tag{2. 12}$$

where $\Gamma_i^h=y^j\Gamma_{ji}^h$ and Γ_{ji}^h are the coefficients of the connection defined in M. The vertical distribution V is spanned by the n vectors

$$e_{i^*}=e_{n+i}=\frac{\partial}{\partial y^i}, \tag{2. 13}$$

so that a vector $X=X^h\partial/\partial x^h$ tangent to M has lifts given by

$$X^H=X^h e_h, \qquad X^V=X^h e_{h^*}.$$

Forms ω^h, ω^{h^*} which are dual to the vectors e_i and e_{i^*} are given by

$$\omega^h = dx^h, \qquad \omega^{h^*} = dy^h + \Gamma^h_i dx^i.$$

The complete lift X^C of a vector has components in both the horizontal and the vertical distributions. These components are readily found to be

$$HX^C = X^H = \begin{pmatrix} X^h \\ -\Gamma^h_i X^i \end{pmatrix} \tag{2.14}$$

and

$$VX^C = \begin{pmatrix} 0 \\ \nabla_0 X^h \end{pmatrix}, \tag{2.15}$$

where

$$\nabla_0 X^h = y^i \nabla_i X^h.$$

3. Connections associated with the metric G^C. Any Riemannian metric G and a skew-symmetric tensor $T(X, Y)$ determine a connection ∇ for which $\nabla G = 0$ with T as torsion in accordance with the following formula

$$\begin{aligned} 2G(Z, \nabla_X Y) &= XG(Y, Z) + YG(Z, X) - ZG(X, Y) + G(Y, [Z, X] + T(Z, X)) \\ &\quad + G(Z, [X, Y] + T(X, Y)) - G(X, [Y, Z] + T(Y, Z)). \end{aligned} \tag{3.1}$$

We shall now take some particular cases of the general formula.

(i) Let us take the case in which (a) the vectors X, Y, and Z are the vectors of the natural basis corresponding to the system of coordinates $\xi^A = (x^h, y^h)$, (b) the torsion is zero, (c) the metric is the complete lift G^C of the metric G of the base manifold. The general formula (3.1) will then give the Christoffel symbols for G^C and we shall have, on letting Γ^h_{ji} represent the corresponding symbols in M, and using a bar to indicate quantities in TM:

$$\bar{\Gamma}^h_{ji} = \bar{\Gamma}^{h^*}_{j^*i} = \bar{\Gamma}^{h^*}_{ji^*} = \Gamma^h_{ji}, \qquad \bar{\Gamma}^{h^*}_{ji} = \partial \Gamma^h_{ji}, \tag{3.2}$$

while the others vanish [10]. Using these we can easily obtain the equations of the geodesics in TM in the form

$$\begin{aligned} &\frac{d^2x^h}{dt^2} + \Gamma^h_{ji} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0, \\ &\frac{\nabla^2 y^h}{dt^2} + R_{kji}{}^h y^k \frac{dx^j}{dt} \frac{dx^i}{dt} = 0 \end{aligned} \tag{3.3}$$

and we may state [10]

PROPOSITION 3.1. *If a point $(x(t), y(t))$ in TM describes a geodesic relative to the metric G^C, the projection $x(t)$ to M describes a geodesic in M and $y(t)$ gives a Jacobi field along this geodesic of M.*

If we compute the components of the curvature tensor of TM relative to G^C, we obtain the following non-vanishing components

$$\bar{R}_{kji}{}^{h}=\bar{R}_{kji^*}{}^{h^*}=\bar{R}_{kj^*i}{}^{h^*}=\bar{R}_{k^*ji}{}^{h^*}=R_{kji}{}^{h}, \qquad \bar{R}_{kji}{}^{h^*}=\partial R_{kji}{}^{h}. \tag{3.4}$$

The components of the Ricci tonsor are

$$\bar{R}_{ji}=2R_{ji}, \qquad \bar{R}_{j^*i}=0, \qquad \bar{R}_{ji^*}=0, \qquad \bar{R}_{j^*i^*}=0. \tag{3.5}$$

From these expressions we may conclude

PROPOSITION 3. 2. *The tangent bundle TM with the complete lift as metric* (a) *is flat if and only if M is flat,* (b) *is an Einstein space if and only if M is Einstein with vanishing scalar curvature,* (c) *has vanishing scalar curvature.*

(ii) We take the vectors and the metric as before, but we take the torsion tensor to have only one set of components determined in terms of the curvature tensor of the base space M by

$$2T_{ji}{}^{h^*}=R_{jik}{}^{h}y^{k}=R_{ji0}{}^{h}.$$

This metric space with torsion will have the non-vanishing connection components

$$\bar{\Gamma}^{h}_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\bar{\Gamma}^{h^*}_{j^*i}=\Gamma^{h}_{ji}, \qquad \bar{\Gamma}^{h^*}_{ji}=\partial\Gamma^{h}_{ji}-R_{0ji}{}^{h}. \tag{3.6}$$

This set of connection coefficients has appeared as the "horizontal lift" of the connection in M [9].

(iii) We next take the case in which (a) the torsion is zero, (b) the vectors X, Y, and Z are the base vectors e_i and e_{i^*} of the distribution H and V in which case, using relations (1. 3) and (1. 4)

$$\begin{aligned} &G^C(e_j, e_i)=0, \qquad G^C(e_{j^*}, e_{i^*})=0, \\ &G^C(e_j, e_{i^*})=G^C(e_{j^*}, e_i)=g(\partial_j, \partial_i)=g_{ji}. \end{aligned} \tag{3.7}$$

Also for the square brackets we shall have, corresponding to the relations (2. 11)

$$\begin{aligned} &[e_{j^*}, e_{i^*}]=0, \\ &[e_j, e_{i^*}]=(\Gamma^{h}_{ji}\circ\pi)e_{h^*}, \\ &[e_j, e_i]=-(R_{ji0}{}^{h}\circ\pi)e_{h^*} \end{aligned} \tag{3.8}$$

which will give us as coefficients of connection

$$\bar{\Gamma}^{h}_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\Gamma^{h}_{ji}\circ\pi, \qquad \bar{\Gamma}^{h^*}_{ji}=R_{0ji}{}^{h}\circ\pi. \tag{3.9}$$

(iv) Finally in considering connections deducible from the metric we consider the case in which the torsion of the connection is related to the torsion of the almost product structure by the relation

$$2T(X, Y)=S_{H,V}(X, Y)=-2V[HX, HY], \tag{3.10}$$

which means that, in components, relative to the basis e_i, e_{i^*} the only non-vanishing component of the torsion tensor is

$$-V[e_j, e_i]=(R_{jio}{}^h\circ\pi)e_{h^*}. \tag{3.11}$$

The non-vanishing components of the corresponding connection $\bar{\nabla}$ will therefore be

$$\bar{\Gamma}^h_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\Gamma^h_{ji}\circ\pi. \tag{3.12}$$

The fact that $\bar{\Gamma}^h_{ji^*}$ and $\bar{\Gamma}^h_{j^*i^*}$ vanish means that the distribution V is parallel with respect to $\bar{\nabla}$. Similarly the vanishing of $\bar{\Gamma}^{h^*}_{ji}$ and $\Gamma^{h^*}_{j^*i}$ means that the distribution H is parallel with respect to $\bar{\nabla}$. We can therefore state

THEOREM II. *The lifted metric G^C and the torsion $S_{H,V}$ of the almost product structure determine a connection $\bar{\nabla}$ for which* (a) *the fibres and the complementary horizontal distribution are parallel,* (b) *the metric is constant, i.e.* $\bar{\nabla}G^C=0$.

The metric G is not one with respect to which the two distributions are orthogonal.

Returning for the moment to the torsion-free connection listed in (3.9) we notice that it does satisfy the conditions in order that the distribution V and H are relatively parallel with respect to it, which would demand the vanishing of $\bar{\Gamma}^h_{j^*i}$ and $\bar{\Gamma}^{h^*}_{j^*i}$, but it does not satisfy the path parallelism condition for H, which requires $\bar{\Gamma}^{h^*}_{ji}+\bar{\Gamma}^{h^*}_{ij}=0$. It was pointed out in the preceding section however that a torsion-free connection $\mathring{\nabla}$ satisfying the conditions can always be constructed by the formula (2.7). The application of the formula in this case gives

$$\mathring{\Gamma}^h_{ji}=\mathring{\Gamma}^{h^*}_{j^*i}=\Gamma^h_{ji}\circ\pi,\qquad 2\mathring{\Gamma}^{h^*}_{ji}=-2\mathring{\Gamma}^{h^*}_{ij}=-R_{jio}{}^h\circ\pi. \tag{3.13}$$

This connection is of course not metrical, since the only torsion-free connection for which the metric G^C is constant is the one given in (3.9). To deduce from $\mathring{\nabla}$ a connection $\bar{\nabla}$ (with torsion) with respect to which both distributions are parallel we need only use formula (2.10). In this case it is to be pointed out that

$$A(X, Y)(\mathring{\nabla})=V[HX, HY] \tag{3.14}$$

so that although the A tensor is constructed from the $\mathring{\nabla}$, it is independent of any connection. The torsion of the conection $\bar{\nabla}$, the torsion of the almost product structure, and the A tensor, are now related by

$$2T(X, Y)=S_{H,V}(X, Y)=-2A(X, Y)(\mathring{\nabla}). \tag{3.15}$$

The connection $\bar{\nabla}$ is then given by

$$\bar{\Gamma}^h_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\Gamma^h_{ji}\circ\pi. \tag{3.16}$$

We notice that although this connection has been deduced from the non-metrical $\mathring{\nabla}$ connection, $\bar{\nabla}$ coincides with the metrical connection with torsion appearing in (3.12).

4. The metric G^L. Having defined the horizontal and vertical lifts of vectors to the tangent bundle, let us introduce a metric G^L by demanding

$$G^L(X^H, Y^H)=G^L(X^V, Y^V)=g(X, Y)\circ\pi, \quad G^L(X^H, Y^V)=0. \tag{4.1}$$

In particular since $e_i=(\partial_i)^H$ and $e_{i^*}=(\partial_i)^V$ are lifts of the base vectors we have

$$G^L(e_j, e_i)=G^L(e_{j^*} e_{i^*})=g_{ji}\circ\pi, \qquad G^L(e_j, e_{i^*})=0, \tag{4.2}$$

so that with respect to the lifted base vectors the components of G^L are

$$G^L: \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix}. \tag{4.3}$$

This metric is already well known and its components with respect to the induced coordinate system given by Sasaki [5] as

$$G^L: \begin{pmatrix} g_{ji}+\Gamma^t_j\Gamma^s_i g_{ts} & \Gamma^s_j g_{si} \\ \Gamma^s_i g_{sj} & g_{ji} \end{pmatrix} \tag{4.4}$$

who also calculated the corresponding Christoffel symbols, which appear also in a recent paper by Tashiro [6] as

$$\begin{aligned} &2\bar{\Gamma}^h_{ji}=2\Gamma^h_{ji}+R_{0sj}{}^h\Gamma^s_i+R_{0si}{}^h\Gamma^s_j, \qquad 2\bar{\Gamma}^h_{ji^*}=R_{0ji}{}^h, \\ &2\bar{\Gamma}^h_{j^*i}=R_{0ij}{}^h, \\ &2\bar{\Gamma}^{h^*}_{ji}=2\partial\Gamma^h_{ji}-R_{0ji}{}^h-R_{0ij}{}^h-(R_{0sj}{}^t\Gamma^s_i+R_{0si}{}^t\Gamma^s_j)\Gamma^h_t, \\ &2\bar{\Gamma}^{h^*}_{j^*i}=2\Gamma^h_{ji}-R_{0ji}{}^t\Gamma^h_t. \end{aligned} \tag{4.5}$$

If we apply the general formula (3.1) to the metric in the form (4.3), using the simplifications that arise from the orthogonality of the V and H with respect to this metric, we obtain for the case of zero torsion

$$\begin{aligned} &\bar{\Gamma}^h_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\Gamma^h_{ji}\circ\pi, \\ &2\bar{\Gamma}^h_{j^*i}=2\bar{\Gamma}^h_{ji^*}=R_{0ji}{}^h, \\ &2\bar{\Gamma}^{h^*}_{ji}=-R_{ji0}{}^h. \end{aligned} \tag{4.6}$$

For the unique torsion-free connection associated with the metric G^L therefore we notice that both distributions are path parallel but the conditions of relative parallelism are not satified, since $\Gamma^h_{ji^*}$ does not vanish.

If we now take the same general formula (3.1), and take for torsion the expression given by (3.15) in terms of the torsion of the almost product structure, the corresponding set of connection coefficients simplify to the following:

$$\bar{\Gamma}^h_{ji} = \bar{\Gamma}^{h*}_{ji*} = \Gamma^h_{ji} \circ \pi \tag{4.7}$$

and the others all vanish. We remark that the connection coefficients so obtained coincide with the coefficients obtained from G^C and appearing in (3.12) and (3.16). We may now state

THEOREM III. *The metric G^L and the torsion $S_{H,V}$ of the almost product structure determine connections such that*

(i) *the distributions are orthogonal with respect to G^L.*

(ii) *the distributions are path parallel with respect to the torsion-free connection given in* (4.6).

(iii) *the distributions are parallel with respect to the connection given in* (4.7).

(iv) *the G^L is constant with respect to the connection* (4.7).

5. The metric G^L deducible from a Finsler metric on M. The present authors have shown elsewhere [8] how a function L defined on TM possessing the properties necessary to serve as the fundamental function for a Finsler space can be used to define a Riemannian metric in TM which is Hermitian. Let us take a Riemannian metric $'G$ on TM such that

$$'G(X^H, X^H) = 'G(X^V, X^V) = L^2(x, X) \circ \pi,$$

where X is a vector tangent to M and X^H and X^V its lifts to the tangent bundle. If $\bar{X}$ denotes the vector whose components in the distributions V and H are respectively X^V and X^H then we define

$$G^L(\bar{X}, \bar{Y}) = 'G(X^H, Y^H) + 'G(X^V, Y^V),$$

which will have a coordinate representation

$$G^L: \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix} \quad \text{with} \quad g_{ji} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^j \partial y^i}. \tag{5.1}$$

If we apply the general formula (3.1) to this case in which the G is the G^L and the torsion is zero, we can use the following to simplify the expressions:

(i) the fibres and the horizontal distribution are orthogonal with respect to G^L:

$$G^L(X^V, Y^H) = 0, \tag{5.2}$$

(ii) the fibres are integrable so that

$$H[X^V, Y^V] = 0, \tag{5.3}$$

(iii) the bracket $[X^H, Y^V]$ is veritical

$$H[X^H, Y^V] = 0. \tag{5.4}$$

If we use the abbreviation HHV for $G^L(Z^H, \nabla_{X^H} Y^V)$ with corresponding meanings for the other combinations of the letters H and V, we can write for the coefficients of the connection determined by G^L the following:

$$
(5.5)\quad
\begin{aligned}
&\text{(a)}\quad 2HHV = Y^V G^L(Z^H, X^H) + G^L(Y^V, [Z^H, X^H]),\\
&\text{(b)}\quad 2HVH = X^V G^L(Y^H, Z^H) - G^L(X^V, [Y^H, Z^H]),\\
&\text{(c)}\quad 2HVV = -Z^H G^L(X^V, Y^V) + G^L(Y^V, [Z^V, X^H]) + G^L(X^V, [Z^H, Y^V]),\\
&\text{(d)}\quad 2VHH = -Z^V G^L(X^H, Y^H) + G^L(Z^V, [X^H, Y^H]),\\
&\text{(e)}\quad VHV = X^H G^L(Y^V, Z^V) + G^L(Y^V, [Z^V, X^H]) + G^L(Z^V, [X^H, Y^V]),\\
&\text{(f)}\quad VVH = Y^H G^L(Z^V, X^V) + G^L(Z^V, [X^V, Y^H]) - G^L(X^V, [Y^H, Z^V]),\\
&\text{(g)}\quad VVV = X^V G^L(Y^V, Z^V) + Y^V G^L(Z^V, X^V) - Z^V G^L(X^V, Y^V).
\end{aligned}
$$

If instead of taking the torsion of the connection to be zero we take it to be determined by

$$(5.6)\qquad T(X, Y) = -V[X^H, Y^H]$$

then some of the above experessions simplify and we have

$$
(5.7)\quad
\begin{aligned}
&\text{(a)}\quad HHV = Y^V G^L(Z^H, X^H),\\
&\text{(b)}\quad HVH = X^V G^L(Y^H, Z^H),\\
&\text{(c)}\quad 2VHH = -Z^V G^L(X^H, Y^H).
\end{aligned}
$$

6. Relation to the theory of Finsler spaces. Referring now to the classical theory of Finsler spaces as presented in Cartan [1] we recall that if γ^h_{ji} denote the Christoffel symbols formed from the g_{ji} defined in (5.1) we write, using $\partial_i = \partial/\partial x^i$, $\dot\partial_i = \partial/\partial y^i$

$$(6.1)\qquad 2G^h = \gamma^h_{ji} y^j y^i, \quad G^h_i = \dot\partial_i G^h, \quad G^h_{ji} = \dot\partial_j G^h_i$$

and recall that ([1], *p.* 19)

$$(6.2)\qquad G^h_{ji} = \overset{*}{\Gamma}{}^h_{ji} + \nabla_0 A_{ji}{}^h = \overset{*}{\Gamma}{}^h_{ji} + y^k \partial_k A_{ji}{}^h.$$

The horizontal and vertical lifts of the base vectors ∂_i tangent to M can now be written

$$(6.3)\qquad (\partial_i)^H = e_i = \partial_i - G_i{}^t \partial_t, \qquad (\partial_i)^V = e_{i*} = \dot\partial_i$$

and the brackets corresponding to (3.8) now become

$$
(6.4)\quad
\begin{aligned}
&[e_{j*}, e_{i*}] = 0,\\
&[e_j, e_{i*}] = G^t_{ji} e_{t*} = (\overset{*}{\Gamma}{}^t_{ji} + \nabla_0 A_{ji}{}^t) e_{t*},\\
&[e_j, e_i] = -R_{jit}{}^s y^t e_{s*} = -R_{ji0}{}^s e_{s*}.
\end{aligned}
$$

The coefficients of the torsion-free connection associated with the function L

and the Riemannian metric G^L on TM are therefore obtained from (5. 5) on taking the vectors X, Y and Z to be the natural base vectors associated with the coordinate system in M, and recalling the coordinate expression for G^L and its inverse, we get

$$
\begin{aligned}
&\bar{\Gamma}^h_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\overset{*}{\Gamma}{}^h_{ji},\\
&\bar{\Gamma}^h_{j^*i^*}=-\bar{\Gamma}^{h^*}_{i^*j}=\nabla_0 A_{ji}{}^h,\\
(6.\,5)\qquad &2\bar{\Gamma}^h_{ji^*}=2C_{ji}{}^h-R^h{}_{j0i},\\
&2\bar{\Gamma}^h_{j^*i}=2C_{ji}{}^h+R^h{}_{j0i},\\
&2\bar{\Gamma}^{h^*}_{ji^*}=-2C_{ji}{}^h-R_{ji0}{}^h,\\
&\bar{\Gamma}^{h^*}_{j^*i^*}=C_{ji}{}^h,
\end{aligned}
$$

expressions which have already appeared [8]. If we now calculate the corresponding coefficient of a metric space with the torsion determined by (5. 6) the corresponding table becomes

$$
\begin{aligned}
&\bar{\Gamma}^h_{ji}=\bar{\Gamma}^{h^*}_{ji^*}=\overset{*}{\Gamma}{}^h_{ji},\\
(6.\,6)\qquad &\bar{\Gamma}^h_{ji^*}=\bar{\Gamma}^h_{j^*i}=-\Gamma^{h^*}_{ji}=\bar{\Gamma}^{h^*}_{j^*i^*}=C_{ji}{}^h,\\
&\bar{\Gamma}^h_{j^*i^*}=-\Gamma^{h^*}_{j^*i}=\nabla_0 A_{ji}{}^h.
\end{aligned}
$$

If we denote by F the almost complex structure given by

$$
(6.\,7)\qquad F_B{}^A:\ \begin{pmatrix} G_i{}^h & \delta^h_i \\ -\delta^h_i-G_i{}^tG_t{}^h & -G_i{}^h \end{pmatrix}
$$

or, with reference to the frame determined by e_i, e_{i^*} as

$$
(6.\,8)\qquad F_B{}^A:\ \begin{pmatrix} 0 & \delta^h_i \\ -\delta^h_i & 0 \end{pmatrix},
$$

we shall find [8] that $\overset{L}{\nabla}F=0$ if and only if $R_{ji0}{}^h=0$, which is the condition that the horizontal distribution H is also integrable, as follows from (6. 4). The conditions listed in (6. 5) and (6. 6) do not satisfy the conditions necessary to make the distributions V and H parallel.

A connection $\overset{\circ}{\nabla}$ which is symmetrical and with respect to which V and H are relatively parallel and path parallel is deducible from (6. 5) by an application of the formula (2. 10) giving the non-vanishing components

$$
\begin{aligned}
&\overset{\circ}{\Gamma}{}^h_{ji}=\overset{\circ}{\Gamma}{}^{h^*}_{ji^*}=\overset{*}{\Gamma}{}^h_{ji},\\
(6.\,9)\qquad &2\overset{\circ}{\Gamma}{}^{h^*}_{ji}=-2\overset{\circ}{\Gamma}{}^{h^*}_{ij}=2\Gamma^h_{j^*i}=-R_{ji0}{}^h,\\
&\overset{\circ}{\Gamma}{}^{h^*}_{j^*i^*}=C_{ji}{}^h.
\end{aligned}
$$

Finally calculating

$$2\nabla_X Y = 2\mathring{\nabla}_X Y + S_{H,V}(X, Y) \tag{6.10}$$

gives the connection ∇ with torsion of non-vanishing components

$$\begin{aligned} &\bar{\Gamma}^h_{ji} = \bar{\Gamma}^{h^*}_{ji^*} = \overset{*}{\Gamma}{}^h_{ji}, \\ &\bar{\Gamma}^h_{j^*i} = -R_{jio}{}^h, \qquad \bar{\Gamma}^{h^*}_{j^*i^*} = C_{ji}{}^h. \end{aligned} \tag{6.11}$$

In this case the vanishing of the coefficients assuring the parallelism of the distributions is assured. We may therefore state

THEOREM IV. *The fundamental function L of a Finsler spce, together with the torsion of the almost product structure represented by V and H, determine an F-connnection which is metric, and also a connection for which V and H are parallel. Their components are given by* (6.6) *and* (6.10) *respectively.*

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