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# A REMARK ON ROYDEN'S COMPACTIFICATION OF RIEMANNIAN SPACES*

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In the development of the theory of Royden's compactification of Riemann surfaces, maximum principles for harmonic functions play a key role. These principles were established by means of the double of a bordered subregion of a Riemann surface (cf. Kusunoki-Mori [4], Nakai [7]). In the case of a higher dimensional Riemannian space, the double of a bordered subregion is, however, not available without losing the smoothness of the metric. Although the theory in Chapter III in the monograph Sario-Nakai [9] can be reproduced for higher dimensional Riemannian spaces with discontinuous metric, including the one concerning doubling, it is also worthwhile to consider the smooth metric. The purpose of this paper is to replace Theorem A below, known for Riemann surfaces, by Theorem B, so that all fundamental theorems in Chapter III of Sario-Nakai [9] (also cf. Nakai [6]), such as the maximum principles and the decomposition theorem, can be extended verbatim to higher dimensional Riemannian spaces. For another approach, see Chang [1], Glasner-Katz [2], and Kwon [5].

Consider an $N$-dimensional ($N \geqq 2$) Riemannian space $R$ with Royden's compactification $R^*$ and harmonic boundary $\Delta$. Denote by $M(R)$ the family of bounded Tonelli functions with finite Dirichlet integrals over $R$. A function $f$ on $R$ is said to be a $BD$-limit of a sequence $\{f_n\}$ of functions on $R$ if $\{f_n\}$ is uniformly bounded, $\lim_n \sup_K |f-f_n|=0$ for any compact set $K$ in $R$, and $\lim_n D_R(f-f_n)=0$ where $D_R(f-f_n)$ is the Dirichlet integral of $f-f_n$ over $R$. Denote by $M_0(R)$ the subfamily of functions in $M(R)$ with compact supports in $R$ and by $M_\Delta(R)$ the $BD$-closure of $M_0(R)$ in $M(R)$.

THEOREM A. *Let $G$ be a subregion of $R$ ($N=2$) with analytic relative boundary $\partial G$ and with $\bar{G} \cap \Delta = \phi$. Then the double $\hat{G}$ of $G$ about $\partial G$ is parabolic.*

Suppose $G$ is a subregion of $R$ ($N \geqq 2$) such that $\partial G$ is smooth. Take a regular exhaustion $\{R_n\}_1^\infty$ of $R$ with $R_1 \cap G \neq \phi$. For each $n \geqq 1$, there exists a function $u_n$ on $\bar{G}$ (cf. Itô [3]) such that

$$u_n \in C(\bar{G}) \cap H(G \cap R_n - \bar{R}_0),$$

$$u_n | \bar{G} - R_n = 1, \qquad u | \bar{R}_0 = 0,$$

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$$*du_n=0 \quad \text{on} \quad \partial G\cap R_n,$$

$$0\leqq u_n\leqq 1 \quad \text{on} \quad \bar{G},$$

with $R_0$ a parametric ball in $R_1\cap G$.

THEOREM B. *Let G be a subregion of R* $(N\geqq 2)$ *with smooth relative boundary* $\partial G$ *and with* $\bar{G}\cap \Delta=\phi$. *Then the functions* $u_n$ *constructed above have the property*

$$0=BD\text{-}\lim_n u_n$$

*on G.*

*Proof.* $\{u_n\}$ is uniformly bounded and $0\leqq u_{n+1}\leqq u_n\leqq 1$. Hence $\{u_n\}$ converges uniformly on each compact set $K$ in $G-\bar{R}_0$ to a function $u$ harmonic on $G-\bar{R}_0$. For $m<n$, by Green's formula

$$D_G(u_m-u_n, u_n)=\int_{\partial(G\cap R_n-\bar{R}_0)}(u_m-u_n)*du_n=\int_{\partial G\cap R_n}(u_m-u_n)*du_n$$
$$+\int_{G\cap\partial R_n}(u_m-u_n)*du_n-\int_{\partial R_0}(u_m-u_n)*du_n=0.$$

This implies that

$$D_G(u_m-u_n)=D_R(u_m)-D_R(u_n)\geqq 0,$$

i.e. $\{u_n\}$ is $D$-Cauchy. Therefore $u=BD\text{-}\lim u_n$ exists on $G$, $u\in C(\bar{G})\cap HBD(G-\bar{R}_0)$ and $u|\bar{R}_0=0$.

Since $\bar{G}\cap\Delta=\phi$, for each $x\in\bar{G}$, there exists a function $f_x\in M_\Delta(R)$ such that $f_x(x)>1$ and $f_x\geqq 0$ on $R$. Let $\Omega_{f_x}=\{y\in\bar{G}|f_x(y)>1\}$. The $\{\Omega_{f_x}\}_{x\in\bar{G}}$ forms an open cover of the compact set $\bar{G}$ and hence there exist a finite number of points $x_1,\cdots,x_p$ in $\bar{G}$ such that $\bar{G}\subset\bigcup_{i=1}^p\Omega_{f_{x_i}}$. Let $f=\sum_{i=1}^p f_{x_i}$. Then $f\in M_\Delta(R)$, $f|\bar{G}>1$, and $f\geqq 0$ on $R$.

Let $g=\min(f,1)$ on $R$. Thus $g\in M_\Delta(R)$, $g|G=1$, and $0\leqq g\leqq 1$ on $R$. There exists a sequence $\{g_n\}$ in $M_0(R)$ such that $g=BD\text{-}\lim_n g_n$. For a fixed $n$, and an $m$ so large that $G\cap\operatorname{supp}(g_n)\subset G\cap\bar{R}_m$,

$$D_G(u_mg_n, u_m)=\int_{\partial(G\cap R_m-\bar{R}_0)}u_mg_n*du_m=\int_{\partial G\cap R_m}u_mg_n*du_m$$
$$+\int_{G\cap\partial R_m}u_mg_n*du_m-\int_{\partial R_0}u_mg_n*du_m=0.$$

Similarly $D_G((u-u_m)g_n, u_m)=0$. Since

$$D_G(ug_n, u)=D_G((u-u_m)g_n+u_mg_n, u-u_m+u_m)$$
$$=D_G((u-u_m)g_n, u-u_m)+D_G(u_mg_n, u-u_m)$$
$$+D_G((u-u_m)g_n, u_m)+D_G(u_mg_n, u_m),$$

and $u=D\text{-}\lim_m u_m$, we have $D_G(ug_n, u)=0$ for all $g_n$. Therefore

$$D_G(u)=D_G(ug, u)=\lim_n D_G(ug_n, u)=0,$$

i.e. $u=$const. on $G$. Since $u\in C(G)$ and $u|\bar{R}_0=0$, it follows that $u\equiv 0$ on $G$ and

$$0=BD\text{-}\lim_n u_n$$

on $G$.

REMARK. For $N=2$, it is readily seen that Theorems A and B are identical. (cf. Royden [8], Sario-Nakai [9]).

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