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SUBMANIFOLDS OF MANIFOLDS WITH AN f-STRUCTURE

BY GERALD D. LUDDEN

Let M^n be an n-dimensional C^∞ manifold and f a tensor of type (1, 1) such that

$$f^3+f=0,$$

and the rank of f is constant, say r, on M^n. We then say that M^n has an *f-structure* of rank r (Cf. [4]). The rank r of f is necessarily even and it is known that if r is maximal, then f is an almost complex structure on M^n if n is even or an almost contact structure on M^n if n is odd (Cf. [4]). Yano and Ishihara [5] have shown that if M^n is an almost complex manifold then a submanifold of M^n satisfying a certain property possesses a natural f-structure. In particular, Tashiro [3] has shown that if the submanifold is a hypersurface then the induced f-structure has maximal rank (i.e. is almost contact). On the other hand, the present author and Prof. D. E. Blair [1] have shown that a hypersurface of an almost contact manifold possesses a natural f-structure, which may not have maximal rank.

The purpose of this paper is to show that if M^n has an f-structure then a submanifold of M^n satisfying the condition of Yano and Ishihara possesses a natural f-structure. In § 3 we examine the meaning of this condition in the special case where the submanifold is a hypersurface. § 4 is devoted to a study of the integrability of the induced f-structure.

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§ 1. Preliminaries.

Let M^n be a given n-dimensional C^∞ manifold. Let f be a given f-structure on M^n of rank r. Then the tensors l and m, where $l=-f^2$ and $m=f^2+I$, are complementary projection operators, i.e.

$$l^2=l, \qquad m^2=m,$$
$$l+m=I, \qquad lm=ml=0. \tag{1.1}$$

Here I denotes the identity operator. Thus, there exist in M^n complementary distributions L and M corresponding to l and m respectively. The dimension of L is r and the dimension of M is $n-r$. If $n=2k$ and $r=2k$ we denote f by J and

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see that $J^2=-I$. Also if $n=2k+1$ and $r=2k$, we denote f by ϕ and in this case there is a vector field ξ and a 1-form η such that $\phi^2=-I+\eta\otimes\xi$, $\eta(\xi)=1$ and $\phi(\xi)=\eta\circ\phi=0$. J is called an *almost complex structure* and (ϕ, ξ, η) is called an *almost contact structure*. Let $[f,f]$ denote the Nijenhuis tensor of f, that is

$$[f,f](X, Y)=[fX,fY]-f[fX, Y]-f[X,fY]+f^2[X, Y]$$

for all vector fields X and Y on M^n. If $[f,f]=0$, the f-structure is said to be *integrable*. It can be shown that there exists a positive definite metric g on M^n such that $g(X, Y)=g(fX,fY)+g(X,mY)$ for all vector fields X and Y on M^n. Such a pair of f and g is called an *(f, g)-structure* on M^n.

Suppose now that there exist global vector fields ξ_x on M^n, where $x=1, 2, \cdots, n-r$, spanning the distribution M. In this case we say that M^n has an f-structure with *complemented frames* (Cf. [2]). Let E^{n-r} denote $(n-r)$-dimensional Euclidean space. Then the tensor field $\tilde{J}$, defined by

$$\tilde{J}=\begin{pmatrix} f & -\xi_x \\ \eta^y & 0 \end{pmatrix}, \tag{1. 2}$$

is an almost complex structure on the product manifold $M^n\times E^{n-r}$ (Cf. [2]). Here the η^y denote $(n-r)$ 1-forms defined on M^n such that $\eta^y(\xi_x)=\delta^y_x$ and $\eta^y(X)=0$ for any X lying in L. $\tilde{J}$ is integrable if and only if $[f,f]+\xi_x\otimes d\eta^x=0$, where $d\eta^x$ is the exterior derivative of η^x.

$ 2. Main Theorem.

For $p\in M^n$, let $T(M^n)_p$ denote the tangent space to M^n at p. Also, let $fT(M^n)_p=\{fX_p\,|\,X_p\in T(M^n)_p\}$. The following theorem is in [5].

THEOREM A. *If J is an almost complex structure on M^n and N^m is a C^∞, m-dimensional submanifold of M^n such that the dimension of $T(N^m)_p\cap JT(N^m)_p$ is constant, say s, for $p\in N^m$, then there is a natural f-structure on N^m of rank s.*

Let B denote the differential of the imbedding of N^m in M^n. Then B is a map of TN^m into T_RM^n, where T_RM^n denotes the restriction of TM^n, the tangent bundle of M^n, to N^m. Then locally we can find $n-m$ linearly independent vector fields C_a such that $C_a\in T_RM^n$ and $C_a\notin TN^m$, and a mapping B^{-1} of T_RM^n into TN^m, and $n-m$ 1-forms C^b defined on N^m such that

$$\begin{aligned} &B^{-1}B=I, \qquad BB^{-1}=I-C_a\otimes C^a, \\ &C^aB=B^{-1}C_a=0, \qquad C^b(C_a)=\delta^b_a. \end{aligned} \tag{2. 1}$$

The meaning of the word 'natural' in the statement of Theorem A is that the f-structure on N^m is given locally by $B^{-1}JB$. We can now state our main theorem as follows:

THEOREM 2. 2. *If M^n has an f-structure f of rank r with complemented frames and N^m is a C^∞, m-dimensional submanifold of M^n such that the dimension of $T(N^m)_p \cap f(T(N^m)_p \cap L_p)$ is constant, say s, for $p \in N^m$, then there is a natural f-structure on N^m of rank s. Here L_p is the subspace of $T(M^n)_p$ in the distribution L.*

Proof. N^m can be identified with the submanifold $\tilde{N}^m \equiv N^m \times \{0\}$ of $M^n \times E^{n-r}$. If we can show that the dimension of $T(\tilde{N}^m)_{\tilde{p}} \cap \tilde{J}\, T(\tilde{N}^m)_{\tilde{p}}$ is s for each $\tilde{p} \in \tilde{N}^m$, where $\tilde{J}$ is given by (1. 2), then Theorem A shows that there is an f-structure on N^m of rank s. This f-structure is natural with respect to the imbedding of $\tilde{N}^m$ in $M^n \times E^{n-r}$. The second part of the proof will show that in fact it is natural with respect to the imbedding of N^m in M^n.

Let $Y \in T(N^m)_p \cap f(T(N^m)_p \cap L_p)$. Then $Y \in T(N^m)_p$ and $Y = fX = f((m+l)X) = flX$ for some X in $T(N^m)_p \cap L_p$. Therefore, since $\eta^x(lX) = 0$ for $x = 1, \cdots, n-r$, we see that

$$\begin{pmatrix} Y \\ 0 \end{pmatrix} = \begin{pmatrix} flX \\ \eta^x(lX) \end{pmatrix} = \tilde{J} \begin{pmatrix} lX \\ 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} Y \\ 0 \end{pmatrix} \in T(\tilde{N}^m)_{\tilde{p}} \cap \tilde{J}\, T(\tilde{N}^m)_{\tilde{p}}.$$

On the other hand, if $\tilde{Y} \in T(\tilde{N}^m)_{\tilde{p}} \cap \tilde{J}\, T(\tilde{N}^m)_{\tilde{p}}$ then

$$\tilde{Y} = \begin{pmatrix} Y \\ 0 \end{pmatrix} = \begin{pmatrix} fX \\ \eta^x(X) \end{pmatrix}$$

for some X and Y in $T(N^m)_p \cap L_p$. Hence, we see that the dimension of $T(\tilde{N}^m)_{\tilde{p}} \cap \tilde{J}\, T(\tilde{N}^m)_{\tilde{p}}$ is s for all $\tilde{p} \in \tilde{N}^m$.

Now let $\tilde{B}$ be the differential of the imbedding of $\tilde{N}^m$ in $M^n \times E^{n-r}$ and let $\tilde{B}^{-1}$ be found as was B^{-1} (here B is the differential of the imbedding of N^m in M^n). Now we see that if

$$\begin{pmatrix} Y \\ 0 \end{pmatrix} \in T(\tilde{N}^m)_{\tilde{p}}$$

then

$$\tilde{B}^{-1}\tilde{J}\tilde{B}\begin{pmatrix} Y \\ 0 \end{pmatrix} = \tilde{B}^{-1}\tilde{J}\begin{pmatrix} BY \\ 0 \end{pmatrix} = \tilde{B}^{-1}\begin{pmatrix} fBY \\ \eta^x(BY) \end{pmatrix} = \begin{pmatrix} B^{-1}fBY \\ 0 \end{pmatrix}$$

so that $B^{-1}fB$ is the natural f-structure on N^m. Therefore the proof is finished.

It is clear that an f-structure of maximal rank has complemented frames. Hence, for almost contact structures we have the following corollary, which is the analogue of Theorem A.

COROLLARY 2. 3. *If (ϕ, ξ, η) is an almost contact structure on M^n and N^m is a C^∞, m-dimensional submanifold of M^n such that the dimension of $T(N^m)_p \cap \phi T(N^m)_p$ is constant, say s, for $p \in N^m$, then there is a natural f-structuɪe on N^m of rank s.*

§ 3. Hypersurfaces.

In this section we suppose that $m=n-1$, that is, N^m is a hypersurface of M^n. Let C be a transversal defined on N^m, that is $C \in T(M^n)_p$ but $C \notin T(N^m)_p$ for all $p \in N^m$. Suppose p is a fixed point of N^m and that C at p (denoted by C_p) is in the distribution M at p. Then we see that $fT(N^m)_p$ is the intersection of $T(N^m)_p$ and the distribution L at p. Thus, the dimension of $T(N^m)_p \cap f(T(N^m)_p \cap L_p)$ is r, the rank of f. On the other hand, if C_p is in the distribution L at p, then there is a vector X_p in $T(N^m)_p$ such that $fX_p = C_p$ and so the dimension of $T(N^m)_p \cap f(T(N^m)_p \cap L_p)$ is $r-2$. Note that we have made use of the fact that f annihalates all vectors in the distribution M. From these observations, the following propositions are evident.

PROPOSITION 3. 1. *Let M^n be a manifold with an f-structure of rank r. A hypersurface N^{n-1} of M^n is such that the dimension of $T(N^{n-1})_p \cap f(T(N^{n-1})_p \cap L_p)$ is constant for all $p \in N^{n-1}$ if a normal of N^{n-1} can be found that is everywhere or nowhere in the distribution L, that is $C_p \in L$ for all $p \in N^{n-1}$ or $C^p \in M$ for all $p \in N^{n-1}$. This dimension is r if C is in L and it is $r-2$ if C is in M.*

PROPOSITION 3. 2. *If (ϕ, ξ, η) is an almost contact structure on M^n and N^{n-1} is a hypersurface of M^n, then N^{n-1} possesses a natural almost complex structure if ξ is nowhere tangent to N^{n-1} or N^{n-1} possesses a natural f-structure of rank $n-3$ if ξ is everywhere tangent to N^{n-1}* (see [1]).

We will close this section by showing that the hypothesis that the f-structure on M^n have complimented frames in Theorem 2. 2 is not necessary if $m=n-1$. Therefore, let B, B^{-1}, C and C^* satisfy (locally) the following equations (the special case of equations (2. 1)

$$B^{-1}B=I, \qquad BB^{-1}=I-C^* \otimes C,$$

$$C^*B=B^{-1}C=0 \quad \text{and} \quad C^*(C)=1.$$

Let F be defined locally on N^{n-1} by $F=B^{-1}fB$. Then

$$F^2X=B^{-1}fBB^{-1}fBX=B^{-1}f(I-C^* \otimes C)f(BX)$$

$$=B^{-1}f^2(BX)-C^*(fBX)B^{-1}fC.$$

If C is in the distribution M, then $fC=0$ so we see that

$$(F^3+F)X=B^{-1}fBB^{-1}f^2BX+B^{-1}fBX$$
$$=B^{-1}f(I-C^*\otimes C)f^2BX+B^{-1}fBX=B^{-1}((f^3+f)BX)=0$$

for all X. On the other hand suppose that C is in the distribution L. Then

$$(F^3+F)X=-C^*(f^2BX)B^{-1}fC-C^*(fBX)B^{-1}fBB^{-1}fC$$
$$=-C^*(f^2BX)B^{-1}fC-C^*(fBX)B^{-1}f^2C+C^*(fBX)C^*(fC)B^{-1}fC.$$

Now $f^2C=-C$ and we can assume that $C^*(fC)=0$. Also $C^*(f^2BX)=C^*(-BX+(f^2+1)BX)$. So, since we can assume that C^* annihalates all vectors in the distribution M, we have that $(F^3+F)X=0$ for all vector fields on N^{m-1}.

§ 4. Integrability.

In this section we assume that M^n and f are as in Theorem 2. 2 and N^m has the naturally induced f-structure $B^{-1}fB$. As before, let F denote $B^{-1}fB$. Then we see that

$$[F,F](X,Y)=[FX,FY]-F[FX,Y]-F[X,FY]+F^2[X,Y]$$
$$=B^{-1}[BB^{-1}fBX,BB^{-1}fBY]-B^{-1}f[BB^{-1}fBX,BY]$$
$$-B^{-1}f[BX,BB^{-1}fBY]+B^{-1}fBB^{-1}f[BX,BY]$$
$$=B^{-1}([f,f](BX,BY)-[C^a(fBX)C_a,fBY]-[fBX,C^a(fBY)C_a]$$
$$+[C^a(fBX)C_a,C^b(fBY)C_b]-f[C^a(fBX)C_a,BY]$$
$$-f[BX,C^a(fBY)C_a]-C^a(f[BX,BY])fC_a),$$

where we have used the fact that $B[X,Y]=[BX,BY]$ for vector fields X and Y on N^m. Also from 2. 1, we use the fact that locally $BB^{-1}=I-C_a\otimes C^a$. If the transversal C_a lies in the distribution M then the corresponding 1-form C^a can be chosen so that $C^af=0$. Hence we have the following theorem.

THEOREM 4. 1. *Let M^n and N^m be as in Theorem 2. 2 and suppose f is integrable. If, locally, transversals to N^m can be found that lie in the distribution M then the induced f-structure on N^m is integrable.*

Proposition 3. 2 and Theorem 4. 1 then give the following corollary.

COROLLARY 4. 2. *If (ϕ,ξ,η) is an integrable almost contact structure on M^n and N^{m-1} is a hypersurface of M^n such that ξ is a transversal of N^{n-1}, then the induced almost complex structure on N^{n-1} is complex* (see Theorem 3. 3 in [1]).

Let g be a metric on M^n such that f and g give an (f, g)-structure on M^n. Then we assume that the transversals C_a are orthogonal to N^m. Define a metric G on N^m by

$$G(X, Y)=g(BX, BY).$$

If $\bar{m}$ is the projection operator on N^m corresponding to m, then

$$\bar{m}=(F^2+I)X=B^{-1}fBB^{-1}fBX+B^{-1}BX=B^{-1}mBX+C^a(fBX)BfC_a.$$

Also,

$$\begin{aligned}G(FX, FY)&=G(B^{-1}fBX, B^{-1}fBY)=g(BB^{-1}fBX, BB^{-1}fBY)\\&=g(fBX, fBY)-g(C^a(fBX)C_a, fBY)-g(fBX, C^a(fBY)C_a)\\&\quad+g(C^a(fBX)C_a, C^b(fBY)C_b).\end{aligned}$$

Now assume that the C_a's are all in the distribution M so that $C^af=0$ for all a. Then

$$\begin{aligned}G(FX, FY)=g(fBX, fBY)&=g(BX, BY)-g(BX, mBY)\\&=g(BX, BY)-g(BX, (BB^{-1}+C_a\otimes C^a)mBY)\\&=G(X, Y)-G(X, \bar{m}Y).\end{aligned}$$

Therefore F and G form an (f, g)-structure on N^m. We state this as

Theorem 4.3. *If M^n has an (f, g)-structure with complemented frames and N^m is as in Theorem* 2.2 *and the normals* (*with respect to g*) *are in the distribution M, then N^m possesses a natural (f, g)-structure.*

Suppose now that the C_a's can be chosen to be global vector fields defined along N^m orthogonal (with respect to g) to N^m, and hence B^{-1} and the C^a's are globally defined. Then, since the f-structure f on M^n has complemented frames, we have that $m=\eta^x\otimes\xi_x$. Then $\bar{m}X=B^{-1}mBX=B^{-1}\eta^x\otimes\xi_xBX=\eta^x(BX)B^{-1}\xi_x$. If $\xi_\alpha, \alpha=1, \cdots, m-s$ are tangent to N^m while the rest of the ξ_x's are transversal to N^m then we have $\bar{m}=\eta^\alpha B\otimes B^{-1}\xi_\alpha$ and hence the f-structure on N^m has complemented frames. These frames could perhaps be used to investigate the integrability of the almost complex structure on $N^m\times E^{m-s}$.

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DEPARTMENT OF MATHEMATICS,
MICHIGAN STATE UNIVERSITY.