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ON INFINITESIMAL DEFORMATIONS OF CLOSED HYPERSURFACES

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§ 1. Introduction.

In the present paper we study the effect of infinitesimal deformations of (1) a closed orientable hypersurface in an orientable Riemannian manifold and (2) a closed hypersurface in a Euclidean space on some integrals.

Let M be an $(n+1)$-dimensional orientable Riemannian manifold and M' be a closed orientable hypersurface in M whose equations are given by

$$x^h = x^h(u^a)$$

in local coordinates. We use indices h, i, j, k for M and a, b, c, d for M', hence h, i, j, k run over the range $\{1, \cdots, n+1\}$ and a, b, c, d over the range $\{\dot{1}, \cdots, \dot{n}\}$. As usual $B_a{}^h$ means $\partial_a x^h$ where $\partial_a = \partial/\partial u^a$. $g_{ba} = B_b{}^i B_a{}^h g_{ih} = B_{ba}^{ih} g_{ih}$ are the components of the first fundamental tensor of M'. The unit normal vector is denoted by N^h and the reciprocal of the matrix $(B_a{}^h, N^h)$ by $(B^a{}_h, N_h)$. ∇ means the Van der Waerden-Bortolotti differential operator, hence $\nabla_b B_a{}^h = h_{ba} N^h$, $\nabla_b N^h = -h_b{}^a B_a{}^h$ where $h_b{}^a = h_{bc} g^{ca}$. h_{ba} are the components of the second fundamental tensor of M'.

§ 2. Infinitesimal deformations.

Let $\mathcal{M}'$ be a set of hypersurfaces $M'(t)$, $0 \leqq t < \varepsilon$, where ε is a sufficiently small positive number and $M'(0) = M'$. We assume that the local coordinates of the points of $M'(t)$ are given by

$$x^h = x^h(u^a, t)$$

in M. We also assume that $x^h(u^a, t)$ are C^∞ functions and the mapping $\varphi(t)$: $M'(0) \to M'(t)$ induced by

$$x^h(u^a, 0) \to x^h(u^a, t) \tag{2.1}$$

is diffeomorphic, u^a being local coordinates of $M'(t)$ in $U \cap M'(t)$ for some neighborhood U of M and for all $t \in [0, \varepsilon)$. $\varphi(t)$ is a deformation of M'.

We define $\xi^h(u^a)$ by

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$$\xi^h(u^a)=(\partial_t x^h(u^a, t))_0 \tag{2.2}$$

where $\partial_t=\partial/\partial t$ and $(\ \ )_0$ means $(\ \ )_{t=0}$. $\xi^h(u^a)$ is the vector field of an infinitesimal deformation.

The infinitesimal deformation Dg_{ba} of the metric of M' associated with the vector field ξ^h will be denoted by D_{ba}. If $g_{ba}(u, t)$ is the metric tensor of $M'(t)$, D_{ba} is given by

$$D_{ba}=(\partial_t g_{ba}(u, t))_0 \tag{2.3}$$

for we have (2. 1). Since $g_{ba}(u, t)$ is given by

$$g_{ba}(u, t)=\partial_b x^i(u, t)\partial_a x^h(u, t)g_{ih}(x(u, t)),$$

we obtain

$$D_{ba}=\partial_b\xi^i B_a{}^h g_{ih}+B_b{}^i\partial_a\xi^h g_{ih}+B_{ba}^{ih}\xi^j\partial_j g_{ih},$$

hence

$$D_{ba}=\nabla_b\xi^i B_a{}^h g_{ih}+\nabla_a\xi^i B_b{}^h g_{ih}. \tag{2.4}$$

If $N^h(u, t)$ is the unit normal vector field of $M'(t)$, we have

$$\partial_a x^i(u, t)g_{ih}(x(u, t))N^h(u, t)=0,$$

hence

$$\partial_a\xi^i g_{ih}N^h+B_a{}^i\xi^j\partial_j g_{ih}N^h+B_a{}^i g_{ih}(\partial_t N^h)_0=0$$

on M'. From this we obtain

$$N_i\nabla_a\xi^i+B_a{}^i g_{ih}P^h=0 \tag{2.5}$$

where P^h is defined by

$$P^h=(\partial_t N^h)_0+\left\{ {h \atop kj} \right\}\xi^k N^j$$

on M' and will be called the infinitesimal deformation of the unit normal vector. From (2. 5) and $N_iP^i=0$ we obtain

$$P^h=-B_a{}^h N^i\nabla^a\xi_i. \tag{2.6}$$

§ 3. Deformation of some integrals on a hypersurface in a Riemannian manifold.

Let us first consider the total volume of M',

$$V=\int_{M'}dV$$

where $dV=(\det(g_{ba}))^{1/2}du^1\cdots du^n$. It is known that the infinitesimal deformation of this integral, namely,

$$D\int_{M'}dV\overset{\text{def}}{=}\left[\frac{d}{dt}\int_{M'(t)}dV\right]_0$$

is given by

$$\frac{1}{2}\int_{M'}g^{ba}D_{ba}dV. \tag{3. 1}$$

Substituting (2. 4) into (3. 1) and using Green's theorem, we get

$$\begin{aligned}\int_{M'}B_a{}^i\nabla^a\xi_i dV&=-\int_{M'}\xi_i\nabla^a B_a{}^i dV\\&=-\int_{M'}h_a{}^a\xi_i N^i dV,\end{aligned}$$

hence

$$D\int_{M'}dV=-\int_{M'}h_a{}^a N_i\xi^i dV. \tag{3. 2}$$

Thus we obtain the following proposition.

PROPOSITION 3. 1. *Let M' be a closed orientable hypersurface in an orientable Riemannian manifold. A necessary and sufficient condition that the total volume of M' be critical for every infinitesimal deformation such that*

$$\int_{M'}N_i\xi^i dV=0 \tag{3. 3}$$

is that the mean curvature be constant on M'.

Let us calculate the deformation Dh_{ba} of the second fundamental tensor. Let $h_{ba}(u, t)$ be the second fundamental tensor of $M'(t)$. Since we have

$$h_{ba}(u, t)=-B_b{}^i g_{ih}\nabla_a N^h$$

in $M'(t)$, we get

$$\partial_t h_{ba}(u, t)=-(\nabla_b\partial_t x^i)g_{ih}\nabla_a N^h-B_b{}^i g_{ih}\left(\partial_t\nabla_a N^h+\left\{{h \atop kj}\right\}\partial_t x^k\nabla_a N^j\right). \tag{3. 4}$$

By straightforward calculation we get

$$\partial_t \nabla_a N^h = \partial_t \left(\partial_a N^h + \left\{ {h \atop j\,i} \right\} B_a{}^j N^i \right)$$

$$= \partial_a \partial_t N^h + \partial_t x^k \partial_k \left\{ {h \atop j\,i} \right\} B_a{}^j N^i + \left\{ {h \atop j\,i} \right\} \partial_a \partial_t x^j N^i + \left\{ {h \atop j\,i} \right\} B_a{}^j \partial_t N^i$$

$$= \nabla_a \left(\partial_t N^h + \left\{ {h \atop j\,i} \right\} \partial_t x^j N^i \right) + K_{kji}{}^h \partial_t x^k B_a{}^j N^i - \left\{ {h \atop j\,i} \right\} \partial_t x^j \left(\partial_a N^i + \left\{ {i \atop l\,k} \right\} B_a{}^l N^k \right)$$

where $K_{kji}{}^h$ is the curvature tensor of M. Thus we have

$$(3.5) \qquad (\partial_t \nabla_a N^h)_0 = \nabla_a P^h + K_{kji}{}^h \xi^k B_a{}^j N^i - \left\{ {h \atop j\,i} \right\} \xi^j \nabla_a N^i$$

on M'. From (3.4) and (3.5) we get

$$(3.6) \qquad Dh_{ba} = -\nabla_b \xi^i g_{ih} \nabla_a N^h - B_b{}^h (\nabla_a P_h + K_{kjih} \xi^k B_a{}^j N^i).$$

From (3.6) we get

$$D(h_e{}^e) = D(h_{ba} g^{ba}) = (Dh_{ba}) g^{ba} - h^{ba} D_{ba}$$

$$= -\nabla_a \xi^i \nabla^a N_i - B^a{}_i \nabla_a P^i + K_{ji} \xi^j N^i - h_{ba} D^{ba},$$

hence

$$(3.7) \qquad Dh = \frac{1}{n} [-\nabla^a N_i \nabla_a \xi^i + K_{ji} N^j \xi^i + B^a{}_j \nabla_a (B_b{}^j N_i \nabla^b \xi^i) - 2h_b{}^a B^b{}_i \nabla_a \xi^i]$$

where h is the mean curvature.

Let us calculate deformation of the integral of the mean curvature over the hypersurface M', namely,

$$DH = D \int_{M'} h dV$$

$$= \int_{M'} \left(Dh + \frac{1}{2} h g^{ba} D_{ba} \right) dV.$$

Substituting (2.4) and (3.7) into the last member and using Green's theorem, we get

$$DH = \frac{1}{n} \int_{M'} [-\nabla^a N_i \nabla_a \xi^i + K_{ji} N^j \xi^i + B^a{}_j \nabla_a (B_b{}^j N_i \nabla^b \xi^i)$$

$$-2h_b{}^a B^b{}_i \nabla_a \xi^i + h_b{}^b B^a{}_i \nabla_a \xi^i] dV$$

$$= \frac{1}{n} \int_{M'} [(\nabla_a \nabla^a N_i) \xi^i + K_{ji} N^j \xi^i - (\nabla_a B^a{}_j) B_b{}^j N_i \nabla^b \xi^i$$

$$+ 2\nabla_a (h_b{}^a B^b{}_i) \xi^i - \nabla_a (h_b{}^b B^a{}_i) \xi^i] dV.$$

As we have

$$\nabla_a \nabla^a N_i = -(\nabla_a h_b{}^a) B^b{}_i - h_b{}^a h_a{}^b N_i,$$

we get after some calculation

$$DH = \frac{1}{n} \int_{M'} [\nabla_b h_a{}^b - \nabla_a h_b{}^b) B^a{}_i \xi^i - \{(h_a{}^a)^2 - h_b{}^a h_a{}^b\} N_i \xi^i + K_{ji} N^j \xi^i] dV.$$

Substituting the equations

$$\nabla_b h_a{}^b - \nabla_a h_b{}^b = -K_{ji} N^j B_a{}^i,$$

which are derived from the equations of Codazzi, into the last member, we get

$$DH = \frac{1}{n} \int_{M'} [K_{kj} N^k N^j - (h_a{}^a)^2 + h_b{}^a h_a{}^b] N_i \xi^i dV. \tag{3.8}$$

If we use the scalar curvature $'K$ of M', we can write (3. 8) in the form

$$DH = \frac{1}{n} \int_{M'} (K - 'K - K_{kj} N^k N^j) N_i \xi^i dV, \tag{3.9}$$

for the equations of Gauss state that

$$'K = K - 2K_{kh} N^k N^h + (h_a{}^a)^2 - h_b{}^a h_a{}^b.$$

Thus we obtain the following proposition.

PROPOSITION 3. 2. *A necessary and sufficient condition that the integral of the mean curvature over an orientable closed hypersurface M' in an orientable Riemannian manifold M be critical for any infinitesimal deformation such that*

$$\int_{M'} N_i \xi^i dV = 0$$

is that $K - 'K - K_{kj} N^k N^j$ be constant on M'. A necessary and sufficient condition that the integral of the mean curvature over M' be critical for any infinitesimal deformation is that the following equation be satisfied on M',

$$K - 'K - K_{kj} N^k N^j = 0.$$

If M is an Einstein space, K and $K_{kj} N^k N^j$ are constant. Hence we obtain the

COROLLARY. *A necessary and sufficient condition that the integral of the mean curvature over an orientable closed hypersurface M' in an orientable Einstein space M be critical for any infinitesimal deformation such that*

$$\int_{M'} N_i \xi^i dV = 0$$

is that the scalar curvature $'K$ of M' be constant.

§ 4. Deformation of a closed hypersurface in Euclidean space.

Let us consider the case where M is a Euclidean space E^{n+1}.

Let $H(\lambda)$ and $H_b{}^a(\lambda)$ be defined by

(4. 1) $$H(\lambda)=\det(h_b{}^a-\lambda\delta_b^a),$$

(4. 2) $$H_b{}^a(\lambda)(h_c{}^b-\lambda\delta_c^b)=\delta_c^a H(\lambda)$$

and put

(4. 3) $$H(\lambda)=H_n+\lambda H_{n-1}+\cdots+\lambda^{n-1}H_1+(-\lambda)^n.$$

From (4. 2) we get

$$(\nabla_a H_b{}^a(\lambda))(h_c{}^b-\lambda\delta_c^b)+H_b{}^a(\lambda)\nabla_a h_c{}^b=\nabla_c H(\lambda).$$

As we have $\nabla_a h_c{}^b=\nabla_c h_a{}^b$ by virtue of the Codazzi equations and as we have

$$\nabla_c H(\lambda)=H_b{}^a(\lambda)\nabla_c h_a{}^b,$$

we get

$$(\nabla_a H_b{}^a(\lambda))(h_c{}^b-\lambda\delta_c^b)=0,$$

hence

(4. 4) $$\nabla_a H_b{}^a(\lambda)=0.$$

Differentiating $H_b{}^a(\lambda)(h_a{}^c-\lambda\delta_a^c)=\delta_b^c H(\lambda)$ covariantly, we also obtain

$$(\nabla^b H_b{}^a(\lambda))(h_a{}^c-\lambda\delta_a^c)+H_b{}^a(\lambda)\nabla^b h_a{}^c=\nabla^c H(\lambda),$$

hence

(4. 5) $$\nabla^b H_b{}^a(\lambda)=0$$

by virtue of the Codazzi equations and $\nabla^c H(\lambda)=H_b{}^a(\lambda)\nabla^c h_a{}^b$.

Now let us calculate deformation of the integral of $H(\lambda)$ over M'.

As we have (2. 4) and

$$DH(\lambda)=H_a{}^b(\lambda)Dh_b{}^a$$
$$=H_a{}^b(\lambda)(g^{ca}Dh_{bc}-h_b{}^d g^{ca}D_{dc}),$$

we get

$$D\int H(\lambda)dV$$

$$=\int H(\lambda)\frac{1}{2}g^{dc}D_{dc}dV$$

$$+\int H_a{}^b(\lambda)(g^{ca}Dh_{bc}-h_b{}^d g^{ca}D_{dc})dV$$

$$=\int H(\lambda)B^c{}_i\nabla_c\xi^i dV$$

$$+\int H_a{}^b(\lambda)(-\nabla^a\xi^i\nabla_b N_i-B^a{}_i\nabla_b P^i-h_b{}^d\nabla_d\xi^i B^a{}_i-h_{bd}\nabla^a\xi^i B^d{}_i)dV$$

by virtue of (3. 6) and $K_{kjih}=0$. By Green's theorem the last member becomes

$$=\int[-(\nabla_c H(\lambda))B^c{}_i\xi^i-h_c{}^c H(\lambda)N_i\xi^i]dV$$

$$+\int[H_a{}^b(\lambda)(\nabla^a\nabla_b N_i)\xi^i+H_a{}^b(\lambda)h_b{}^a N_i P^i+(\nabla_d H_a{}^b(\lambda))h_b{}^d B^a{}_i\xi^i$$

$$+H_a{}^b(\lambda)(\nabla_d h_b{}^d)B^a{}_i\xi^i+H_a{}^b(\lambda)h_b{}^d h_d{}^a N_i\xi^i+H_a{}^b(\lambda)(\nabla^a h_{bd})B^d{}_i\xi^i$$

$$+H_a{}^b(\lambda)h_{bd}h^{ad}N_i\xi^i]dV$$

where we have used (4. 4) and (4. 5).

As we have

$$\nabla^a\nabla_b N_i=-(\nabla^a h_{bc})B^c{}_i-h_{bc}h^{ac}N_i,$$

$$N_i P^i=0,$$

we get

$$D\int H(\lambda)dV$$

$$=\int[-\nabla_c H(\lambda)-H_a{}^b(\lambda)\nabla^a h_{bc}+(\nabla_d H_c{}^b(\lambda))h_b{}^d$$

$$+H_c{}^b(\lambda)\nabla_d h_b{}^d+H_a{}^b(\lambda)\nabla^a h_{bc}]B^c{}_i\xi^i dV$$

$$+\int[-h_c{}^c H(\lambda)-H_a{}^b(\lambda)h_{bc}h^{ac}+H_a{}^b(\lambda)h_b{}^d h_d{}^a+H_a{}^b(\lambda)h_{bd}h^{ad}]N_i\xi^i.$$

We easily get

$$\begin{aligned}
&(\nabla_d H_c{}^b(\lambda))h_b{}^d\\
&=\nabla_d(H_c{}^b(\lambda)h_b{}^d)-H_c{}^b(\lambda)\nabla_d h_b{}^d\\
&=\nabla_d\{H_c{}^b(\lambda)(h_b{}^d-\lambda\delta_b^d)+\lambda H_c{}^d(\lambda)\}-H_c{}^b(\lambda)\nabla_d h_b{}^d\\
&=\nabla_d(\delta_c^d H(\lambda))-H_c{}^b(\lambda)\nabla_d h_b{}^d\\
&=\nabla_c H(\lambda)-H_c{}^b(\lambda)\nabla_d h_b{}^d,
\end{aligned}$$

and the first integral containing $B^c{}_i\xi^i$ vanishes.

Thus we obtain

$$D\int H(\lambda)dV=\int[-h_c{}^cH(\lambda)+H_a{}^b(\lambda)h_{bc}h^{ac}]N_i\xi^i dV. \tag{4. 6}$$

On the other hand we have

$$\begin{aligned}
&-h_c{}^cH(\lambda)+H_a{}^b(\lambda)h_b{}^ch_c{}^a\\
&=-h_c{}^cH(\lambda)+H_a{}^b(\lambda)[(h_b{}^c-\lambda\delta_b^c)(h_c{}^a-\lambda\delta_c^a)+2\lambda(h_b{}^a-\lambda\delta_b^a)+\lambda^2\delta_b^a]\\
&=-h_c{}^cH(\lambda)+(h_c{}^c-n\lambda)H(\lambda)+2n\lambda H(\lambda)+\lambda^2H_a{}^a(\lambda)\\
&=n\lambda H(\lambda)+\lambda^2H_a{}^a(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{d\lambda}H(\lambda)&=H_b{}^a(\lambda)\frac{d}{d\lambda}(h_a{}^b-\lambda\delta_a^b)\\
&=-H_a{}^a(\lambda),
\end{aligned}$$

hence

$$\begin{aligned}
&-h_c{}^cH(\lambda)+H_a{}^b(\lambda)h_{bc}h^{ac}\\
&=n\lambda H(\lambda)-\lambda^2\frac{d}{d\lambda}H(\lambda)\\
&=\sum_{m=0}^{n}(n-m)H_{n-m}\lambda^{m+1}
\end{aligned}$$

where we have used (4. 3).

Substituting this identity into (4. 6) we get

$$\begin{aligned}
&\sum_{m=0}^{n}D\int H_{n-m}dV\lambda^m\\
&=\sum_{m=0}^{n}(n-m)\int H_{n-m}N_i\xi^i dV\lambda^{m+1},
\end{aligned}$$

hence

$$D\int H_n dV=0, \tag{4.7}$$

$$D\int H_{n-m}dV=(n-m+1)\int H_{n-m+1}N_i\xi^i dV \qquad (m=1, \cdots, n). \tag{4.8}$$

Thus we obtain the following theorem.

THEOREM 4.1. *Let M' be a closed hypersurface in a Euclidean space E^{n+1} and let $H(\lambda)$ and H_m be defined by* (4.1) *and* (4.3). *If ξ^h is a vector field of an infinitesimal deformation of the hypersurface M' and D is the symbol of the deformation induced by ξ^h, then $H_0, H_1, \cdots, H_n$ satisfy* (4.7) *and* (4.8).

(4.7) states that the integral of $\det(h_b{}^a)$ over the hypersurface M' is a topological invariant. This is a well-known fact, the integral being equal to the volume integral of S^n multiplied by the degree of mapping of the Gauss map φ: $M'\to S^n$ induced by the unit normal vector field N^h.

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