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ON THE SOLUTION OF THE FUNCTIONAL EQUATION $f\circ g(z)=F(z)$, III

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In our previous papers [3], [4] we discussed transcendental entire solutions of the functional equation $f\circ g(z)=F(z)$ and gave several transcendental unsolvability criteria, which based upon the existence of a Picard exceptional value, perfectly branched values, finite asymptotic paths and so on. All the criteria proved there do not work when F is an entire function of order less than 1/2 and even when $F(z)$ is $1/\Gamma(z)$. In this note we shall give a very useful criterion, which is based upon an elegant theorem due to Edrei [2] and which does work to some entire functions of order less than 1/2 and to $1/\Gamma(z)$ and the n-th Bessel function $J_n(z)$. And we shall give certain variants of this result. Further we shall give several criteria based upon Denjoy-Carleman-Ahlfors theorem.

Let $f(z)$ be an entire function and $M_f(r)$ its maximum modulus on $|z|=r$. We shall use the following notations:

$$\rho_f=\varlimsup_{r\to\infty}\frac{\log\log M_f(r)}{\log r},\qquad \lambda_f=\varliminf_{r\to\infty}\frac{\log\log M_f(r)}{\log r}$$

and

$$\hat{\rho}_f=\varlimsup_{r\to\infty}\frac{\log\log\log M_f(r)}{\log r},\qquad \hat{\lambda}_f=\varliminf_{r\to\infty}\frac{\log\log\log M_f(r)}{\log r}.$$

LEMMA 1. [4]. *$\rho_f<\infty$ implies $\hat{\rho}_{f\circ g}\leqq\rho_g$.*

LEMMA 2. *$\lambda_f>0$ implies $\hat{\rho}_{f\circ g}\geqq\rho_g$ and $\hat{\lambda}_{f\circ g}\geqq\lambda_g$.*

Proof. By Pólya's method we have

$$M_{f\circ g}(r)\geqq M_f\circ\left(d\,M_g\left(\frac{r}{2}\right)\right)$$

for a constant d, $0<d<1$. For a sufficiently small positive number ε there is an r_0 such that for $r\geqq r_0$

$$\log\log M_f(r)>(\lambda_f-\varepsilon)\log r$$

and there is a sequence $\{r_n\}$ of radii such that

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$$\log \log M_g\left(\frac{r_n}{2}\right) \geqq (\rho_g-\varepsilon) \log \frac{r_n}{2}$$

for $n \geqq n_0$. Hence

$$\log \log M_{f\circ g}(r_n) \geqq (\lambda_f-\varepsilon)\left[\log d+\left(\frac{r_n}{2}\right)^{\rho_g-\varepsilon}\right].$$

Thus

$$\begin{aligned}\hat{\rho}_{f\circ g} &\geqq \overline{\lim_{n\to\infty}} \frac{\log\log\log M_{f\circ g}(r_n)}{\log r_n} \\ &\geqq \overline{\lim_{n\to\infty}} \frac{\log(\lambda_f-\varepsilon)+\log[\log d+(r_n/2)^{\rho_g-\varepsilon}]}{\log r_n} \\ &= \rho_g-\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have the desired result: If $\lambda_f>0$, $\hat{\rho}_{f\circ g} \geqq \rho_g$.

Again by Pólya's method and by

$$\log M_f(r) > r^{\lambda_f-\varepsilon} \quad \text{and} \quad \log M_g(r) > r^{\lambda_g-\varepsilon}$$

for $r \geqq r_0$ and for a given sufficiently small positive number ε,

$$\log \log M_{f\circ g}(r) > (\lambda_f-\varepsilon)\left[\log d+\left(\frac{r}{2}\right)^{\lambda_g-\varepsilon}\right],$$

which implies the second desired result.

LEMMA 3. *Let $f(z)$ be* $\exp(L(z))$ *with an entire function $L(z)$, then $\lambda_f \geqq 1$.*

Proof. By Pólya's method

$$M_f(r) \geqq \exp\left(d\, M_L\left(\frac{r}{2}\right)\right) \geqq \exp(dcr)$$

with two constants $d(0<d<1)$ and $c>0$. Therefore $\lambda_f \geqq 1$.

LEMMA 4. [5]. *Let $F(z)$ be an entire function of finite order. Assume that the functional equation $f\circ g(z)=F(z)$ holds for two transcendental entire functions f and g. Then $\rho_f=0$ and $\rho_g \leqq \rho_F$.*

LEMMA 5. [2]. *Let $f(z)$ be an entire function. Assume that there exists an unbounded sequence $\{a_n\}_{n=1}^{\infty}$ such that all the roots of the equations*

$$f(z)=a_n \qquad (n=1, 2, \cdots)$$

are real. Then $f(z)$ is a polynomial of degree at most two.

THEOREM 1. *Let $F(z)$ be an entire function of finite order for which $F(z)=A$ for some A has only real roots. Then the functional equation $f\circ g(z)=F(z)$ does not admit any pair of transcendental entire solutions f and g.*

Proof. By Lemma 4 f must be a transcendental entire function of order zero. Therefore the equation $f(w)=A$ has an infinite number of roots $\{w_n\}$. Consider the equations $g(z)=w_n$, $n=1, 2, \cdots$. Then all the roots must be real, since they are the roots of $F(z)=A$. Hence $g(z)$ satisfies the assumptions of Lemma 5, whence follows that $g(z)$ is a polynomial. This contradicts the transcendency of $g(z)$.

Applications. Theorem 1 can apply to the following functions:

$z^p \sin^q z$ (p, q: integers $q\geqq 1$, $p\geqq -q$); $1/\Gamma(z)$;

the n-th Bessel function $J_n(z)$; $P_\rho(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{n^\rho}\right)(\rho>1)$;

$\prod_{n=1}^{\infty}\left(1-\frac{z}{e^n}\right)$; $P(z)(e^z-\gamma)(e^z-\gamma e^{\imath a})$ (P: a polynomial, a: real).

THEOREM 2. *Let $F(z)$ be an entire function of finite order for which $F'(z)=0$ has only real zeros. Then the functional equation $f\circ g(z)=F(z)$ does not admit any pair of transcendental entire solutions f and g.*

Proof. Consider the derived functional equation $f'\circ g(z)\cdot g'(z)=F'(z)$. Since f is of order zero and transcendental, $f'(w)=0$ has an infinite number of roots $\{w_n\}$ and $g(z)=w_n$ has only real roots for each n. Hence by Lemma 5 $g(z)$ must be a polynomial, which contradicts the transcendency of $g(z)$.

Applications. Theorem 2 can apply to the primitive functions of the functions already listed.

When $F(z)$ is of infinite order, we need some modifications in the above theorems.

THEOREM 3. *Let $F(z)$ be an entire function of infinite order, all of whose A-points for some A lie on the real axis. Assume further that the order of $N(r; A, F)$ is greater than $\hat{\rho}_F$. Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g.*

Proof. When $f(w)=A$ has an infinite number of roots, the same method as in theorem 1 works and then we have a contradiction. If $f(w)=A$ has a finite number of roots, then

$$f(w)=A+P(w)e^{L(w)},\qquad f\circ g(z)=A+P\circ g(z)e^{L\circ g(z)}$$

with a polynomial P and an entire function L. By Lemma 3 $\lambda_f\geqq 1$. Hence $\rho_g\leqq\hat{\rho}_{f\circ g}$ by Lemma 2. On the other hand by its form

$$\rho_{N(r;A,F)}=\rho_{N(r;0,P\circ g)}\leqq\rho_g.$$

This implies an absurdity relation $\hat{\rho}_F<\rho_{N(r;A,F)}\leqq\hat{\rho}_F$.

THEOREM 4. *Let $F(z)$ be an entire function of infinite order. Assume that*

$F'(z)$ has only real zeros. Further assume that the order of $N(r; 0, F')$ is greater than $\hat{\rho}_{F'}$. Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g.

Proof. If $f'(w)=0$ has an infinite number of roots, then the same procedure as in theorem 2 is applicable and we have a contradiction. If $f'(w)=0$ has a finite number of roots, then

$$f'(w)=P(w)e^{L(w)}, \qquad f'\circ g(z)\cdot g'(z)=P\circ g(z)e^{L\circ g(z)}g'(z)$$

with a polynomial P and an entire function L. Evidently $\lambda_{f'}\geqq 1$ and hence $\rho_g\leqq\hat{\rho}_{f'\circ g}$ by Lemma 2. Thus $\rho_g=\rho_{g'}\leqq\hat{\rho}_{f'\circ g\cdot g'}$. On the other hand

$$\rho_{N(r;0,F')}=\rho_{N(r;0,P\circ g\cdot g')}\leqq\rho_g\leqq\hat{\rho}_{F'},$$

which is a contradiction.

We shall give another result based upon a different principle.

THEOREM 5. *Let $F(z)$ be an entire function of finite hyper-order $\hat{\rho}_F$. Assume further that the order of $N(r; A, F)$ is less than $\hat{\rho}_F$ for some A and $F(z)=A$ has either at least two roots for the same A or one root which is not a Fatou exceptional value of F. Then there is no entire solution f of the functional equation $f\circ f(z)=F(z)$.*

Proof. Evidently f must be transcendental. If $f(w)=A$ has no root, $f\circ f(z)=A$ has no root, which is a contradiction. If $f(w)=A$ has only one zero w_1, then $f(w)$ has the form

$$A+(w-w_1)^n e^{L(w)},$$

where n is an integer >0 and L is an entire function. We, then, have

$$F(z)=f\circ f(z)=A+(A-w_1+(z-w_1)^n e^{L(z)})^n e^{L\circ(A+(z-w_1)^n e^{L(z)})}.$$

Assume that $A=w_1$. Then

$$F(z)=A+(z-A)^{n^2}e^{nL(z)+L\circ(A+(z-A)^n e^{L(z)})},$$

which shows that A is a Fatou exceptional value of F. This contradicts our assumption. Assume that $A\neq w_1$. Then

$$\rho_{N(r;A,F)}=\rho_{e^L}=\rho_f.$$

By Lemma 3 $\lambda_f\geqq 1$. Hence $\rho_f\leqq\hat{\rho}_F<\infty$ and then $\hat{\rho}_F\leqq\rho_f$ by Lemma 1. Thus we have

$$\hat{\rho}_F=\rho_f=\rho_{N(r;A,F)},$$

which is a contradiction.

If $f(w)=A$ has at least two roots w_1 and w_2, then

$$N(r; A, F) \geqq N(r; w_1, f)+N(r; w_2, f)$$
$$\geqq m(r, f)-O(\log rm(r, f))$$

by the second fundamental theorem for f. We, then, have

$$\rho_{N(r; A, F)} \geqq \rho_f.$$

Hence $\rho_f < \hat{\rho}_F < \infty$, whence follows $\hat{\rho}_F \leqq \rho_f$ by Lemma 1. This is again a contradiction.

A corresponding result for $f\circ g(z)=F(z)$ may be stated in the following form

THEOREM 6. *Let $F(z)$ be a transcendental entire function of finite hyperorder $\hat{\rho}_F$. Assume that the order of $N(r; A, F)$ is less that $\hat{\rho}_F$ for some A and $F(z)=A$ has at least one root for the same A. Then the functional equation $f\circ g(z)=F(z)$ has no pair of transcendental entire solutions f and g which satisfy the following conditions*: (a) *f is of finite order and* (b) *g is of finite order and has no Borel exceptional value.*

Proof. It should be remarked that $\hat{\rho}_F \leqq \rho_g$ when ρ_f is finite. Firstly $f(w)=A$ has at least one root. If $f(w)=A$ has only one root w_1, $f(w)$ has the form:

$$f(w)=A+(w-w_1)^n e^{L(w)}$$

with a positive integer n and a polynomial $L(w)$. Thus

$$F(z) \equiv f\circ g(z)=A+(g(z)-w_1)^n e^{L\circ g(z)}.$$

By lemma 3, $\lambda_f \geqq 1$ and hence $\hat{\rho}_F \geqq \rho_g$. Hence g is of finite order. Then

$$\hat{\rho}_F > \rho_{N(r; A, F)} = \rho_{N(r; w_1, g)},$$

which is equal to ρ_g, since g does not have any Borel exceptional value. Thus we have $\hat{\rho}_F > \rho_g$, which is a contradiction.

If $f(w)=A$ has at least two roots w_1, w_2, we have $\rho_{N(r; A, F)} \geqq \rho_g$ by the second fundamental theorem. This leads us to an absurdity relation $\rho_g < \hat{\rho}_F \leqq \rho_g$.

Baker [1] proved the following two results:

i) Let $f(z)$ be an entire function with $\hat{\rho}_{f\circ f} \leqq A$, $0 \leqq A < \infty$. Then $\lambda_f=0$ unless $\rho_f \leqq A$.

ii) Let $f(z)$ be an entire function with $\hat{\rho}_{f\circ f} \leqq A$, $0 \leqq A < \infty$. Then $f\circ f(z)$ has at most $2[2A]$ different finite asymptotic values.

i) is an immediate corollary of Lemma 2. Baker's proof for i) is not straightforward. We shall extend ii) to $f\circ g(z)$.

THEOREM 7. *Let $n_{f\circ g}$ be the number of finite different asymptotic values of*

$f\circ g(z)$. *Then*

$$n_{f\circ g}\leqq[2\lambda_f]+[2\lambda_g].$$

Proof. It should be firstly remarked that the cluster set of a transcendental entire function along a path, which extends to infinity, is a continuum, unless it is a point which may be a finite value or ∞. Let Γ_j be an asymptotic path of $f\circ g(z)$ along which $f\circ g(z)$ tends to A_j. By the remark mentioned above we only have two possibilities: (a) $g(z)$ has a finite asymptotic value a_j along Γ_j or (b) $g(z)$ tends to ∞ along Γ_j and $f(w)$ tends to A_j along $g(\Gamma_j)$. Since $A_1, \cdots, A_p$ $(p=n_{f\circ g})$ are different with each other, all the possible finite $\{a_j\}$ are different and all the possible paths $g(\Gamma_j)$ are non-contiguous. By the Denjoy-Carleman-Ahlfors theorem

$$n_{f\circ g}\leqq[2\lambda_g]+[2\lambda_f].$$

Now Baker's result ii) is easy to prove.

If $n_{f\circ g}$ is replaced by the number of finite non-contiguous asymptotic paths in theorem 7, the result does not hold in general. Baker remarked this fact already in the case $f\circ f$. However, if $\lambda_f<1/2$, we can replace the $n_{f\circ g}$ by that of the wider sense. This fact have been proved in [4] already and is very useful. In this connection we can prove the following two results, which are slight extensions of our results in [4].

THEOREM 8. *Let F be a transcendental entire function of finite order which has p non-contiguous finite asymptotic paths. Further assume that the lower order of $N(r; A, F)$ for an A is less than $p/2$. Then there is no pair of transcendental entire functions satisfying the functional equation $f\circ g(z)=F(z)$.*

THEOREM 9. *Let F be the same as in theorem 8. Further assume that the lower order of $N(r; 0, F')$ is less than $p/2$. Then the same conclusion holds as in theorem 8.*

We do not give any proofs of these theorems.

THEOREM 10. *Let F be an entire function of infinite order such that $\lambda_{N(r; A, F)}>0$ and $2[2\lambda_{N(r; A, F)}]<n_F$, where n_F is the number of different finite asymptotic values of F. Then the functional equation $f\circ f(z)=F(z)$ has no solution f.*

Proof. Evidently f must be transcendental. By Baker's result or by theorem 7 we have

$$n_F\leqq 2[2\lambda_f].$$

If $f(w)=A$ does not have any root, then $\lambda_{N(r; A, F)}=0$, which is a contradiction. If $f(w)=A$ has only one solution, then

$$f(w)=A+(w-w_1)^n e^{L(w)}$$

and

$$F(z)\equiv f\circ f(z)=A+(A-w_1+(z-w_1)^n e^{L(z)})^n e^{L\circ f(z)}.$$

Assume $A=w_1$. Then the lower order of $N(r; A, F)=0$, which is a contradiction. Hence $A\neq w_1$. In this case

$$N(r; A, F)=nN(r; A-w_1, (z-w_1)^n e^{L(z)}), \qquad \lambda_{N(r; A, F)}<\infty.$$

Hence $L(z)$ is a polynomial and then

$$\lambda_{N(r; A, F)}=\rho_{e^L}=\lambda_{e^L}=\lambda_f.$$

This implies that

$$n_F\leqq 2[2\lambda_f]=2[2\lambda_{N(r; A, F)}]<n_F,$$

which is a contradiction. If $f(w)=A$ has at least two roots, we have

$$N(r; A, F)\geqq m(r, f)(1-\varepsilon), \qquad \lim_{r\to\infty}\varepsilon=0$$

by the second fundamental theorem, and hence

$$\lambda_{N(r; A, F)}\geqq\lambda_f.$$

Therefore

$$2[2\lambda_f]<2[2\lambda_{N(r; A, F)}]<n_F\leqq 2[2\lambda_f],$$

which is a contradiction.

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