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CONJUGATE CLASSES OF ORISPHERICAL SUBALGEBRAS IN REAL SEMISIMPLE LIE ALGEBRAS

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Introduction.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and let $\mathcal{G}$ be a Lie group whose Lie algebra is $\mathfrak{g}$. Let $g(t)=\exp(tX)$ be a one-parameter subgroup in $\mathcal{G}$ generated by X in $\mathfrak{g}$. We define an orispherical subgroup $\mathcal{Z}$ relative to X (or $g(t)$) as follows.

DEFINITION.

$$\mathcal{Z}=\left\{z\in\mathcal{G};\ \lim_{t\to\infty} g(t)zg(-t)=e\right\}.$$

$\mathcal{Z}$ is a connected closed subgroup of $\mathcal{G}$ [5], and its Lie algebras is equal to

$$\mathfrak{z}=\left\{Z\in\mathfrak{g};\ \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}Z=0\right\}.$$

This Lie algebra is called an orispherical subalgebra relative to X. I. M. Gel'fand and M. I. Graev [2] showed that these subgroups play an important role in the theory of group representations. Moreover, in the theory of unitary representations in homogeneous spaces with discrete stationary subgroups, Gel'fand and Pyatetskii-Shapiro [4] gave an effective process for isolating the discrete spectrum from the continuous spectrum. This process was the method of orispheres, the orbit of orispherical subgroups. In connection with this, the same authors [3] remarked and used the fact that, in the case of $\mathrm{SL}(n, R)$, there exist as many conjugate orispherical subgroups as there are representations of n in the form of a sum of positive summands $n=k_1+k_2+\cdots+k_s$.

In this note we wish to show that there is a one to one correspondence between conjugate classes of orispherical subalgebras and the set of faces of a Weyl chamber of $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}$. (for the definitions, see below). Our method and results are somewhat similar to the case of Cartan subalgebras, but are more simple. We shall use frequently notations and results appeared in [6].

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§ 1. Definition of orispherical subalgebras.

DEFINITION. A subalgebra $\mathfrak{z}$ of a real semisimple Lie algebra $\mathfrak{g}$ is called *an orispherical subalgebra relative to an element X of* $\mathfrak{g}$, if $\mathfrak{z}$ is the set of all $Z \in \mathfrak{g}$ for which

$$\lim_{t\to\infty} \exp(t \operatorname{ad} X)Z = 0.$$

As is well known, X can be decomposed into a unique sum $X = H + N$, where $H, N \in \mathfrak{g}$ satisfy the following conditions: (i) $[H, N] = 0$; (ii) $\operatorname{ad} H$ is semisimple and all of its eigenvalues are real; (iii) $\operatorname{ad} N$ has only pure imaginary eigenvalues. In this situation, we can assert

PROPOSITION 1. 1. *The orispherical subalgebra $\mathfrak{z}$ relative to X is equal to the orispherical subalgebra relative to H. $\mathfrak{z}$ is the direct sum of all eigenspaces in $\mathfrak{g}$ of $\operatorname{ad} H$ corresponding negative eigenvalues.*

Proof. Let $\mathfrak{g}^C$ be the complexification of $\mathfrak{g}$. $\operatorname{ad} X$, $\operatorname{ad} H$, $\operatorname{ad} N$ can be considered as complex linear transformations on $\mathfrak{g}^C$. Let $a_1, \cdots, a_r$ be different eigenvalues of $\operatorname{ad} H$, and let $\mathfrak{g}_1, \cdots, \mathfrak{g}_r$ be the corresponding eigenspaces of $\operatorname{ad} H$ in $\mathfrak{g}^C$. $\mathfrak{g}_i$ is invariant under $\operatorname{ad} X$. We put $\mathfrak{g}_i(\lambda) = \{Y \in \mathfrak{g}_i, (\operatorname{ad} X - \lambda)^p Y = 0$ for some $p = 1, 2, \cdots\}$. There are only a finite number of λ's, say $\lambda_1^{(i)}, \cdots, \lambda_{n_i}^{(i)}$, for which $\mathfrak{g}_i(\lambda) \neq \{0\}$. Then the subspaces $\mathfrak{g}_i(\lambda_j^{(i)})$, $i = 1, \cdots, r$; $j = 1, \cdots, n_i$, are linearly independent, and $\mathfrak{g}^C = \sum \mathfrak{g}_i(\lambda_j^{(i)})$. In $\mathfrak{g}_i(\lambda_j^{(i)})$, we can choose a basis such that $\operatorname{ad} X$ has the form

$$\begin{pmatrix} \lambda_j^{(i)} & & * \\ & \lambda_j^{(i)} & \\ 0 & & \ddots \quad \lambda_j^{(i)} \end{pmatrix} = \lambda_j^{(i)} E + T.$$

From the conditions on eigenvalues of $\operatorname{ad} H$, $\operatorname{ad} N$, we have $\lambda_j^{(i)} = a_j + \sqrt{-1}\, b_{ij}$ where a_i, b_{ij} are real. For any $Y \in \mathfrak{g}_i(\lambda_j^{(i)})$, $\exp(t \operatorname{ad} X) Y = e^{t\lambda_j^{(i)}} \exp(tT) Y$. Since the matrix components of $\exp(tT)$ are polynomials in t, $\exp(t \operatorname{ad} X) Y$ converges to 0 $(t \to \infty)$, if and only if $a_i < 0$, i.e., if and only if $\exp(t \operatorname{ad} H) Y$ converges to 0. For any $Z \in \mathfrak{g}$ we have $Z = \sum Z_{ij}$, $Z_{ij} \in \mathfrak{g}_i(\lambda_j^{(i)})$. $\exp(t \operatorname{ad} X) Z = \sum \exp(t \operatorname{ad} X) Z_{ij}$ $(\exp(t \operatorname{ad} H) Z = \sum \exp(t \operatorname{ad} H) Z_{ij})$ converges to 0 $(t \to \infty)$ if and only if all $\exp(t \operatorname{ad} X) Z_{ij}$ $(\exp(t \operatorname{ad} H) Z_{ij})$ converges to 0. This proves the first part.

If $Z \in \mathfrak{g}$, then $Z = \sum Z_i$ where $Z_i \in \mathfrak{g} \cap \mathfrak{g}_i$. $Z \in \mathfrak{z}$ if and only if $\exp(t \operatorname{ad} H) Z_i = e^{ta_i} Z_i \to 0$ $(t \to \infty)$. Hence $Z_i \neq 0$ if and only if $a_i < 0$. This proves the second part.

Now, H belongs to some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, since $\operatorname{ad} H$ is semisimple. Let $\mathfrak{h}^+, \mathfrak{h}^-$ be subalgebras of $\mathfrak{h}$ defined by

$$\mathfrak{h}^+ = \{Y \in \mathfrak{h};\ \text{all eigenvalues of } \operatorname{ad} Y \text{ are pure imaginary}\},$$

$$\mathfrak{h}^- = \{Y \in \mathfrak{h};\ \text{all eigenvalues of } \operatorname{ad} Y \text{ are real}\},$$

then $H \in \mathfrak{h}^-$.

As is known (see for example [6]), for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a Cartan decomposition of $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ such that $\mathfrak{h}^{-}\subset\mathfrak{p}$, $\mathfrak{h}^{+}\subset\mathfrak{k}$.

Now we take a new Cartan subalgebras as follows. Let $\mathfrak{m}$ be a maximal abelian subalgebra of $\mathfrak{p}$, containing H. Maximal abelian subalgebra $\mathfrak{h}_0$ in $\mathfrak{g}$ containing $\mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{g}$. We have $\mathfrak{h}_0^{-}=\mathfrak{h}_0\cap\mathfrak{p}=\mathfrak{m}\ni H$.

Let $\mathfrak{h}_0^C$ be the complexification of $\mathfrak{h}_0$ in the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. We denote by Δ the root system of non zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{h}_0^C$. We have the root decomposition of $\mathfrak{g}^C$ as follows:

$$\mathfrak{g}^C=\mathfrak{h}_0^C+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha.$$

For any $\alpha\in\Delta$, $\alpha(H)$ is real. Eigenvalues of $\operatorname{ad}H$ are the numbers $\alpha(H)$ as α runs through Δ with the right multiplicity. Hence the eigenspace $\mathfrak{g}_i$ of $\operatorname{ad}H$, belonging to the eigenvalue a_i, is $\mathfrak{g}_i=\sum_{\alpha(H)=a_i}\mathfrak{g}_\alpha$. Being a_i real, $\mathfrak{g}_i\cap\mathfrak{g}$ is the eigenspace in $\mathfrak{g}$ of $\operatorname{ad}H$.

Hence, the orispherical subalgebra $\mathfrak{z}$ relative to H is given by

$$\mathfrak{z}=\Big(\sum_{\alpha(H)<0}\mathfrak{g}_\alpha\Big)\cap\mathfrak{g}=\sum_{a_i<0}(\mathfrak{g}_i\cap\mathfrak{g}).$$

Thus we have

PROPOSITION 1. 2. *Any orispherical subalgebra is given as follows. We decompose* $\mathfrak{g}$ *into a Cartan decomposition* $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. *Choose any element* $H\in\mathfrak{p}$. $\operatorname{ad}H$ *is semisimple.* (*In a suitable basis of* $\mathfrak{p}$, $\operatorname{ad}H$ *is symmetric.*) *Let* $\mathfrak{m}$ *be a maximal abelian subspace of* $\mathfrak{p}$ *containing* H. *A maximal abelian subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *containing* $\mathfrak{m}$ *is a Cartan subglgebra of* $\mathfrak{g}$. *Denote by* $\mathfrak{g}^C, \mathfrak{h}^C$ *the complexifications of* $\mathfrak{g}, \mathfrak{h}$ *respectively, and let the root decomposition of* $\mathfrak{g}^C$ *with respect to* $\mathfrak{h}^C$ *be*

$$\mathfrak{g}^c=\mathfrak{h}^c+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha.$$

Then

$$\mathfrak{z}=\Big(\sum_{\alpha(H)<0}\mathfrak{g}_\alpha\Big)\cap\mathfrak{g}$$

is an orispherical subalgebra relative to H.

§ 2. Canonical form of orispherical subalgebas.

In the followings, we consider the adjoint group G of $\mathfrak{g}$ frequently.

PROPOSITION 2. 1. *Let* H *and* H' *be two elements of* $\mathfrak{g}$, *and let* $\operatorname{ad}H$, (*or* $\operatorname{ad}H'$) *be semisimple whose eigenvalues are all real. Assume that there exists a* $g\in G$ *such that* $H'=gH$. *Denote by* $\mathfrak{z}, \mathfrak{z}'$ *the orispherical subalgebras relative to* H, H' *resp. Then we have* $\mathfrak{z}'=g\mathfrak{z}$.

Proof is obvious.

Let $\mathfrak{g}=\mathfrak{k}_0+\mathfrak{p}_0$ be a Certan decompositition. In what follows, we fix this Cartan decomposition. Consider another Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and let $H\in\mathfrak{p}$. It is well known that there exists a $g\in G$ such that

$$g\mathfrak{k}=\mathfrak{k}_0, \qquad g\mathfrak{p}=\mathfrak{p}_0.$$

Then $gH=H'\in\mathfrak{p}_0$. By Props. 1. 2 and 2. 1, we have

PROPOSITION 2. 2. *Any orispherical subalgebra is conjugate under the adjoint group G to an orispherical algebra relative to an element in $\mathfrak{p}_0$, where $\mathfrak{g}=\mathfrak{k}_0+\mathfrak{p}_0$ is a fixed Cartan decomposition.*

Now we consider two maximal abelian subspaces $\mathfrak{m}$, $\mathfrak{m}'$ in $\mathfrak{p}_0$. It is known [6], then, that $\mathfrak{m}$ and $\mathfrak{m}'$ are conjugate under the group G. More precisely, let K_0 be the analytic subgroup of G generated by the subalgebra $\mathfrak{k}_0$. Then there exists a $k\in K_0$ such that $k\mathfrak{m}=\mathfrak{m}'$ (of course $k\mathfrak{k}_0=\mathfrak{k}_0$, $k\mathfrak{p}_0=\mathfrak{p}_0$). From these facts, we can deduce the following

PROPOSITION 2. 3. *Let $\mathfrak{g}=\mathfrak{k}_0+\mathfrak{p}_0$ be a fixed Cartan decomposition and let $\mathfrak{m}_0$ be a fixed maximal abelian subspace of $\mathfrak{p}_0$. Then any orispherical subalgebra is conjugate under the gorup G to an orispherical subalgebra relative to an element in $\mathfrak{m}_0$.*

Now we fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}$, which contains $\mathfrak{m}_0$. Δ be the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}_0^c$. Root decomposition

$$\mathfrak{g}^c=\mathfrak{h}_0^c+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$$

is also fixed. Denote by Δ_0 the subset of all $\alpha\in\Delta$ whose restrictions to $\mathfrak{m}_0$ are not identically 0. Let $H\in\mathfrak{m}_0$. Then $\alpha(H)<0$, $\alpha\in\Delta$, implies $\alpha\in\Delta_0$. By prop. 1. 2, the orispherical subalgebra relative to H is

$$\mathfrak{z}=\Big(\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}\mathfrak{g}_\alpha\Big)\cap\mathfrak{g}. \tag{1}$$

For later use, we give a necessary and sufficient condition for an element Z to be contained in $\mathfrak{z}$. For this purpose, we consider the conjugation σ of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$, and define for any $\alpha\in\Delta$, $\alpha^\sigma(H)=\overline{\alpha(\sigma H)}$, $H\in\mathfrak{h}_0^c$. It is obvious that $\alpha^\sigma\in\Delta_0$ if $\alpha\in\Delta_0$. Let X_α be any element in $\mathfrak{g}_\alpha$.

PROPOSITION 2. 4.

$$\mathfrak{z}=\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}\mathcal{R}(X_\alpha+\sigma X_\alpha)+\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}\mathcal{R}\sqrt{-1}(X_\alpha-\sigma X_\alpha).$$

Here sum is not necessarily direct.

Proof. Any element Z of

$$\mathfrak{z}=\left(\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}\mathfrak{g}_\alpha\right)\cap\mathfrak{g}$$

can be written

$$Z=\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}\tau_\alpha X_\alpha.$$

This element belongs to $\mathfrak{z}$ if and only if $Z=\sigma Z$. Hence

$$Z=\frac{1}{2}(Z+\sigma Z)=\frac{1}{2}\sum_{\substack{\alpha(H)<0\\ \alpha\in\Delta_0}}(\tau_\alpha X_\alpha+\bar{\tau}_\alpha\sigma X_\alpha)$$
$$=\sum r_\alpha(X_\alpha+\sigma X_\alpha)+\sum s_\alpha\sqrt{-1}(X_\alpha-\sigma X_\alpha)$$

where $\tau_\alpha=r_\alpha+\sqrt{-1}\,s_\alpha$.

Since $\alpha^\sigma(H)=\alpha(H)$ for $H\in\mathfrak{m}_0$, the converse assertion is also evident.

From this proposition, we get

PROPOSITION 2. 5. *Let H_1, H_2 be in $\mathfrak{m}_0$. The two orispherical subalgebras relative to H_1 and H_2 coincide when and only when $\alpha(H_1)<0$ is equivalent to $\alpha(H_2)<0$ for all $\alpha\in\Delta_0$.*

Proof. The " when " part is trivial. Let $\mathfrak{z}$ be orispherical subalgebra relative to $H\in\mathfrak{m}_0$. Another part of this proposition will follow if we show that $\beta(H)<0$ is equivalent to $T_\beta=X_\beta+\sigma X_\beta$ and $S_\beta=\sqrt{-1}(X_\beta-\sigma X_\beta)\in\mathfrak{z}$ for $\beta\in\Delta_0$. If $\beta(H)<0$, then T_β, $S_\beta\in\mathfrak{z}$ by prop. 2. 4. Conversely let $T_\beta, S_\beta\in\mathfrak{z}$. Both of these elements are eigenvectors of $\operatorname{ad}H$ belonging to the eigenvalue $\beta(H)$. Either of the two elements is not zero. By the definition of $\mathfrak{z}$, $\exp(t\operatorname{ad}H)T_\beta=e^{t\beta(H)}T_\beta$ $(\exp(t\operatorname{ad}H)S_\beta=e^{t\beta(H)}S_\beta)\to 0$. Hence $\beta(H)<0$.

§ 3. Canonical form of orispherical subalgebras (continued).

Let $\mathfrak{k}_0, \mathfrak{p}_0, \mathfrak{m}_0, \mathfrak{h}_0$ be the same as in § 2. It is classical that $\mathfrak{h}_0\cap\mathfrak{p}_0=\mathfrak{m}_0=\mathfrak{h}_0^-$, $\mathfrak{h}_0\cap\mathfrak{k}_0=\mathfrak{h}_0^+$. Let $\mathfrak{g}^c, \mathfrak{h}_0^c$ be the complexifications of $\mathfrak{g}$, $\mathfrak{h}_0$ resp. Δ be the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}_0^c$. We have a root decomposition of $\mathfrak{g}^c=\mathfrak{h}_0^c+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$. Killing form of $\mathfrak{g}^c$ will be denoted by $B(\ ,\ )$. $B(\ ,\ )$ is non-degenerate on $\mathfrak{h}_0^c\times\mathfrak{h}_0^c$. Hence we have elements $H_\alpha, \alpha\in\Delta$ such that $B(H_\alpha, H)=\alpha(H)$ hold for all $H\in\mathfrak{h}_0^c$. Then as is known, $\alpha(H_\beta)$ are real for all $\alpha, \beta\in\Delta$ and $\mathfrak{h}^*=\sum_{\alpha\in\Delta}\mathcal{R}H_\alpha\supset\mathfrak{m}_0$, where $\mathcal{R}$ denote the field of real numbers.

In the vector space $\mathfrak{h}^*$ over $\mathcal{R}$, we select a basis $X_1,\cdots,X_n$ such that $X_1,\cdots,X_m$ $(m<n)$ is a basis of $\mathfrak{m}_0$. By this basis, we introduce the lexicographic ordering of Δ as follows. For an $\alpha\in\Delta$, $\alpha>0$ means by definition that the first nonzero component of $(\alpha(X_1),\cdots,\alpha(X_n))$ is positive. For $\alpha,\beta\in\Delta$, $\alpha>\beta$ is equivalent to $\alpha-\beta>0$. By this ordering, Δ is a linearly ordered set.

Let Δ^+ be the set of all positive roots in this order. We denote by P the subset of all $\alpha \in \Delta^+$ such that α is not identically zero on $\mathfrak{m}_0$. Then we have an Iwasawa decomposition of $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}_0+\mathfrak{m}_0+\mathfrak{n}_0, \qquad \mathfrak{n}_0=\left(\sum_{\alpha \in P} \mathfrak{g}_\alpha\right) \cap \mathfrak{g}.$$

Let K_0 be the analytic subgroup of adjoint group G of $\mathfrak{g}$, generated by $\mathfrak{k}_0$. We set

$$N=\{k \in K_0;\ k\mathfrak{m}_0=\mathfrak{m}_0\}.$$

Then for any $k \in N$, restriction $\varphi(k)$ of k to $\mathfrak{m}_0$ is a linear transformation on $\mathfrak{m}_0$. We denote the totality of $\varphi(k)$ by W, i.e.

$$W=\{\varphi(k);\ k \in N\}.$$

W is the Weyl group of G/K_0.

Each root $\alpha \in P$ defines a hyperplane $\pi_\alpha=\{H \in \mathfrak{m}_0; \alpha(H)=0\}$ in the vector space $\mathfrak{m}_0$. These hyperplanes divide the space $\mathfrak{m}_0$ into finitely many connected components, called Weyl chambers. It is well known that each $\varphi(k) \in W$ permutes the Weyl chambers, and W acts simply transitively on the set of Weyl chambers of $\mathfrak{m}_0$.

The restriction of Killing form $B(\ ,\ )$ to $\mathfrak{m}_0 \times \mathfrak{m}_0$ is positive definite. With respect to this bilinear form, we consider the orthogonal reflection s_α of $\mathfrak{m}_0$ in the hyperplane π_α. Then, the Weyl group W is generated by reflections s_α, $\alpha \in P$. On these facts stated above, the reader may refer to, for example [1].

Now, consider the following subset:

(1) $$C_0=\{H \in \mathfrak{m}_0;\ \alpha(H)<0 \text{ for all } \alpha \in P\}.$$

We assert that C_0 is *not empty*, and hence C_0 is a Weyl chamber.

In fact, let us assume that P contains k roots $\alpha_1, \alpha_2, \cdots, \alpha_k$, whose order is $\alpha_1<\alpha_2<\cdots<\alpha_k$. Then we can find an element $X \in \mathfrak{m}_0$ for which all $\alpha_i(X)>0$. This can be seen by induction on k. If $k=1$, $\alpha_1(X_i)>0$ for some $i=1, \cdots, m$. ($X_1, \cdots, X_m$ is the basis of $\mathfrak{m}_0$, with respect to which Δ_0 is ordered.) Assume that we can find an $X=\sum_{i=1}^m x_i X_i$ for which $\alpha_1(X), \cdots, \alpha_{k-1}(X)>0$. We may assume $\alpha_k(X_1)>0$. By the definition of ordering, $\alpha_1(X_1), \cdots, \alpha_{k-1}(X_1) \geqq 0$. If we increase x_1, $\alpha_1(X), \cdots, \alpha_{k-1}(X)$ remain positive. For sufficiently large x_1, $\alpha_k(X)$ becomes positive. Thus $C_0 \ni -X$.

Now let $H \in \mathfrak{m}_0$. Then there is a Weyl chamber C of $\mathfrak{m}_0$ for which $H \in \bar{C}$, where $\bar{C}$ denotes the closure of C. On the other hand, for some element $\varphi(k)$ of W, we have $kC=C_0$, $k \in N \subset G$. Then $kH=H' \in \bar{C}_0$. Thus we have

PROPOSITION 3.1. *Any orispherical subalgebra is conjugate under the group G to a subalgebra which is relative to an element H in $\bar{C}_0$, where C_0 is the Weyl chamber given by* (1).

In what follows, we consider, for a Weyl chamber C, the set of all faces of various dimensions of C in the sense of algebraic topology.

For each element $H\in\bar{C}_0$ we can find a face $B(H)$ of C_0 which contains H. If $H\in C_0$, we put $B(H)=C_0$. We denote by $\mathscr{B}(C_0)$ the set of all $B(H)$, $H\in\bar{C}_0$.

We remark that if $H\in\bar{C}_0$, $\alpha(H)<0$ occurs only for $\alpha\in P$.

Thus orispherical subalgebra $\mathfrak{z}$ relative to H is

$$\mathfrak{z}=\Big(\sum_{\substack{\alpha(H)<0\\ \alpha\in P}}\mathfrak{g}_\alpha\Big)\cap\mathfrak{g} \tag{2}$$

by (1), § 2. It is obvious, that if $B(H_1)=B(H_2)$, $H_1, H_2\in\bar{C}_0$, then the two orispherical subalgebras relative to H_1 and H_2 are equal. Conversely, if $H_1, H_2\in\bar{C}_0$, and if the two orispherical subalgebras relative to H_1, H_2 are equal, then prop. 2. 5 and the above remark imply $B(H_1)=B(H_2)$. Hence we have

PROPOSITION 3. 2. *Let $H_1, H_2\in\bar{C}_0$. Then the two orispherical subalgebras relative to H_1 and H_2 are equal if and only if $B(H_1)=B(H_2)$.*

Now consider the orispherical subalgebra $\mathfrak{z}$ given by (2). In particular, if $H\in C_0$ then $\mathfrak{z}$ is equal to $\mathfrak{n}_0$:

$$\mathfrak{n}_0=\Big(\sum_{\alpha\in P}\mathfrak{g}_\alpha\Big)\cap\mathfrak{g}.$$

Of course $\mathfrak{n}_0$ is a maximal dimensional orispherical subalgebra.

PROPOSITION 3. 3. *$\mathfrak{z}$ is an ideal in $\mathfrak{n}_0$, i.e., $[\mathfrak{n}_0, \mathfrak{z}]\subseteq\mathfrak{z}$. Also, $[\mathfrak{m}_0, \mathfrak{z}]\subseteq\mathfrak{z}$ holds.*

Proof. By prop. 2. 4, $\mathfrak{z}$ and $\mathfrak{n}_0$ are given by

$$\mathfrak{z}=\sum_{\substack{\alpha(H)<0\\ \alpha\in P}}\mathscr{R}(X_\alpha+\sigma X_\alpha)+\sum_{\substack{\alpha(H)<0\\ \alpha\in P}}\mathscr{R}\sqrt{-1}(X_\alpha-\sigma X_\alpha),$$

$$\mathfrak{n}_0=\sum_{\alpha\in P}\mathscr{R}(X_\alpha+\sigma X_\alpha)+\sum_{\alpha\in P}\mathscr{R}\sqrt{-1}(X_\alpha-\sigma X_\alpha) \qquad (X_\alpha\in\mathfrak{g}_\alpha)$$

where sums are not necessarily to be direct. Let $\alpha, \beta\in P$. If $\alpha(H)<0$ then $(\alpha+\beta)(H)<0$ since $H\in\bar{C}_0$. Moreover if $\beta\in P$, $\beta^\sigma\in P$. These facts combining with

$$[\tau_\alpha X_\alpha+\bar{\tau}_\alpha\sigma X_\alpha, \eta_\beta X_\beta+\bar{\eta}_\beta\sigma X_\beta]$$
$$=\tau_\alpha\eta_\beta[X_\alpha, X_\beta]+\overline{\tau_\alpha\eta_\beta}\sigma[X_\alpha, X_\beta]+\tau_\alpha\bar{\eta}_\beta[X_\alpha, \sigma X_\beta]+\bar{\tau}_\alpha\eta_\beta\sigma[X_\alpha, \sigma X_\beta],$$

$$[X_\alpha, X_\beta]\in\mathfrak{g}_{\alpha+\beta}, \qquad [X_\alpha, \sigma X_\beta]\in\mathfrak{g}_{\alpha+\sigma\beta},$$

where $\mathfrak{g}_{\alpha+\beta}$ may be 0, imply the first part. Similarly for the second part.

§ 4. Conjugacy under the Weyl group.

We now prove the following

PROPOSITION 4. 1. *Let H_1 and H_2 be in $\bar{C}_0$, and let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be the orispherical subalgebras relative to H_1 and H_2 resp. Then $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are conjugate under the adjoint group G, if and only if there exists an element s of the Weyl group W of G/K_0, for which $sH_2 \in \bar{C}_0$ and $B(H_1)=B(sH_2)$.*

For the proof of this proposition, we now appeal to the following theorem stated in [6].

THEOREM. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and G the adjoint group of $\mathfrak{g}$. Suppose two subalgebras $\mathfrak{n}_1$, $\mathfrak{n}_2$ of $\mathfrak{g}$ generate toroidal groups N_1, N_2 respectively in G. If there exists a compact subgroup L in G, such that*

$$(1) \qquad [l\mathfrak{n}_1, \mathfrak{n}_2] \subset \mathfrak{l} \quad \textit{for all } l \in L$$

($\mathfrak{l}$ is the Lie algebra of L), then there exists an element l_0 in L satisfying following two conditions:

1) *Every element of $l_0 N_1 l_0^{-1}$ commutes with each element of N_2;*
2) $[l_0\mathfrak{n}_1, \mathfrak{n}_2]=0$.

Since in the followings, this theorem plays an essential role, we reproduce the proof of it.

Proof. First, we prove that for any $X \in \mathfrak{n}_1$, $Y \in \mathfrak{n}_2$, there exists an l_0 in L such that $[l_0X, Y]=0$.

As $f(l)=B(lX, Y)$ is a continuous function on L, f attains its maximal value at some point l_0 in L. We define the function $g(t)$ of real variable t as $g(t)=B(\exp(t \operatorname{ad} Z)l_0X, Y)$ for any Z in $\mathfrak{l}$. Then $g(t)$ attains its maximum at $t=0$. So we have

$$(2) \qquad 0=g'(0)=B([Z, l_0X], Y)=B(Z, [l_0X, Y]).$$

The relations (1), (2) and the fact that B is negative definite on $\mathfrak{l}$, prove the equality $[l_0X, Y]=0$. Now, there exists an X_i in $\mathfrak{n}_i$ $(i=1,2)$ such that one parameter subgroup $\{\exp \operatorname{ad} tX_i, -\infty<t<\infty\}$ is everywhere dense in N_i.

For such X_1 and X_2, we can apply the first part of this proof. There exists an element l_0 in L satisfying $[l_0X_1, X_2]=0$. Consequently, for any two real numbers s and t, $\exp(\operatorname{ad}(l_0tX_1))=l_0(\exp \operatorname{ad}(tX_1))l_0^{-1}$ and $\exp(\operatorname{ad}(sX_2))$ commute with each other. This proves 1), and 2) is a direct consequence of 1).

Proof of prop. 4. 1. Suppose that $\mathfrak{z}_1=g\mathfrak{z}_2$, where $g \in G$. According to Iwasawa decomposition $G=K_0A_0N_0$, where K_0, A_0, N_0 are analytic subgroups in G generated by subalgebras $\mathfrak{k}_0$, $\mathfrak{m}_0$, $\mathfrak{n}_0$ respectively, we decompose g into $g=kan$, $k \in K_0$, $a \in A_0$, $n \in N_0$.

By prop. 3. 3, we have $an\mathfrak{z}_2=\mathfrak{z}_2$. Hence $k\mathfrak{z}_2=\mathfrak{z}_1$.

Now, consider the compact real form $\mathfrak{g}_u=\mathfrak{k}_0+\sqrt{-1}\mathfrak{p}_0$, and let G_u be the adjoint group of $\mathfrak{g}_u$.

We can regard both G and G_u as subgroups of adjoint group G^c of $\mathfrak{g}^c$. Then we have $K_0=G\cap G_u$.

If we put

$$L=\{l\in K_0;\ l\mathfrak{z}_1=\mathfrak{z}_1\},$$

then L is a closed subgroup of K_0, and hence L is a compact subgroup of G_u. The Lie algebra of L is given by

$$\mathfrak{l}=\{X\in\mathfrak{k}_0\colon [X,\mathfrak{z}_1]=\mathfrak{z}_1\}.$$

On the other hand, if we put $k\sqrt{-1}\mathfrak{m}_0=\mathfrak{n}_1$, $\sqrt{-1}\mathfrak{m}_0=\mathfrak{n}_2$, then $\mathfrak{n}_i\subset\mathfrak{g}_u$ generate toroidal subgroups in G_u. This can be proved as follows.

Let σ be the conjugation of $\mathfrak{g}^c$ with respect to $\mathfrak{g}$. The restriction σ' of σ to $\mathfrak{g}_u$ is an involutive automorphism of $\mathfrak{g}_u$. Then $\sigma'X=-X$ for $X\in\mathfrak{g}_u$ means $X\in\sqrt{-1}\mathfrak{p}_0$. Let τ be the automorphism of G_u whose differential $d\tau$ is identical with σ'.

We denote by N_i the analytic subgroup generated by $\mathfrak{n}_i$ in G_u. Then the closure $\bar{N}_i$ of N_i is compact, connected and abelian. For each $h\in N_i$, $\tau(h)=h^{-1}$. It follows that $\tau(h)=h^{-1}$ for each $h\in\bar{N}_i$. Hence the Lie algebra $\mathfrak{n}_i'$ of $\bar{N}_i$ has the property that for each $H\in\mathfrak{n}_i'$ $\sigma'H=-H$. This implies $\mathfrak{n}_i'\subset\sqrt{-1}\mathfrak{p}_0$. Then $\sqrt{-1}\mathfrak{n}_i'\subset\mathfrak{p}_0$ and $k\mathfrak{m}_0\subset\sqrt{-1}\mathfrak{n}_1'$, $\mathfrak{m}_0\subset\sqrt{-1}\mathfrak{n}_2'$. By maximality of $\mathfrak{m}_0$, we have $k\mathfrak{m}_0=\sqrt{-1}\mathfrak{n}_1'$, $\mathfrak{m}_0=\sqrt{-1}\mathfrak{n}_2'$ i.e. $\mathfrak{n}_i'=\mathfrak{n}_i$. This proves that N_i is compact, connected and abelian.

Next we prove

$$[l\mathfrak{n}_1,\mathfrak{n}_2]\subset\mathfrak{l}\quad\text{for all } l\in L.$$

In fact, we have

$$\begin{aligned}[][[l\mathfrak{n}_1,\mathfrak{n}_2],\mathfrak{z}_1]&=[[lk\mathfrak{m}_0,\mathfrak{m}_0],\mathfrak{z}_1]\\&=[[lk\mathfrak{m}_0,\mathfrak{z}_1],\mathfrak{m}_0]+[lk\mathfrak{m}_0,[\mathfrak{m}_0,\mathfrak{z}_1]].\end{aligned}$$

The first term of the last expression is equal to

$$\begin{aligned}[][l[k\mathfrak{m}_0,l^{-1}\mathfrak{z}_1],\mathfrak{m}_0]&=[l[k\mathfrak{m}_0,\mathfrak{z}_1],\mathfrak{m}_0]\\&=[l[k\mathfrak{m}_0,k\mathfrak{z}_2],\mathfrak{m}_0]=[lk[\mathfrak{m}_0,\mathfrak{z}_2],\mathfrak{m}_0]\\&\subset[lk\mathfrak{z}_2,\mathfrak{m}_0]=[l\mathfrak{z}_1,\mathfrak{m}_0]=[\mathfrak{z}_1,\mathfrak{m}_0]\subset\mathfrak{z}_1.\end{aligned}$$

The second term is contained in

$$\begin{aligned}[][lk\mathfrak{m}_0,\mathfrak{z}_1]&=l[k\mathfrak{m}_0,l^{-1}\mathfrak{z}_1]=l[k\mathfrak{m}_0,\mathfrak{z}_1]\\&=l[k\mathfrak{m}_0,k\mathfrak{z}_2]=lk[\mathfrak{m}_0,\mathfrak{z}_2]\subset lk\mathfrak{z}_2=l\mathfrak{z}_1=\mathfrak{z}_1.\end{aligned}$$

We can now apply the above theorem [6], and we have an element $l_0 \in L$ such that

$$[l_0 \mathfrak{n}_1, \mathfrak{n}_2]=0,$$

i.e.

$$[l_0 k \mathfrak{m}_0, \mathfrak{m}_0]=0.$$

Since $l_0 k \mathfrak{m}_0 \subset \mathfrak{p}_0$, and $\mathfrak{m}_0$ being maximal abelian, we have

$$l_0 k \mathfrak{m}_0 = \mathfrak{m}_0.$$

If we put $s=\varphi(l_0 k)$ where φ is the restriction to $\mathfrak{m}_0$, then s is an element of Weyl group W, and we have

$$s\mathfrak{z}_2 = l_0 k \mathfrak{z}_2 = l_0 \mathfrak{z}_1 = \mathfrak{z}_1.$$

This last relation implies that two elements $sH_2(=l_0 k H_2)$ and H_1 are related to the same orispherical subalgebra. Then by prop. 2. 4, $\alpha(sH_2)<0$ is equivalent to $\alpha(H_1)<0$ for any $\alpha \in \Delta_0$. By the fact that $-\alpha \in \Delta_0$ if $\alpha \in \Delta_0$, we see also that $\alpha(sH_2)>0$ is equivalent to $\alpha(H_1)>0$ for any $\alpha \in \Delta_0$. It follows consequently that $\alpha(sH_2)=0$ is equivent to $\alpha(H_1)=0$ for any $\alpha \in \Delta_0$. These three relations imply $sH_2 \in \bar{C}_0$, and by prop. 3. 2 we have $B(sH_2)=B(H_1)$.

Converse assertion of prop. 4. 1 can be proved as follows. If $H_1, sH_2 \in \bar{C}_0$ and if $B(H_1)=B(sH_2)$, then by prop. 3. 2 $\mathfrak{z}_1 = k\mathfrak{z}_2$ where $\mathfrak{z}_i$ is the orispherical subalgebra relative to H_i, and $k \in K_0$ is such that $s=\varphi(k)$. Hence $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are conjugate under the group G.

Thus, all of our assertions are proved.

Now we need the following

LEMMA. *Let C_0 be the Weyl chamber of $\mathfrak{m}_0$ defined as above, and let $H \in \bar{C}_0$. If there exists an $s \in W$ for which $sH \in \bar{C}_0$, then $sH=H$.*

Proof. If $H \in C_0$, then $sH \in sC_0$. Thus $\bar{C}_0 \cap sC_0 \ni sH$. If we suppose $sC_0 \neq C_0$, then we have $\bar{C}_0 \cap sC_0 = \phi$ since sC_0 is also a Weyl chamber. Hence we have $sC_0=C_0$. This implies $s=e$ and $sH=H$.

If $H \in \bar{C}_0 - C_0$, then $sH \in s\bar{C}_0 - sC_0$. Of course, $sH \in \bar{C}_0 - C_0$. Thus, $B(sH)$, the face of C_0 containing sH, is a common face to C_0 and sC_0.

Now, it is obvious that we can find a sequence of Weyl chambers $C_0, C_1, \cdots, C_p = sC_0$ which satisfy the following conditions:

(1) C_{i-1} and C_i have a common face, say B_i of dimension $\dim \mathfrak{m}_0 - 1$;

(2) The face B_i lies on a hyperplane π_{α_i} in which $B(sH)$ is contained.

By (1) and (2), we have $s_{\alpha_i} C_i = C_{i+1}$ and hence $sC_0 = s_{\alpha_p} \cdots s_{\alpha_1} C_0$ where s_{α_i} means the reflection of $\mathfrak{m}_0$ in the hyperplane π_{α_i}. This implies $s=s_{\alpha_p} \cdots s_{\alpha_1}$. By (2), $s_{\alpha_i} sH = sH$ for all i. Then $H=s^{-1}sH = s_{\alpha_1} \cdots s_{\alpha_p} sH = sH$.

Combining prop. 4. 1 with the above lemma, we get the following

THEOREM 4.2. *Let $H_1, H_2 \in \bar{C}_0$ and let $\mathfrak{z}_1, \mathfrak{z}_2$ be orispherical subalgebras relative to H_1, H_2 respectively. Then $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are conjugate under the adjoint group of $\mathfrak{g}$, if and only if $B(H_1)=B(H_2)$.*

Thus we can conclude

THEOREM 4.3. *There is a one to one correspondence between conjugate classes of orispherical subalgebras and faces of a Weyl chamber C_0 (including C_0 itself).*

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