ON THE EQUIVALENCE OF LOCAL HOLOMORPHY AND LOCAL HOLOMORPHIC CONVEXITY IN TWO-DIMENSIONAL NORMAL COMPLEX SPACES

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A subdomain (or an open subset) X' of a complex space X is called a *subdomain* (an *open subset*) *of holomorphy of* X *of a holomorphic function* f *in* X', or shortly a *subdomain* (an *open subset*) *of holomorphy of* X, if f can not be analytically continued in any boundary point of X' relative to X. A subdomain X' of a complex space X is called a *subdomain of local holomorphy* (a *locally holomorphically convex subdomain*) if there exists a neighbourhood U for any boundary point of X' relative to X such that $U \cap X'$ is an open subset of holomorphy (a holomorphically convex open subset) of X. In the present paper we shall prove that a subdomain X' of a two-dimensional normal complex space X is a subdomain of local holomorphy of X if and only if it is a locally holomorphically convex subdomain of X.

§ 1. Application of Weyl's theorem.

Let K be a subset of a complex space Y. We shall denote by $\tilde{K}$ the set of all $x \in Y$ satisfying the following condition:

$$|f(x)| \leqq \sup_{y \in K} |f(y)|$$

for all holomorphic functions f in Y.

$\tilde{K}$ is called the *envelope of holomorphy of* K with respect to Y. If the envelope of holomorphy of any compact subset of a complex space Y is compact, Y is called *holomorphically convex.*

LEMMA 1. *Let* K *be a subset of a complex space* Y *with* $\tilde{K} \subsetneqq Y$ *and* $\Delta = \{x_1, x_2, \cdots, x_s\}$ *be a finite subset of* $Y - \tilde{K}$. *Then there exists a holomorphic function* f *in* Y *satisfying the following condition:*

$$\sup_{y \in K} |f(y)| < 1 < \inf_{y \in \Delta} |f(y)|.$$

Proof. For any $j = 1, 2, \cdots, s$, there exists a holomorphic function f_j in Y satisfying the following condition:

$$\sup_{y \in K} |f_j(y)| < 1/s < 2 < |f_j(x_j)|$$

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for any $j=1, 2, \cdots, s$. If we put

$$F(y)=\sum_{j=1}^{s}(f_j(y))^{\alpha}$$

for any positive integer α, F is a holomorphic function in Y with

$$\sup_{y\in K}|F(y)|<1.$$

Making use of Weyl [6] we shall prove that

$$|F(x_j)|>1 \qquad (j=1, 2, \cdots, s)$$

for a suitable choice of α. We put

$$f_j(x_k)=r_{jk}\exp(2\pi\sqrt{-1}\theta_{jk}) \quad (r_{jk}\geqq 0,\ \theta_{jk} \text{ is real})$$

for $j, k=1, 2, \cdots, s$. There exists a positive integer p such that each $p\theta_{jk}$ is either an integer or an irrational number. There exists a subset $\{\theta_1, \theta_2, \cdots, \theta_a\}$ of $\{\theta_{jk};\ j, k=1, 2, \cdots, s\}$ satisfying the following conditions:

(1) $p(\alpha_1\theta_1+\alpha_2\theta_2+\cdots+\alpha_a\theta_a)\equiv 0 \pmod 1$ implies $\alpha_1=\alpha_2=\cdots=\alpha_a=0$ for integers $\alpha_1, \alpha_2, \cdots$ and α_a.

(2) There exist integers $\alpha_1^{(j,k)}, \alpha_2^{(j,k)}, \cdots$ and $\alpha_a^{(j,k)}$ such that

$$p\theta_{jk}\equiv p(\alpha_1^{(j,k)}\theta_1+\alpha_2^{(j,k)}\theta_2+\cdots+\alpha_a^{(j,k)}\theta_a) \pmod 1$$

for any $j, k=1, 2, \cdots, s$.

Since $\{(\alpha p\theta_1, \alpha p\theta_2, \cdots, \alpha p\theta_a);\ \alpha=0, \pm1, \pm2, \cdots\}$ is everywhere dense mod 1 from Weyl [6], there exists a positive integer α' such that

$$0<|\alpha' p\theta_m|<1/B \pmod 1$$

for $m=1, 2, \cdots, a$ where $B=6\sum|\alpha_m^{(j,k)}|$ $(m=1, 2, \cdots, a;\ j, k=1, 2, \cdots, s)$. Then we have

$$0\leqq|\alpha' p\theta_{jk}|<1/6 \qquad \pmod 1$$

for $j, k=1, 2, \cdots, s$. Hence we have

$$\cos(2\pi\alpha' p\theta_{jk})>1/2$$

for $j, k=1, 2, \cdots, s$. This means that

$$\mathrm{Re}(F(x_k))>1$$

for $k=1, 2, \cdots, s$ and $\alpha=\alpha' p$. Therefore we have a desired function F in Y.

§ 2. Analytic ramified covering.

Let τ be a proper nowhere degenerating holomorphic mapping of a normal complex space X onto a normal complex space Y satisfying the following condition:

There exists an analytic set B in Y such that τ is a locally biholomorphic mapping of $X-\tau^{-1}(B)$ onto $Y-B$ and $\tau^{-1}(B)$ separates X nowhere.

Then (X, τ) is called an *analytic ramified covering over* Y. A point x of X is called to *lie over a point* y of Y if $\tau(x)=y$. If Y is connected, there exists a positive integer

s such that just s points $x_1, x_2, \cdots$ and x_s of $X-\tau^{-1}(B)$ lie over a point y of $Y-B$. s is called the *number of ramification of* (X, τ). If τ is a locally biholomorphic mapping of X onto Y, (X, τ) is called *unramified.* Let φ be a holomorphic mapping of a normal complex space X in a normal complex space Y such that for a suitable neighbourhood U of any point of X (U, φ) is an analytic ramified covering over $\varphi(U)$. Then (X, φ) is called a *ramified domain over* Y. If each (U, φ) is unramified (X, φ) is called an *unramified domain over* Y.

Let (X, τ) be an analytic ramified covering over a connected normal complex space Y and s be the number of ramification of (X, τ). Let f be a holomorphic function in X. We put

$$F_m(y)=\sum f(x_{j_1})f(x_{j_2}) \cdots f(x_{j_m})$$

for all mutually distinct $j_1, j_2, \cdots, j_m=1, 2, \cdots, s$ $(m=1, 2, \cdots, s)$ where $x_1, x_2, \cdots$ and x_s are points over $y \in Y-B$. Each F_m is a single-valued holomorphic function in $Y-B$ and is bounded in a neighbourhood of each point of B. Since Y is normal, each F_m can be analytically continued in Y. The analytic continuation, which is denoted by the same symbol F_m, of F_m in Y is called the *m-th symmetric function of f with respect to* (X, τ).

LEMMA 2. *Let (X, τ) be an analytic ramified covering over a normal complex space Y. If F is a holomorphic function in a subdomain D of Y such that F can not be analytically continued in a boundary point y_0 of D, $f=F\circ\tau$ can not be analytically continued in any point over y_0.*

Proof. Suppose that f can be analytically continued in a neighbourhood U' of a point x_0 over y_0. There exists a connected neighbourhood $U \subset U'$ and V of x_0 and y_0 respectively such that U is a connected component of $\tau^{-1}(V)$. Let s be the number of ramification of (U, τ) and F_1 be the first symmetric function of the analytic continuation of f in U with respect to (U, τ). Then F_1/s is an analytic continuation of F in V. This is a contradiction. Therefore f can not be analytically continued in any point over y_0.

LEMMA 3. *Let (X, τ) be an analytic ramified covering over a separable normal complex space Y. Let D be a subdomain of Y and D_1 be a connected component of $\tau^{-1}(D)$. If D_1 is a subdomain of holomorphy of a holomorphic function f in D_1, D is a subdomain of holomorphy of a linear combination of symmetric functions $F_1, F_2, \cdots$ and F_s of f with respect to (D_1, τ) whose number of ramification is denoted by s.*

Proof. At first f is a solution of the equation

$$\Lambda=w^s+F_1(\tau(x))w^{s-1}+\cdots+F_s(\tau(x))=0$$

in D_1. If all F_j's can be analytically continued in a neighbourhood U of $y_0 \in \partial D$ with $U \cap E=\phi$ where E is the zero-surface of the discriminant of Λ, f can be analytically continued in a connected component U_1 of $\tau^{-1}(U)$ with $U_1 \cap D_1 \neq \phi$. Let y_0 be a point of ∂D with a neighbourhood U' such that $U' \cap E \supset U' \cap \partial D$. If all F_j's can be analytically continued in a neighbourhood U of $y_0 \in \partial D$ with $U \cap E \supset U \cap \partial D$, f can be analytically continued to a function which is bounded and holomorphic in $U_1-\tau^{-1}(E)$ where U_1 is a connected component of $\tau^{-1}(U)$ with $U_1\cap D_1\neq\phi$. Since X is normal, f can be analytically continued in U_1.

Since Y is separable, there exists a countable subset $\Delta=\{x_k;\ k=1, 2, 3, \cdots\}$ of D satisfying the following conditions:

Each x_k is either a point of $\partial D-E$ or has a neighbourhood U with $U\cap E\supset U\cap\partial D$ even if all F's are analytically continuable in x_k. Δ is dense in ∂D.

From the above argument, one of F_j's can not be analytically continued in x_k for any k. Let C^s_k be the set of all s-pls $(a_1, a_2, \cdots, a_s)$ of complex numbers such that

$$a_1F_1+a_2F_2+\cdots+a_sF_s$$

can be analytically continued in x_k. C^s_k is a proper subspace of C^s and is nowhere dense in C^s for any k. Hence there exists $(b_1, b_2, \cdots, b_s)\in C^s-\bigcup_{k=1}^{\infty}C^s_k$. Then

$$F=b_1F_1+b_2F_2+\cdots+b_sF_s$$

can not be analytically continued in any point of Δ. Therefore D is a subdomain of holomorphy of F.

LEMMA 4. *Let (X, τ) be an analytic ramified covering over a normal complex space Y. Let D be a subdomain of Y and D_1 be a connected component of $\tau^{-1}(D)$. Then D is holomorphically convex if and only if D_1 is holomorphically convex.*

Proof. Suppose that D_1 is holomorphically convex. Let K be a compact subset of D. We put

$$K_1=\tau^{-1}(K)\cap D_1.$$

Since D_1 is holomorphically convex, the envelope $\tilde{K}_1$ of holomorphy of K_1 with respect to D_1 is compact. Let y_0 be a point of $D-\tau(\tilde{K}_1)$. We put $\Delta=\tau^{-1}(y_0)\cap D_1$. From Lemma 1 there exists a holomorphic function f in D_1 such that

$$\sup_{x\in K_1}|f(x)|<1<\inf_{x\in\Delta}|f(x)|.$$

Let s be the number of ramification of (D_1, τ). Then the s-th symmetric function F_s of f with respect to (D_1, τ) satisfies

$$\sup_{y\in K}|F_s(y)|<1<|F_s(y_0)|.$$

Therefore the envelope of holomorphy of K with respect to D is contained in a compact subset $\tau(\tilde{K}_1)$ of D. Hence D is holomorphycally convex.

Conversely suppose that D is holomorphically convex. Let K_1 be a compact subset. We put $K=\tau(K_1)$. The envelope $\tilde{K}$ of holomorphy of the compact set K with respect to D is compact. Let x_0 be a point of $D_1-\tau^{-1}(\tilde{K})$. There exists a holomorphic function F in D such that

$$\sup_{y\in K}|F(y)|<|F(\tau(x_0))|.$$

Then $f=F\circ\tau$ satisfies

$$\sup_{x\in K_1}|f(x)|<|f(x_0)|.$$

The envelope of holomorphy of K_1 with respect to D_1 is contained in a compact subset $\tau^{-1}(\tilde{K})$ of D_1. Hence D is holomorphically convex.

§3 Riemann's domain of the function $\sqrt[p]{z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}}$.

PROPOSITION 1. *Let (X, τ) be an analytic ramified covering over a normal complex space Y such that X is a Stein manifold. Then Y is a Stein space and for a subdomain D of Y the following assertions are equivalent*:

(1) *D is holomorphically convex.*

(2) *D is a subdomain of holomorphy of Y.*

(3) *There exists a holomorphic function F in D for any boundary point y_0 of D such that f can not be analytically continued in y_0.*

(4) *D is a subdomain of local holomorphy of Y.*

(5) *For any boundary point y_0 of D there exists a neighbourhood U of y_0 such that $U \cap D$ satisfies the condition in* (3).

Proof. If D is holomorphically convex, it is a subdomain of holomorphy from Cartan-Thullen [2]. It is easy to see that (2) implies (3) and (4), which imply (5). Hence it suffices to prove that (5) implies (1). Let D be a subdomain of Y such that there exists a neighbourhood U for any boundary point y_0 of D satisfying the following conditions:

For any boundary point y of $U \cap D$ there exists a holomorphic function F in $U \cap D$ which can not be analytically continued in y. $\tau^{-1}(U)$ is a holomorphically convex local coordinate neighbourhood of each point of $\tau^{-1}(y_0)$.

From Lemma 2 $\tau^{-1}(U) \cap \tau^{-1}(D)$ posesses the same property as $U \cap D$ does. Hence $\tau^{-1}(U) \cap \tau^{-1}(D)$ is an open set of holomorphy. From Docquier-Grauert [3] $\tau^{-1}(D)$ is holomorphically convex. From Lemma 4 D is holomorphically convex.

Let X be a Stein manifold with a finite group of automorphisms. We shall denote by $Y = X/G$ the factor space of X by the equivalence relation defined by G. Y is a Hausdorff space. From Cartan [1] there exists a complex structure in Y such that a continuous function F in a subdomain D of Y is holomorphic in D if and only if $F \circ \tau$ is holomorphic in a connected component of $\tau^{-1}(D)$ and that the canonical mapping τ of X onto Y is holomorphic. τ is a nowhere degenerating proper mapping of X onto Y satisfying the following condition:

There exists an analytic set B in Y such that τ is a locally biholomorphic mapping of $X - \tau^{-1}(B)$ onto $Y - B$ and $\tau^{-1}(B)$ separates X nowhere.

Even if Y is not normal, we can prove Lemmas 1, 2 and 4 for this mapping τ making use of the properties of the complex structure of Y cited above. Hence Proposition 1 is valid for this Y.

Let $n, p_1, p_2, \cdots, p_n$ and p are positive integers such that $p_1, p_2, \cdots$ and p_n are coprime. Then

$$Y = \{y = (z, w);\ w^p = z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}\}$$

is a normal analytic set in the space C^{n+1} of $(n+1)$ complex variables $z = (z_1, z_2 \cdots,$

z_n) and w. If we put $\varphi(y)=z$ for $y \in Y$, (Y, φ) is an analytic ramified covering over C^n and is called the *Riemann's domain of the function* $\sqrt[p]{z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}}$. If we put

$$z_j = t_j^p \ (j=1, 2, \cdots, n),\ w = t_1^{p_1} t_2^{p_2} \cdots t_n^{p_n}$$

for $t=(t_1, t_2, \cdots, t_n) \in C^n$, then $\tau(t)=(z, w)$ is a proper holomorphic mapping of C^n onto Y. We put $\omega = \exp(2\pi\sqrt{-1}/p)$. We denote by G the finite group of automorphisms

$$(t_1, t_2, \cdots, t_n) \to (\omega^{\nu_1} t_1, \omega^{\nu_2} t_2, \cdots, \omega^{\nu_n} t_n)$$

for all integers $\nu_1, \nu_2, \cdots$ and ν_n with

$$\nu_1 p_1 + \nu_2 p_2 + \cdots + \nu_n p_n \equiv 0 \pmod{p}.$$

Then we have $Y = C^n/G$. Hence we have

COROLLARY OF PROPOSITION 1. *For a subdomain D of the Riemann's domain of the function $\sqrt[p]{z_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}}$ the five assertions in Proposition* 1 *are equivalent where $p_1, p_2, \cdots$ and p_n are coprime.*

Let Y be a pure two-dimensional normal complex space. If y_0 is a uniformizable point of Y, there exists a biholomorphic mapping of a neighbourhood U in C^2. If y_0 is not a uniformizable point of Y, there exists a biholomorphic mapping of a neighbourhood U of y_0 in the Riemann's domain of the function $\sqrt[p]{z_1 z_2{}^{p-q}}$ from Jung [5] and Grauert-Remmert [4]. Therefore the five assertions in Proposition 1 are equivalent for a subdomain of U from Corollary of Proposition 1. Roughly speaking, any two-dimensional normal complex space has a neighbourhood for any point of it where the Levi problem is affirmatively solved. Thus we have

PROPOSITION 2. *A subdomain of a pure two-dimensional normal complex space Y is a subdomain of local holomorphy of Y if and only if it is a locally holomorphically convex subdomain of Y.*

§ 4. Sequence of holomorphic functions which converges uniformly to 0.

If we put

$$f_n(z) = z^n/n,$$

then $\{f_n;\ n=1, 2, 3, \cdots\}$ is a sequence of holomorphic functions in the complex plane C which converges uniformly to 0 in $\bar{D}$ and any subsequence of which does not converge pointwise in any domain D' with $D' \supsetneqq D$ where $D=\{z;\ |z|<1\}$. We shall give a necessary and sufficient condition for the existence of such a sequence.

PROPOSITION 3. *A subdomain D of a separable complex space Y coincides with a connected component of the open kernel of its envelope $\tilde{D}$ of holomorphy with respect to Y if and only if there exists a sequence $\{f_n;\ n=1, 2, 3, \cdots\}$ of holomorphic functions f_n in Y satisfying the following conditions*:

(1) *$\{f_n;\ n=1, 2, 3, \cdots\}$ converges uniformly to* 0 *in $\bar{D}$ in the strict sense.*

(2) *Any subsequence of $\{f_n;\ n=1, 2, 3, \cdots\}$ does not converge pointwise in any subdomain D' of Y with $D' \supsetneqq D$.*

Proof. Suppose that D is a connected component of the open kernel of $\tilde{D}$. Since Y is separable, there exists a sequence $\{x_n;\ n=1, 2, 3, \cdots\}$ of points x_n of $Y-\tilde{D}$ such that any point of $Y-\tilde{D}$ is its accumulation point. From Lemma 1 there exists a holomorphic function f_n in Y such that

$$\sup_{y\in D}|f_n(y)|<1/n<n<|f_n(x_m)| \qquad (m=1, 2, \cdots, n)$$

for $n=1, 2, 3, \cdots$. Obviously $\{f_n\}$ satisfies the conditions (1) and (2) in our Proposition.

Conversely suppose that $\{f_n;\ n=1, 2, 3, \cdots\}$ is such a sequence. There exists a positive integer n_0 such that

$$\sup_{y\in D}|f_n(y)|\leqq 1$$

for $n>n_0$. Let D' be a connected component of $\tilde{D}$ containg D. Then we have

$$\sup_{y\in D'}|f_n(y)|\leqq 1$$

for $n>n_0$. Therefore $\{f_n\}$ is a normal sequence in D'. Hence we have $D'=D$.

It is easy to prove

COROLLARY OF PROPOSITION 3. *Let D be a bounded domain in the complex plane C. Then there exists a sequence of polynomials satisfying the conditions in Proposition* 3 *if and only if $C-\bar{D}$ is connected and D is the open kernel of $\bar{D}$.*

A subdomain D of a complex space Y is called *holomorphically convex with respect to* Y if the intersection $\tilde{K}(Y)$ of D and the envelope of holomorphy of any compact subset K of D with respect to Y is a compact subset of D.

PROPOSITION 4. *If a subdomain D of a separable complex space Y is holomorphically convex with respect to Y, there exists a sequence $\{f_n;\ n=1, 2, 3, \cdots\}$ of holomorphic functions f_n in Y satisfying the following conditions*:

(1) *$\{f_n;\ n=1, 2, 3, \cdots\}$ converges uniformly to* 0 *in any compact subset of D.*

(2) *Any subsequence of $\{f_n;\ n=1, 2, 3, \cdots\}$ does not converge uniformly in a compact subset of any subdomain D' of Y with $D'\supsetneqq D$.*

Conversely if there exists such a sequence $\{f_n;\ n=1, 2, 3, \cdots\}$ for a subdomain D of an unramified domain (Y, φ) over C^p, D is holomorphically convex with respect to Y.

Proof. If D is a holomorphically convex subdomain with respect to a separable complex space Y, there exists a sequence $\{K_n;\ n=1, 2, 3, \cdots\}$ of compact subsets K_n of D satisfying the following conditions:

$$K_n=\tilde{K}_n(Y),\ K_n\subset K_{n+1} \quad (n=1, 2, 3, \cdots) \text{ and } D=\bigcup_{n=1}^{\infty} K_n.$$

There exists a countable subset $\{y_n;\ n=1, 2, 3, \cdots\}$ of ∂D which is dence in ∂D as Y is separable. Let $\{U_{n,m};\ m=1, 2, 3, \cdots\}$ be a filtre of neighbourhoods of y_n satsfying the following conditions for any n:

$$\{U_{n,m};\ m=1, 2, 3, \cdots\} \text{ converges to } y_n,\ U_{n,m+1}\subset U_{n,m} \ (m=1, 2, 3, \cdots),$$

$$\bigcup_{n=1}^{m} U_{n,m} \cap D \subset D - K_m.$$

Let $x_{n,m} \in U_{n,m} \cap (D-K_m)$ $(n=1, 2, \cdots, m)$. From Lamma 1 there exists a holomorphic function f_m in Y such that

$$\sup_{y \in K_m} |f_m(y)| < 1/m < m < |f_m(x_{n,m})| \quad (n=1, 2, \cdots, m).$$

$\{f_m;\ m=1, 2, 3, \cdots\}$ satisfies the conditions (1) and (2) in our Proposition.

Conversely suppose that $\{f_m;\ m=1, 2, 3, \cdots\}$ is such a sequence for a subdomain D of an unramified domain (Y, φ) over C^p. Let K be a compact subset of D. We put

$$\rho = \inf \left(\min_{1 \leq j \leq p} |\zeta_j - y_j| \right)$$

for $\zeta \in \partial D$ and $y \in K$. There exists a positive integer m_0 such that

$$\sup_{y \in K} |f_m(y)| \leqq 1$$

for $m > m_0$. Let

$$D_r = \{y;\ |\zeta_j - y_j| > r, \text{ for all } \zeta \in \partial D,\ y \in D\}.$$

Suppose that $y' \in \tilde{K}(Y) - D_r$ for some $r < \rho$. From Cartan-Thullen [2] we have

$$\sup_{y \in U} |f_m(y)| \leqq 1$$

for $m > m_0$ and $U = \{y;\ |y_j - y'_j| < \rho;\ j=1, 2, \cdots, p\}$. Hence $\{f_m;\ m=1, 2, 3, \cdots\}$ is a normal sequence in U. But this is a contradiction. Therefore we have

$$\tilde{K}(Y) \subset \bar{D}_\rho.$$

Since $\tilde{K}(Y)$ is bounded, it is a compact subset of D. Hence D is holomorphically convex with respect to Y.

COROLLARY OF PROPOSITION 4. *Let D be a domain in the complex plane C. Then there exists a sequence of polynomials satisfying the conditions in Proposition 4 if and only if D is simply connected.*

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