ON THE SYSTEM OF NON-LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

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§ **1.** Recently the author has investigated the behaviour of the solution of the non-linear differential equation

$$\frac{dy}{dx} = \sum_{k=1}^{\infty} f_k(x)\, y^k$$

where $f_k(x)$ are uniform and holomorphic in the domain $0 < |x| < r$, and obtained an analytical expression of the solution valid around $x = 0$.[1]

The method of proof used there can easily be generalized for the system of non-linear differential equations

$$\text{(A)} \qquad \frac{dy_j}{dx} = \sum_{k_1+\cdots+k_n \geqq 1} f_{j,k_1\cdots k_n}(x)\, y_1^{k_1} \cdots y_n^{k_n}, \qquad j = 1, \cdots, n,$$

with $f_{j,k_1\cdots k_n}(x)$ uniform and holomorphic in $0 < |x| < r$, or, what is the same thing, for the system

$$\text{(B)} \qquad \frac{dx_j}{dt} = \sum_{k_1+\cdots+k_n \geqq 1} a_{j,k_1\cdots k_n}(t)\, x_1^{k_1} \cdots x_n^{k_n}, \qquad j = 1, \cdots, n,$$

with $a_{j,k_1\cdots k_n}(t)$ periodic in t.

In the present paper, we consider the system (B), and establish the analytical expression of its solutions.

§ **2.** Let the system of differential equations

$$(1) \qquad \frac{dx_j}{dt} = \sum_{k=1}^{n} a_{j,k}(t)\, x_k + \sum_{k_1+\cdots+k_n \geqq 2} a_{j,k_1\cdots k_n}(t)\, x_1^{k_1} \cdots x_n^{k_n}, \qquad j = 1, \cdots, n,$$

be given, where $k_1, \cdots, k_n$ are non-negative integers, $a_{j,k}(t)$ and $a_{j,k_1\cdots k_n}(t)$ are periodic functions of t with period 1 holomorphic for $-\infty < t < \infty$, and the power series in the right-hand members are convergent for

$$-\infty < t < \infty, \quad |x_j| < \rho, \quad \rho > 0, \qquad j = 1, \cdots n.$$

Without loss of generality, we may suppose that the matrix $\|a_{j,k}(t)\|$ is of the following form:

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$$\left\| \begin{array}{cccc} \underbrace{\left.\begin{array}{|cccc|} \hline \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \\ \hline \end{array}\right\}}_{n_1 \text{ columns}} n_1 \text{ rows} & & & \\ & \underbrace{\left.\begin{array}{|cccc|} \hline \lambda_2 & & & 0 \\ 1 & \lambda_2 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_2 \\ \hline \end{array}\right\}}_{n_2 \text{ columns}} n_2 \text{ rows} & & \\ & & \ddots & \\ & & & \underbrace{\begin{array}{|cccc|} \hline \lambda_p & & & \\ 1 & \lambda_p & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_p \\ \hline \end{array}}_{n_p \text{ columns}} \end{array} \right\| \left.\vphantom{\begin{array}{c} a\\a\\a\\a \end{array}}\right\} n_p \text{ rows}$$

where $\lambda_1, \cdots, \lambda_p$ are complex constants.

We denote by

$$x_j(t) = x_j(t,\ t_0,\ x_{10}, \cdots, x_{n0}), \qquad j = 1, \cdots, n,$$

the solutions of (1) such that

$$x_j = x_{j0}, \qquad j = 1, \cdots, n, \qquad \text{for} \qquad t = t_0.$$

In what follows, we always consider these solutions in the fixed interval

$$(2) \qquad t_0 - N \leqq t \leqq t_0 + N$$

where N is any (arbitrarily large) positive number. Then, for any $\varepsilon > 0$ however small, we can find $\delta > 0$ such that

$$|x_j(t)| < \varepsilon, \qquad j = 1, \cdots, n,$$

in the interval (2), if

$$|x_{j0}| < \delta, \qquad j = 1, \cdots, n.$$

Therefore, if $|x_{10}|, \cdots, |x_{n0}|$ are chosen sufficiently small, $x_j(t)$ are holomorphic in $x_{10}, \cdots, x_{n0}$, and, moreover, $x_j(t)$ identically vanish whenever their initial values $x_{10}, \cdots, x_{n0}$ all vanish. Hence they admit the following power series expression in the interval (2):

$$(3) \qquad x_j(t) = \sum_{k=1}^{n} U_{j,k}(t,\ t_0) x_{k0} + \sum_{k_1+\cdots+k_n \geqq 2} U_{j,k_1\cdots k_n}(t,\ t_0) x_{10}^{k_1} \cdots x_{n0}^{k_n},$$
$$j = 1, \cdots, n.$$

§ **3.** We now investigate the relations between the solutions $x_j(t)$ and

$$X_j(t) = X_j(t, t_0, x_{10}, \cdots, x_{n0}) = x_j(t+1, t_0, x_{10}, \cdots, x_{n0}), \qquad j = 1, \cdots, n.$$

Since

$$X_j(t, t_0, x_{10}, \cdots, x_{n0}) = x_j(t+1, t, x_1(t), \cdots, x_n(t)), \qquad j = 1, \cdots, n,$$

we can write

$$(4) \quad \begin{cases} X_j(t) = \sum_{k=1}^{n} U_{j,k}(t+1, t)\, x_k(t) \\ \qquad + \sum_{k_1+\cdots+k_n \geqq 2} U_{j,k_1\cdots k_n}(t+1, t)\{x_1(t)\}^{k_1} \cdots \{x_n(t)\}^{k_n}, \\ \qquad\qquad j = 1, \cdots, n, \end{cases}$$

where the power series in $x_1(t), \cdots, x_n(t)$ in the right-hand members are convergent (in the interval (2)) whenever $|x_{10}|, \cdots, |x_{n0}|$ are sufficiently small.

From the uniqueness theorem of the solutions of differential equations and periodicity of the coefficients $a_{j,k}(t)$, $a_{j,k_1\cdots k_n}(t)$, the relation

$$x_j(t+1, t_0+1, x_{10}, \cdots, x_{n0}) = x_j(t, t_0, x_{10}, \cdots, x_{n0})$$

holds for any $x_{10}, \cdots, x_{n0}$ with sufficiently small absolute values. Thus we obtain, making use of the expression (3),

$$(5) \qquad U_{j,k}(t+1, t_0+1) = U_{j,k}(t, t_0), \quad U_{j,k_1\cdots k_n}(t+1, t_0+1) = U_{j,k_1\cdots k_n}(t, t_0).$$

Therefore, if we put

$$U_{j,k}(t+1, t) = u_{j,k}(t), \qquad U_{j,k_1\cdots k_n}(t+1, t) = u_{j,k_1\cdots k_n}(t),$$

it follows directly from (5) that

$$u_{j,k}(t+1) = u_{j,k}(t), \qquad u_{j,k_1\cdots k_n}(t+1) = u_{j,k_1\cdots k_n}(t),$$

and we obtain the following conclusion:

If $|x_{10}|, \cdots, |x_{n0}|$ are chosen sufficiently small, $X_j(t)$ can be written as

$$(6) \qquad X_j(t) = \sum_{k=1}^{n} u_{j,k}(t)\, x_k(t) + \sum_{k_1+\cdots+k_n \geqq 2} u_{j,k_1\cdots k_n}(t)\{x_1(t)\}^{k_1} \cdots \{x_n(t)\}^{k_n}, \qquad j = 1, \cdots, n,$$

in the interval

$$t_0 - N \leqq t \leqq t_0 + N,$$

where the coefficients $u_{j,k}(t)$, $u_{j,k_1\cdots k_n}(t)$ are periodic functions of t with period 1.

§ **4.** Next we determine explicitly the coefficients $u_{j,k}(t)$, $j, k = 1, \cdots n$. For that purpose, it suffices to determine the coefficients $U_{j,k}(t, t_0)$ in the expressin (3). Substituting (3) into (1) and equating the terms of the same degree in $x_{10}, \ldots, x_{n0}$ on both sides of the equations, we obtain the following system of linear differential equations:

$$(7)\quad \begin{cases} \dfrac{\partial U_{j_r+1,k}}{\partial t} = \lambda_r U_{j_r+1,k}, \\ \dfrac{\partial U_{j_r+2,k}}{\partial t} = U_{j_r+1,k} + \lambda_r U_{j_r+2,k}, \\ \quad\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dfrac{\partial U_{j_r+n_r,k}}{\partial t} = U_{j_r+n_r-1,k} + \lambda_r U_{j_r+n_r,k}, \\ \qquad j_1 = 0, \quad j_{r+1} = j_r + n_r, \quad r = 1, \cdots, p, \quad k = 1, \cdots, n. \end{cases}$$

As we have supposed that

$$x_j(t_0) = x_{j0}, \qquad j = 1, \cdots, n,$$

$U_{j,k}(t, t_0)$ must satisfy the initial condition

$$U_{j,k}(t_0, t_0) = \delta_{jk}, \qquad j, k = 1, \cdots, n.$$

Solving the linear system (7) under this condition, we have

$$U_{j_r+s,k}(t, t_0) = \Big\{\delta_{j_r+s,k} + \delta_{j_r+s-1,k}(t - t_0) + \delta_{j_r+s-2,k}\frac{(t-t_0)^2}{2!} + \cdots + \delta_{j_r+1,k}\frac{(t-t_0)^{s-1}}{(s-1)!}\Big\}e^{\lambda_r(t-t_0)},$$

$$s = 1, \cdots, n_r, \qquad r = 1, \cdots, p.$$

Consequently

$$\begin{aligned} u_{j_r+s,k}(t) &= U_{j_r+s,k}(t+1, t) \\ &= \Big\{\delta_{j_r+s,k} + \delta_{j_r+s-1,k} + \frac{\delta_{j_r+s-2,k}}{2!} + \cdots + \frac{\delta_{j_r+1,k}}{(s-1)!}\Big\}e^{\lambda_r}, \end{aligned}$$

$$s = 1, \cdots, n_r, \qquad r = 1, \cdots, p,$$

and the matrix $\| u_{j,k}(t) \|$ is of the form

$$\| u_{j,k}(t) \| = \begin{Vmatrix} \boxed{A_1} & & & \\ & \boxed{A_2} & & \\ & & \ddots & \\ & & & \boxed{A_p} \end{Vmatrix}$$

where

$$A_r = e^{\lambda_r}\left\|\begin{matrix} 1 & & & & \\ 1 & 1 & & 0 & \\ \frac{1}{2!} & 1 & \ddots & & \\ \vdots & \vdots & & \ddots & \\ \frac{1}{(n_r-1)!} & \frac{1}{(n_r-2)!} & \cdots & 1 & 1 \end{matrix}\right\|.$$

§ **5.** From what we have hitherto shown, the relation between $X_j(t) = x_j(t+1)$ and $x_j(t)$ can be written as follows:

$$(8)\quad \begin{cases} x_{j_r+s}(t+1) \\ = e^{\lambda_r}\left\{\frac{1}{(s-1)!}x_{j_r+1}(t) + \frac{1}{(s-2)!}x_{j_r+2}(t) + \cdots + x_{j_r+s}(t)\right\} \\ \quad + \sum_{k_1+\cdots+k_n\geqq 2} u_{j_r+s,k_1\cdots k_n}(t)\{x_1(t)\}^{k_1}\cdots\{x_n(t)\}^{k_n}, \\ \qquad j_1 = 0,\quad j_{r+1} = j_r + n_r,\quad s = 1,\cdots,n_r,\quad r = 1,\cdots,p. \end{cases}$$

Our final step is to solve this system of functional equations and to determine the explicit forms of $x_j(t)$.

For that purpose, we add an assumption that

$$(9)\qquad |e^{\lambda_r}| > 1 \qquad (\text{i.e. } \Re\lambda_r > 0), \qquad r = 1,\cdots,p.$$

Moreover we may suppose, without loss of generality, that $\lambda_1,\cdots,\lambda_p$ are so arranged that

$$(10)\qquad 0 < \Re\lambda_1 \leqq \Re\lambda_2 \leqq \cdots \leqq \Re\lambda_p.$$

It is known that,[2)] under the condition (9), we can find an analytic transformation

$$x_j(t) \to y_j(t) \equiv \varphi_j(x_1(t),\ \cdots,\ x_n(t);\ t)$$

defined by the power series, convergent for sufficiently small values of $|x_1(t)|$, $\cdots, |x_n(t)|$ (i.e. for sufficiently small values of $|x_{10}|,\cdots,|x_{n0}|$),

$$(11)\quad \begin{cases} y_j(t) = \varphi_j(x_1(t),\ \cdots,\ x_n(t);\ t) \\ \quad = \sum_{k=1}^{n} p_{jk}x_k(t) + \sum_{k_1+\cdots+k_n\geqq 2} w_{j,k_1\cdots k_n}(t)\{x_1(t)\}^{k_1}\cdots\{x_n(t)\}^{k_n}, \\ \qquad\qquad j = 1,\cdots,n, \end{cases}$$

where p_{jk} are constants with $\det|p_{jk}| \neq 0$, and $w_{j,k_1\cdots k_n}(t)$ are polynomials of $u_{j,k_1\cdots k_n}(t)$ (hence periodic in t) such that the system (8) is transformed into

$$(12)\quad \begin{cases} \varphi_{j_r+s}(x_1(t+1),\ \cdots,\ x_n(t+1);\ t) \\ = e^{\lambda_r}\varphi_{j_r+s}(x_1(t),\ \cdots,\ x_n(t);\ t) + \delta_s\varphi_{j_r+s-1}(x_1(t),\ \cdots,\ x_n(t);\ t) \\ \quad + \sum_{k_1+\cdots+k_n\geqq 2} v_{j_r+s,k_1\cdots k_n}(t)\{\varphi_1(x_1(t),\ \cdots,\ x_n(t);\ t)\}^{k_1}\cdots \\ \qquad\qquad \cdot\{\varphi_n(x_1(t),\ \cdots,\ x_n(t);\ t)\}^{k_n}, \\ \qquad j_1 = 0,\quad j_{r+1} = j_r + n_r,\quad s = 1,\cdots,n_r,\quad r = 1,\cdots,p, \end{cases}$$

where $v_{j_r+p,k_1\cdots k_n}(t)$ are periodic functions of t with period 1, and

$$v_{j_r+s,k_1\cdots k_n}(t) \equiv 0 \qquad \text{if} \qquad e^{\lambda_r} \neq e^{k_1\lambda_1+\cdots+k_n\lambda_p},$$

$$\delta_s = \begin{cases} 0, & s = 1, \\ 1, & s = 2, \cdots, n_r. \end{cases}$$

From the periodicity of $w_{j,k_1\cdots k_n}(t)$, we have

$$\varphi_j(x_1(t+1), \cdots, x_n(t+1); t) = y_j(t+1).$$

So the system (12) can be rewritten as

$$\text{(13)} \qquad \begin{cases} y_{j_r+s}(t+1) = e^{\lambda_r} y_{j_r+s}(t) + \delta_s y_{j_r+s-1}(t) \\ \qquad\qquad + \sum_{k_1+\cdots+k_n \geqq 2} v_{j_r+s,k_1\cdots k_n}(t)\{y_1(t)\}^{k_1}\cdots\{y_n(t)\}^{k_n}. \end{cases}$$

As we have supposed that $\Re\lambda_r$ are all positive, the relation

$$\text{(14)} \qquad e^{\lambda_r} = e^{k_1\lambda_1+\cdots+k_n\lambda_p}$$

can be realized for only a finite number of combinations of non-negative integers $k_1, \cdots, k_n$ with $k_1 + \cdots + k_n \geqq 2$. Hence the right-hand members of the equations (13) are all polynomials in $y_j(t)$. Furthermore, owing to the supplementary condition (10), $k_{j_{r+1}}, \cdots, k_n$ must all vanish for the relation (14) to hold. Consequently, in the expressions

$$\sum_{k_1+\cdots+k_n \geqq 2} v_{j_r+s,k_1\cdots k_n}(t)\{y_1(t)\}^{k_1}\cdots\{y_n(t)\}^{k_n}$$

in the right-hand members of the equations (13), the functions $y_{j_{r+1}}(t), \cdots, y_n(t)$ can never appear. Thus we can rewrite the system (13) in the following form:

$$\text{(15)} \qquad \begin{cases} y_{j_r+s}(t+1) = e^{\lambda_r} y_{j_r+s}(t) + \delta_s y_{j_r+s-1}(t) \\ \qquad\qquad + \sum_{k_1+\cdots+k_{j_r} \geqq 2} v_{j_r+s,k_1\cdots k_{j_r}0\cdots 0}(t)\{y_1(t)\}^{k_1}\cdots\{y_{j_r}(t)\}^{k_{j_r}}, \\ \qquad\qquad j_1 = 0, \quad j_{r+1} = j_r + n_r, \quad s = 1, \cdots, n_r, \quad r = 1, \cdots, p. \end{cases}$$

§ **6.** The system of functional equations (15) are divided into p groups according to the value of r.

For $r = 1$, these equations will be written as follows:

$$\text{(16–1)} \qquad y_1(t+1) = e^{\lambda_1} y_1(t),$$

$$\text{(16–2)} \qquad y_2(t+1) = e^{\lambda_1} y_2(t) + y_1(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\text{(16–}n_1\text{)} \qquad y_{n_1}(t+1) = e^{\lambda_1} y_{n_1}(t) + y_{n_1-1}(t).$$

From (16–1), we can immediately see that $y_1(t)$ must be of the form

$$y_1(t) = e^{\lambda_1 t}\Phi_1(t)$$

where $\Phi_1(t)$ is a periodic function of t with period 1. Next we put

$$y_2(t) = e^{\lambda_1 t}\{e^{-\lambda_1} t\Phi_1(t) + P(t)\}.$$

Then

$$y_2(t+1) = e^{\lambda_1} e^{\lambda_1 t}\{e^{-\lambda_1} t\Phi_1(t) + P(t+1)\} + e^{\lambda_1 t}\Phi_1(t).$$

Comparing this with (16–2)

$$y_2(t+1) = e^{\lambda_1} e^{\lambda_1 t}\{e^{-\lambda_1} t\Phi_1(t) + P(t)\} + e^{\lambda_1 t}\Phi_1(t),$$

we obtain

$$P(t+1) = P(t).$$

Therefore the function $P(t)$ must be periodic in t.

In the same way, the equations (16–3), $\cdots$, (16–n_1) can be solved successively, and $y_1(t), \cdots, y_{n_1}(t)$ must be of the form

$$y_j(t) = e^{\lambda_1 t}\Phi_j(t), \qquad j = 1, \cdots, n,$$

where $\Phi_j(t)$ are polynomials of t whose coefficients are all periodic functions of t with period 1.

We will then show that all $y_j(t)$ can be written in the form

$$y_{j_r+s}(t) = e^{\lambda_r t}\Phi_{j_r+s}(t) \tag{17}$$

where $\Phi_{j_r+s}(t)$ are polynomials of t whose coefficients are periodic functions of t with period 1. The proof can be carried out by induction with respect to r in the following way.

Suppose that $y_1(t), \cdots, y_{j_m}(t)$ have been expressed in the form (17). The next group of functional equations (corresponding to $r = m+1$) will be

$$\text{(18–1)} \quad \begin{cases} y_{j_m+1}(t+1) = e^{\lambda_m} y_{j_m+1}(t) \\ \qquad + \sum_{k_1+\cdots+k_{j_m} \geq 2} v_{j_m+1, k_1\cdots k_{j_m}0\cdots 0}(t)\{y_1(t)\}^{k_1}\cdots\{y_{j_m}(t)\}^{k_{j_m}}, \end{cases}$$

$$\text{(18–2)} \quad \begin{cases} y_{j_m+2}(t+1) = e^{\lambda_m} y_{j_m+2}(t) + y_{j_m+1}(t) \\ \qquad + \sum_{k_1+\cdots+k_{j_m} \geq 2} v_{j_m+2, k_1\cdots k_{j_m}0\cdots 0}(t)\{y_1(t)\}^{k_1}\cdots\{y_{j_m}(t)\}^{k_{j_m}}, \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\text{(18–}n_m\text{)} \quad \begin{cases} y_{j_m+n_m}(t+1) = e^{\lambda_m} y_{j_m+n_m}(t) + y_{j_m+n_m-1}(t) \\ \qquad + \sum_{k_1+\cdots+k_{j_m} \geq 2} v_{j_m+n_m, k_1\cdots k_{j_m}0\cdots 0}(t)\{y_1(t)\}^{k_1}\cdots\{y_{j_m}(t)\}^{k_{j_m}}. \end{cases}$$

Since $v_{j_m+s, k_1\cdots k_{j_m}0\cdots 0}(t) \equiv 0$ for

$$\lambda_m \not\equiv k_1\lambda_1 + \cdots + k_{j_m}\lambda_{m-1} \pmod{2\pi i},$$

and $y_k(t), k = 1, \cdots, j_m$, are supposed to be of the form (17), we can write

$$\sum_{k_1+\cdots+k_{j_m} \geq 2} v_{j_m+s, k_1\cdots k_{j_m}0\cdots 0}(t)\{y_1(t)\}^{k_1}\cdots\{y_{j_m}(t)\}^{k_{j_m}} = e^{\lambda_m t} V_{j_m+s}(t),$$
$$s = 1, \cdots, n_m,$$

where $V_{j_m+s}(t)$ are polynomials of t whose coefficients are all periodic functions of t with period 1. Then the epuations (18–1), $\cdots$, (18–n_m) are rewritten as

(19–1) $$y_{j_m+1}(t+1) = e^{\lambda_m} y_{j_m+1}(t) + e^{\lambda_m t} V_{j_m+1}(t),$$

(19–2) $$y_{j_m+2}(t+1) = e^{\lambda_m} y_{j_m+2}(t) + y_{j_m+1}(t) + e^{\lambda_m t} V_{j_m+2}(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

(19–n_m) $$y_{j_m+n_m}(t+1) = e^{\lambda_m} y_{j_m+n_m}(t) + y_{j_m+n_m-1}(t) + e^{\lambda_m t} V_{j_m+n_m}(t).$$

$V_{j_m+1}(t)$ can be written in the following form:

$$V_{j_m+1}(t) = Q_0(t) + tQ_1(t) + \cdots + t^\nu Q_\nu(t)$$

where $Q_0(t), \cdots, Q_\nu(t)$ are all periodic in t. Then put

(20) $$y_{j_m+1}(t) = e^{\lambda_m t}\{R_0(t) + tR_1(t) + \cdots + t^{\nu+1} R_{\nu+1}(t)\}$$

where $R_1(t), \cdots, R_{\nu+1}(t)$ are determined from the following system of linear algebraic equations:

$$\binom{\nu+1}{1} R_{\nu+1}(t) = e^{-\lambda_m} Q_\nu(t),$$

$$\binom{\nu+1}{2} R_{\nu+1}(t) + \binom{\nu}{1} R_\nu(t) = e^{-\lambda_m} Q_{\nu-1}(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\binom{\nu+1}{\nu-k+1} R_{\nu+1}(t) + \binom{\nu}{\nu-k} R_\nu(t) + \cdots + \binom{k+1}{1} R_{k+1}(t) = e^{-\lambda_m} Q_k(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$R_{\nu+1}(t) + R_\nu(t) + \cdots + R_1(t) = e^{-\lambda_m} Q_0(t).$$

As the determinant constructed from the coefficients of the left-hand members of the above equations is obviously different from zero, $R_1(t), \cdots, R_{\nu+1}(t)$ are uniquely determined as the linear combinations of $Q_0(t), \cdots, Q_\nu(t)$. Hence they are all periodic functions of t with period 1.

Then

$$\begin{aligned} y_{j_m+1}(t+1) &= e^{\lambda_m} e^{\lambda_m t}\{R_0(t+1) + (t+1)R_1(t) + \cdots + (t+1)^{\nu+1} R_{\nu+1}(t)\} \\ &= e^{\lambda_m} e^{\lambda_m t}\{R_0(t+1) + tR_1(t) + \cdots + t^{\nu+1} R_{\nu+1}(t)\} \\ &\quad + e^{\lambda_m} e^{\lambda_m t} \Big[\{R_1(t) + R_2(t) + \cdots + R_{\nu+1}(t)\} \\ &\quad + t\Big\{\binom{2}{1} R_2(t) + \binom{3}{2} R_3(t) + \cdots + \binom{\nu+1}{\nu} R_{\nu+1}(t)\Big\} + \cdots \\ &\quad + t^k\Big\{\binom{k+1}{1} R_{k+1}(t) + \cdots + \binom{\nu-k+1}{\nu+1} R_{\nu+1}(t)\Big\} + \cdots \\ &\quad + t^\nu \binom{\nu+1}{1} R_{\nu+1}(t)\Big] \end{aligned}$$

$$= e^{\lambda_m} e^{\lambda_m t}\{R_0(t+1) + tR_1(t) + \cdots + t^{\nu+1}R_{\nu+1}(t)\}$$
$$+ e^{\lambda_m t}\{Q_0(t) + tQ_1(t) + \cdots + t^{\nu}Q_{\nu}(t)\}.$$

Comparing this with (19–1), we obtain

$$R_0(t+1) = R_0(t).$$

Therefore $y_{j_m+1}(t)$ must be of the form (17).

Substituting the expression of $y_{j_m+1}(t)$ just obtained into (19–2), we have

$$y_{j_m+2}(t+1) = e^{\lambda_m} y_{j_m+2}(t) + e^{\lambda_m t} W_{j_m+2}(t) \tag{21}$$

where $W_{j_m+2}(t)$ is a polynomial of t with periodic coefficients. It is then evident that the equation (21) can be solved by the same method as we have adopted for the equation (19–1), and $y_{j_m+2}(t)$ must also be of the form (17).

Proceeding in this way, we can successively show that $y_{j_m+s}(t)$, $s = 1, \cdots, n_m$ must be of the form (17). Thus we have completed the proof.

§ **7.** Substituting (17) into (11), and solving it with respect to $x_1(t), \cdots, x_n(t)$ we arrive at the desired analytical expression of $x_j(t)$ which can be written as follows:

$$x_j(t) = \sum_{k_1+\cdots+k_n \geqq 1} P_{j,k_1\cdots k_n}(t)\, e^{(k_1\lambda_1+\cdots+k_n\lambda_p)t}, \qquad j = 1, \cdots, n,$$

where $P_{j,k_1\cdots k_n}(t)$ are polynomials of t whose coefficients are periodic functions of t with period 1.

The same conclusion can be obtained if we replace the condition (9) by

$$|e^{\lambda_r}| < 1 \qquad (\text{i. e. } \Re\lambda_r < 0), \qquad r = 1, \cdots, p. \tag{9'}$$

Thus we have established the following

THEOREM. *If the real parts of the characteristic exponents $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the linear part of the system* (1) *are all positive or all negative, the solutions $x_j(t), j = 1, \cdots, n$, of the system with the intial condition*

$$x_j = x_{j0}, \quad j = 1, \cdots, n, \qquad \textit{for} \quad t = t_0$$

can be expressed in the domain

$$t_0 - N \leqq t \leqq t_0 + N, \qquad |x_{j0}| < \varepsilon_N, \qquad j = 1, \cdots, n,$$

in the following form:

$$x_j(t) = \sum_{k_1+\cdots+k_n \geqq 1} P_{j,k_1\cdots k_n}(t)\, e^{(k_1\lambda_1+\cdots+k_n\lambda_n)t}, \qquad j = 1, \cdots, n, \tag{*}$$

where N is an arbitrary positive number, ε_N is a positive number depending upon N, and $P_{j,k_1\cdots k_n}(t)$ are polynomials of t whose coefficients are all periodic functions of t with period 1.[3)]

In praticular, if $\lambda_1, \cdots, \lambda_n$ are all distinct and the relation $\lambda_m \equiv k_1\lambda_1 + \cdots + k_n\lambda_n \pmod{2\pi i}$ can never be realized for any combination of non-negative integers $k_1, \cdots, k_n$ with $k_1 + \cdots + k_n \geqq 2$ and $m = 1, \cdots, n$, the functions $\Phi_j(t)$ in (17) are all periodic functions of t. Hence $P_{j,k_1\cdots k_n}(t)$ in (*) must be all periodic in t. Whence follows the

COROLLARY. *If, besides the condition stated in the Theorem,* $\lambda_1, \cdots, \lambda_n$ *are all distinct and the relation*

$$\lambda_m \equiv k_1\lambda_1 + \cdots + k_n\lambda_n \pmod{2\pi i}$$

can never be realized for any set of non-negative integers $k_1, \cdots, k_n$ *with* $k_1 + \cdots + k_n \geqq 2$ *and* $m = 1, \cdots, n$, $x_j(t)$ *can be expressed in the domain*

$$|e^{\lambda_1 t}| < M, \ \cdots, \ |e^{\lambda_n t}| < M, \quad |x_{j0}| < \varepsilon_M, \qquad j = 1, \cdots, n,$$

in the form (*), *where* M *is an arbitrary positive number,* ε_M *is a positive nnmber depending upon* M, *and* $P_{j,k_1\cdots k_n}(t)$ *are periodic functions of* t *with period* 1.

REFERENCES

1) T. SAITO, Sur les solutions autour d'un point singulier fixe des équations différentielles du premier ordre. Kōdai Math. Sem. Rep. **4** (1953), 121—126. The proof given in this paper contained an error and the conclusion should slightly be modified. The corrected proof was given by Y. SIBUYA in his paper: The functional equations of Schröder and the ordinary differential equations. Monthly Rep. of S. S. S. **2** (1955), 25—33 (in Japanese).

In the present paper the discussion is made along the line of Sibuya's corrected proof. I wish to express my cordial thanks to Mr. Sibuya who indicated me the error of my proof in the above cited paper, and gave me many valuable advices concerning this subject.

2) M. HUKUHARA, On functional equations of Schröder. Rep. Fac. Sci. Kyûshû Imp. Univ. **1** (1945), 190—196 (in Japanese).

3) For the case when $\lambda_1, \cdots, \lambda_m$ are all purely imaginary, cf. Y. SIBUYA, Sur un système des équations différentielles ordinaires non-linéaires à coefficients constants ou périodiques. J. Fac. Sci. Univ. Tokyo, Sec. I, **7** (1954), 19—32, and also M. URABE, Application of majorized group of transformations to ordinary differential equations with periodic coefficients. J. Sci. Hiroshima Univ., Ser. A, **19** (1956), 469—478.

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