

## HOLOMORPHIC ISOMORPHISM WHICH PRESERVES CERTAIN HOLOMORPHIC SECTIONAL CURVATURE

BY MINORU KOBAYASHI AND SUSUMU TSUCHIYA

**1. Introduction.** Let  $(M, g)$  and  $(\bar{M}, \bar{g})$  be two Riemannian manifolds. Denote the corresponding sectional curvatures by  $K$  and  $\bar{K}$  respectively. A diffeomorphism  $f$  from  $M$  to  $\bar{M}$  will be said to be curvature preserving if and only if for every  $p \in M$  and for every 2-plane  $\sigma$  in the tangent space  $T_p(M)$  to  $M$ , we have

$$K(\sigma) = \bar{K}(f_*\sigma).$$

It is natural to ask whether a curvature preserving diffeomorphism is isometric or not. The answer to this question was first given by R.S. Kulkarni as follows;

**THEOREM ([2]).** *If  $M$  is an analytic Riemannian manifold with dimension  $\geq 4$ , then a curvature preserving diffeomorphism  $f: M \rightarrow \bar{M}$  is an isometry except in the case that both  $M$  and  $\bar{M}$  have the same constant curvature.*

In the case where both of  $(M, g)$  and  $(\bar{M}, \bar{g})$  are Kaehlerian manifolds, we may expect that a holomorphic sectional curvature preserving diffeomorphism is a isometry. Indeed he proved

**THEOREM ([4]).** *Let  $M$  and  $\bar{M}$  be connected Kaehlerian manifolds with corresponding holomorphic sectional curvature functions  $H$  and  $\bar{H}$  respectively. Suppose that  $\dim M \geq 2$  and there exists a diffeomorphism  $f: M \rightarrow \bar{M}$  such that  $f^*\bar{H} = H$ . Then either  $H = \bar{H} = \text{const.}$ , or  $f$  is holomorphic or anti-holomorphic isometry.*

On the other hand, in our previous paper ([5]), we defined the  $\theta$ -holomorphic sectional curvature and the  $\tau$ -bisectional curvature and showed that the constancy of the holomorphic sectional curvature is equivalent to that of the  $\theta$ -holomorphic sectional curvature or to that of the holomorphic  $\tau$ -bisectional curvature. It is then quite natural to ask whether a  $\theta$ -holomorphic sectional curvature preserving or a holomorphic  $\tau$ -bisectional curvature preserving diffeomorphism is isometric or not. Concerning this problems, we shall prove the following two theorems. We shall define in Section 3 what are called  $\theta$ -holomorphically isocurved Kaehlerian manifolds and what are called  $\tau$ -bisectionally isocurved Kaehlerian manifolds.

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THEOREM I. *Let two Kaehlerian manifolds  $M$  and  $\bar{M}$  be  $\theta$ -holomorphically isocurved manifolds ( $\theta \neq \frac{\pi}{2}$ ). If the holomorphic sectional curvature  $H$  is not constant,  $M$  and  $\bar{M}$  are isometric.*

THEOREM II. *Let two Kaehlerian manifolds  $M$  and  $\bar{M}$  be  $\theta$ -bisectionally isocurved manifolds ( $\theta \neq \frac{\pi}{2}$ ). If the holomorphic sectional curvature  $H$  is not constant than  $M$  and  $\bar{M}$  are isometric.*

In Section 2, we recall the definition of the angle between two subspaces and, for completeness, prove two fundamental lemmas. Section 3 will be devoted the preparation for the proof of above Theorems.

**2. Then angle between two subspaces.** Let  $U$  be an  $n$ -dimensional real vector space with inner product  $\langle, \rangle$ . Consider two arbitrary  $k$ -dimensional subspaces  $S$  and  $T$ . We are now going to define inductively sets

$$(2.1) \quad \Theta_m = \{\theta_1, \dots, \theta_m; X_1, \dots, X_m; Y_1, \dots, Y_m\} \quad (1 \leq m \leq k),$$

$X_1, \dots, X_m$  and  $Y_1, \dots, Y_m$  being orthonormal respectively in  $S$  and  $T$ , where  $0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_m \leq \frac{\pi}{2}$ . First we put

$$\theta_1 = \inf \{ \sphericalangle(X, Y); X \in S, Y \in T, X \neq 0, Y \neq 0 \},$$

where  $\sphericalangle(X, Y)$  denotes the angle between  $X$  and  $Y$ . Then we can take unit vectors  $X_1 \in S$  and  $Y_1 \in T$  such that with  $\sphericalangle(X_1, Y_1) = \theta_1$ . Next, assuming that  $\Theta_m$  is already defined, we put

$$S_m = \{ X \in S; \langle X, X_i \rangle = 0, \quad (i=1, \dots, m) \},$$

$$T_m = \{ Y \in T; \langle Y, Y_i \rangle = 0, \quad (i=1, \dots, m) \},$$

and define  $\theta_{m+1}$  by

$$\theta_{m+1} = \inf \{ \sphericalangle(X, Y); X \in S_m, Y \in T_m \}.$$

Then we can take unit vectors  $X_{m+1} \in S, Y_{m+1} \in T$  such that  $\sphericalangle(X_{m+1}, Y_{m+1}) = \theta_{m+1}$ . So, we have a set

$$\Theta_{m+1} = \{\theta_1, \dots, \theta_{m+1}; X_1, \dots, X_{m+1}; Y_1, \dots, Y_{m+1}\} \quad (1 \leq m \leq k).$$

LEMMA 2.1. *Put  $\theta = \inf \{ \sphericalangle(X, Y); X \in S, Y \in T \}$ . Let  $X_1, X_2 \in S$  and  $Y_1, Y_2 \in T$  be unit vectors such that  $\sphericalangle(X_1, Y_1) = \sphericalangle(X_2, Y_2) = \theta$ . then*

$$\left\langle \frac{aX_1 + bX_2}{\|aX_1 + bX_2\|}, \frac{aY_1 + bY_2}{\|aY_1 + bY_2\|} \right\rangle = \cos \theta \quad \text{for all } aX_1 + bX_2 \neq 0,$$

where  $\|X\|$  denotes the angle of any vector  $X$ .

*Proof.* When  $\theta = \frac{\pi}{2}$ , the lemma is trivial. When  $\theta \neq \frac{\pi}{2}$ , we have

$$(2.2) \quad \left\langle Y_1, \frac{X_1X_1+X_2X_2}{\|X_1X_1+X_2X_2\|} \right\rangle \leq \cos \theta \quad \text{for any } x_1X_1+x_2X_2 \neq 0,$$

since  $\theta$  is the least angle between vectors in  $S$  and  $T$ . The inequality (2.2) implies

$$x_1^2(\cos^2\theta - \langle Y_1, X_2 \rangle^2) + 2x_1x_2 \cos \theta (\langle X_1, X_2 \rangle \cos \theta - \langle Y_1, X_2 \rangle) \geq 0.$$

Since the inequality holds for all numbers  $x_1, x_2$  such that  $x_1X_1+x_2X_2 \neq 0$ , we have

$$(2.3) \quad \langle X_1X_2 \rangle \cos \theta = \langle Y_1, X_2 \rangle.$$

Exchanging  $Y_1$  for  $Y_2$  in (2.2), we have

$$(2.4) \quad \langle X_1, X_2 \rangle \cos \theta = \langle Y_2, X_1 \rangle.$$

Similarly, the inequality

$$\left\langle X_1, \frac{x_1Y_1+x_2Y_2}{\|x_1Y_1+x_2Y_2\|} \right\rangle \leq \cos \theta \quad \text{for } x_1Y_1+x_2Y_2 \neq 0$$

implies

$$(2.5) \quad \langle Y_1, Y_2 \rangle \cos \theta = \langle X_1, Y_2 \rangle.$$

From (2.4) and (2.5), we have

$$\langle X_1, X_2 \rangle = \langle Y_1, Y_2 \rangle$$

and consequently

$$(2.6) \quad \|aX_1+bX_2\| = \|aY_1+bY_2\|.$$

Hence we have

$$\begin{aligned} & \left\langle \frac{aX_1+bX_2}{\|aX_1+bX_2\|}, \frac{aY_1+bY_2}{\|aY_1+bY_2\|} \right\rangle \\ &= \frac{1}{\|aX_1+bX_2\|^2} \{(a^2+b^2) \cos \theta + ab(\langle X_1, Y_2 \rangle + \langle X_2, Y_1 \rangle)\} \\ &= \frac{1}{\|aX_1+bX_2\|^2} (a^2+b^2+2ab\langle X_1, X_2 \rangle) \cos \theta = \cos \theta, \end{aligned}$$

where we have used (2.3), (2.4) and (2.6). Thus the proof is completed.

Lemma 2.1 shows that the set  $\{X \in S; \text{ there exists } Y \in T \text{ such that } \sphericalangle(X, Y) = \{\theta\} \cup \{0\}\}$  is a vector subspace of  $S$  and hence the set  $\{\theta_1, \dots, \theta_k\}$  is independent of the choice of  $X_1, \dots, X_k \in S$  and  $Y_1, \dots, Y_k \in T$ . Now we shall put

$$\sphericalangle(S, T) = (\theta_1, \dots, \theta_m)$$

LEMMA 2.2. Let  $S, S', T$  and  $T'$  be  $k$ -dimensional subspaces of  $V$  such that  $\sphericalangle(S, T) = \sphericalangle(S', T') = (\theta_1, \dots, \theta_k)$ . Then there is an orthogonal transformation  $u$  of  $V$  such that  $u(S) = S'$  and  $u(T) = T'$ .

*Proof.* Let  $\{X_1, \dots, X_k, Y_1, \dots, Y_k\}$  and  $\{X'_1, \dots, X'_k, Y'_1, \dots, Y'_k\}$  be two sets of vectors constructed in processes taken to define the angle  $\sphericalangle(S, T)$  and  $\sphericalangle(S', T')$  respectively. Now we may assume that  $0 = \theta_1 = \dots = \theta_m < \theta_{m+1} \leq \dots \leq \theta_k$ . We now define  $Z_1, \dots, Z_n$  as

$$Z_i = X_i \quad (i \leq k),$$

$$Z_j = \frac{Y_{m-k+j} - \cos \theta_{m-k+j} X_{m-k+j}}{1 - \cos^2 \theta_{m-k+j}} \quad (k+1 \leq j \leq 2k-m)$$

and take an orthonormal basis  $\{Z_{2k-m+1}, \dots, Z_n\}$  of the orthogonal complement to the subspace spanned by  $Z_1, \dots, Z_{2k-m}$ . Similarly, for  $\{X'_1, \dots, X'_k, Y'_1, \dots, Y'_k\}$  we define  $Z'_1, \dots, Z'_n$  as above. Then  $Z_i$ 's and  $Z'_i$ 's are orthogonal bases of  $V$ . Denote by  $u$  the orthogonal transformation of  $V$  such that  $u(Z_i) = Z'_i$  ( $i=1, \dots, n$ ).

Then we have  $u(X_i) = X'_i$  ( $i \leq k$ ) and hence  $u(S) = S'$ . Moreover, since

$$u\left(\frac{Y_{m-k+i} - \cos \theta_{m-k+i} X_{m-k+i}}{1 - \cos^2 \theta_{m-k+i}}\right) = \frac{Y'_{m-k+i} - \cos \theta_{m-k+i} X'_{m-k+i}}{1 - \cos^2 \theta_{m-k+i}},$$

we have  $u(Y_i) = Y'_i$  ( $i \leq k$ ) and hence  $u(T) = T'$ , which completes the proof.

Since lemmas 2.1 and 2.2 are established, we can call  $\sphericalangle(S, T) = (\theta_1, \dots, \theta_k)$  the angle between  $S$  and  $T$ .

Concerning subspaces of a complex vector space  $V$  with complex structure  $J$  and Hermitian metric  $\langle, \rangle$ , we have proved in [5]

THEOREM 2.3. Let  $S$  and  $T$   $2m$ -dimensional complex subspaces of  $V$ . Then there are  $m$  real numbers  $\theta_1, \dots, \theta_m$  such that

$$\sphericalangle(S, T) = (\theta_1, \theta_1, \dots, \theta_m, \theta_m).$$

THEOREM 2.4. If  $S$  is a  $k$ -dimensional real subspace of  $V$ , then we have

- (i)  $\sphericalangle(S, JS) = \left(\theta_1, \theta_1, \dots, \theta_m, \theta_m, \frac{\pi}{2}\right)$  for  $k=2m+1$ ,
- (ii)  $\sphericalangle(S, JS) = (\theta_1, \theta_1, \dots, \theta_m, \theta_m)$  for  $k=2m$ .

Taking account of Theorem 2.3 we can denote  $\sphericalangle(S, T)$  simply by  $(\theta_1, \theta_2, \dots, \theta_m)$  for complex subspaces  $S$  and  $T$ . Because of Theorem 2.4 we can denote  $\sphericalangle(S, JS)$  simply by  $(\theta_1, \dots, \theta_m)$  for real subspace  $S$ . If  $\sigma, \sigma'$  are  $J$ -invariant 2-planes, then  $\sphericalangle(\sigma, \sigma') = (\theta_1) = \theta_1$ , where  $\theta_1 = \inf \{\sphericalangle(X, X'); X \in \sigma, X' \in \sigma'\}$ . If  $\sigma$  is

a 2-plane, then  $\sphericalangle(\sigma, J\sigma) = (\tau_1) = \tau_1$ , where  $\tau_1 = \inf \{ \sphericalangle(X, X') : X \in \sigma, X' \in J\sigma \}$ .

**3.  $\theta$ -holomorphically isocurved and  $\tau$ -bisectionally isocurved manifolds.**

Let  $(M, \langle, \rangle, J)$  be a Kaehlerian manifold. A plane section  $\sigma$  of the tangent space to  $M$  is called a  $\theta$ -holomorphic section if  $\sphericalangle(\sigma, J\sigma) = \theta$ . The sectional curvature for a  $\theta$ -holomorphic section is called the  $\theta$ -holomorphic sectional curvature for  $\sigma$ . For a  $\theta$ -holomorphic section  $\sigma$ , we have  $\cos \theta = |\langle X, JY \rangle|$  for any  $X$  and  $Y$  forming an orthonormal basis in  $\sigma$ . Let  $\sigma$  and  $\sigma'$  be  $J$ -invariant planes. The holomorphic bisectional curvature  $H(\sigma, \sigma')$  is defined by Goldberg and Kobayashi ([1]) as

$$H(\sigma, \sigma') = \langle R(X, JX)JY, Y \rangle,$$

$R$  being the curvature tensor of  $M$ , where  $X$  is a unit vector in  $\sigma$  and  $Y$  is a unit vector in  $\sigma'$ .  $H(\sigma, \sigma')$  is called holomorphic  $\tau$ -bisectional curvature for  $\sigma$  and  $\sigma'$  if  $\sphericalangle(\sigma, \sigma') = \tau$ . When the angle between  $\sigma$  and  $\sigma'$  is equal to  $\tau$ , we have  $\cos \tau = \langle X, JY \rangle$  for certain unit vectors  $X$  in  $\sigma$  and  $Y$  in  $\sigma'$ .

Let  $(\bar{M}, \langle \bar{\cdot}, \bar{\cdot} \rangle, \bar{J})$  be another Kaehlerian manifold. We shall say that the two Kaehlerian manifolds  $M$  and  $\bar{M}$  are  $\theta$ -holomorphically isocurved if there exists a holomorphic diffeomorphism  $f: M \rightarrow \bar{M}$  preserving  $\theta$ -holomorphic sectional curvatures. Also we shall say that  $M$  and  $\bar{M}$  are  $\tau$ -bisectionally isocurved if there exists a holomorphic diffeomorphism  $f: M \rightarrow \bar{M}$  preserving holomorphic  $\tau$ -bisectional curvatures. We now have (A)  $\langle \bar{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X} \rangle = (\|\bar{X}\|^2 \|\bar{Y}\|^2 - \langle \bar{X}, \bar{Y} \rangle^2)$  for  $\theta$ -holomorphically isocurved Kaehlerian manifolds  $M$  and  $\bar{M}$ , and (B)  $\langle R(X, Y)Y, X \rangle \langle \bar{R}(\bar{X}, \bar{Y})\bar{Y}, \bar{X} \rangle = \|\bar{X}\|^2 \|\bar{Y}\|^2 \langle R(X, JX)JY, Y \rangle$  for  $\tau$ -bisectionally isocurved Kaehlerian manifolds  $M$  and  $\bar{M}$ , where  $\bar{R}$  is the curvature tensor of  $\bar{M}$ ,  $X, Y$  are orthonormal vectors tangent to  $M$  such that  $\langle JX, Y \rangle = \cos \theta$  and  $\bar{X} = f_*X, \bar{Y} = f_*Y$ .

We shall prove an algebraic lemma for later use. To do so, we consider a unitary matrix  $U = (u_{ij}) (u_{ij} \in \mathbb{C}, i, j = 1, 2)$ . Then we have  $UU^t = U^tU = I$ , i.e.,

$$(3.1) \quad \begin{aligned} (a) \quad & |u_{11}|^2 + |u_{12}|^2 = 1, & (b) \quad & |u_{21}|^2 + |u_{22}|^2 = 1, \\ (c) \quad & |u_{11}|^2 + |u_{21}|^2 = 1, & (d) \quad & u_{11}\bar{u}_{21} + u_{12}\bar{u}_{22} = 0, \end{aligned}$$

where  $\bar{u}_{i,j}$  is the complex conjugate of  $u_{i,j}$  and  $|\bar{u}_{i,j}|$  is the absolute value of  $u_{i,j}$ . From (3.1, b) and (3.1, c) we have  $|u_{11}| = |u_{22}|$ , and from (3.1, a) and (3.1, c) we have  $|u_{12}| = |u_{21}|$ . Hence we can put  $|u_{11}| = |u_{22}| = \cos \varphi, |u_{12}| = |u_{21}| = \sin \varphi$  ( $0 \leq \varphi \leq \frac{\pi}{2}$ ) and consequently we may put

$$\begin{aligned} u_{11} &= \cos \varphi e^{\sqrt{-1}\alpha_{11}} = \cos \varphi (\cos \alpha_{11} + \sqrt{-1} \sin \alpha_{11}), \\ u_{12} &= \sin \varphi e^{\sqrt{-1}\alpha_{12}} = \sin \varphi (\cos \alpha_{12} + \sqrt{-1} \sin \alpha_{12}), \end{aligned}$$

$$u_{21} = \sin \varphi e^{\sqrt{-1}\alpha_{21}} = \sin \varphi (\cos \alpha_{21} + \sqrt{-1} \sin \alpha_{21}),$$

$$u_{22} = \cos \varphi e^{\sqrt{-1}\alpha_{22}} = \cos \varphi (\cos \alpha_{22} + \sqrt{-1} \sin \alpha_{22}),$$

where  $\alpha_{11}, \alpha_{12}, \alpha_{21}$  and  $\alpha_{22}$  are real numbers with  $\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22} = (2n+1)\pi$  ( $n$ ; integer) because of (3.1, d). Therefore we have

$$U = A + \sqrt{-1} B,$$

where

$$A = \begin{pmatrix} \cos \varphi \cos \alpha_{11} & \sin \varphi \cos \alpha_{12} \\ \sin \varphi \cos \alpha_{21} & \cos \varphi \cos \alpha_{22} \end{pmatrix}, \quad B = \begin{pmatrix} \cos \varphi \sin \alpha_{11} & \sin \varphi \sin \alpha_{12} \\ \sin \varphi \sin \alpha_{21} & \cos \varphi \sin \alpha_{22} \end{pmatrix}.$$

Thus the real representation  $U'$  of  $U$  is given by

$$U' = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with respect the canonical basis  $e_1, J e_1, e_2$  and  $J e_2$  of  $C^2$ . We now put  $X = e_1$  and  $Y = J e_1 + \sqrt{1-a^2} e_2$  ( $a = \cos \theta$ ). Then  $X$  and  $Y$  span a  $\theta$ -holomorphic section and  $\angle(\{X\}, \{Y\}) = \theta$ , where  $\{X\}$  is a holomorphic plane spanned by  $X$  and  $JX$ . Moreover, we have

$$X' := U'X = p_1X + p_2Y + p'_1JX + p'_2JY,$$

$$Y' := U'Y = q_1X + q_2Y + q'_1JX + q'_2JY,$$

where

$$p_1 = \cos \varphi \cos \alpha_{11} - \frac{a}{b} \sin \varphi \sin \alpha_{21}, \quad p_2 = \frac{1}{b} \sin \varphi \cos \alpha_{21},$$

$$p'_1 = \cos \varphi \sin \alpha_{11} - \frac{a}{b} \sin \varphi \cos \alpha_{21}, \quad p'_2 = \frac{1}{b} \sin \varphi \sin \alpha_{21},$$

$$q_1 = \sin \varphi \left( b \cos \alpha_{12} + \frac{a^2}{b} \cos \alpha_{21} \right) - a \cos \varphi (\sin \alpha_{11} - \sin \alpha_{22}),$$

(3.2)

$$q_2 = \cos \varphi \cos \alpha_{22} - \frac{a}{b} \sin \varphi \sin \alpha_{21},$$

$$q'_1 = \sin \varphi \left( b \sin \alpha_{12} + \frac{a^2}{b} \sin \alpha_{21} \right) + a \cos \varphi (\cos \alpha_{11} - \cos \alpha_{22}),$$

$$q'_2 = \cos \varphi \sin \alpha_{22} + \frac{a}{b} \sin \varphi \cos \alpha_{21}, \quad \text{with } b = \sqrt{1-a^2}.$$

We can prove the following Lemma 3.1 by straightforward computations.

LEMMA 3.1 *Let  $X, Y$  be orthonormal vectors such that  $\langle JX, Y \rangle = a$  and*

$$X' = p_1 X + p_2 Y + p'_1 JX + p'_2 JY, \quad Y' = q_1 X + q_2 Y + q'_1 JX + q'_2 JY$$

be orthonormal in a way that  $\langle JX', X' \rangle = a$ . Then the following equations (3.3)  $\sim$  (3.6) hold.

$$\begin{aligned}
(3.3) \quad & \overline{\langle \bar{R}(\bar{X}', \bar{Y}') \bar{Y}', \bar{X}' \rangle} = (p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)^2 \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{Y}, \bar{X} \rangle} \\
& + (p'_1 q_1 - p_1 q'_1)^2 \overline{\langle \bar{R}(\bar{X}, J\bar{X}) J\bar{X}, \bar{X} \rangle} + (p'_2 q_2 - p_2 q'_2)^2 \overline{\langle \bar{R}(\bar{Y}, J\bar{Y}) J\bar{Y}, \bar{Y} \rangle} \\
& + 2(p'_1 q_1 - p_1 q'_1)(p'_2 q_2 - p_2 q'_2) \overline{\langle \bar{R}(\bar{X}, J\bar{X}) J\bar{Y}, \bar{Y} \rangle} \\
& + 2(p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1)(p_1 q'_1 - p'_1 q_1) \overline{\langle \bar{R}(\bar{X}, J\bar{X}) J\bar{X}, \bar{Y} \rangle} \\
& + 2(p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1)(p_2 q'_2 - p'_2 q_2) \overline{\langle \bar{R}(\bar{Y}, J\bar{Y}) J\bar{Y}, \bar{X} \rangle} \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_1 q'_1 - p'_1 q_1) \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{X}, J\bar{X} \rangle} \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_2 q'_2 - p'_2 q_2) \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{Y}, J\bar{Y} \rangle} \\
& + (p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1)^2 \overline{\langle \bar{R}(\bar{X}, J\bar{Y}) J\bar{Y}, \bar{X} \rangle} \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_1 q'_2 + p_2 q'_1 - p'_1 q_2 - p'_2 q_1) \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{X}, J\bar{Y} \rangle},
\end{aligned}$$

$$\begin{aligned}
(3.4) \quad & \|\bar{X}'\|^2 \|\bar{Y}'\|^2 - \overline{\langle \bar{X}', \bar{Y}' \rangle}^2 = (p'_1 q_1 - p_1 q'_1)^2 \|\bar{X}\|^4 + (p'_2 q_2 - p_2 q'_2)^2 \|\bar{Y}\|^4 \\
& + \{(p_1 q_2 - p_2 q_1)^2 + (p_1 q'_2 - p'_2 q_1)^2 + (p'_1 q_2 - p_2 q'_1)^2 \\
& + (p'_1 q'_2 - p'_2 q'_1)\} \|\bar{X}\|^2 \|\bar{Y}\|^2 \\
& + 2(p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1)(p_1 q'_1 - p'_1 q_1) \|\bar{X}\| \overline{\langle \bar{X}, \bar{Y} \rangle} \\
& + 2(p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1)(p_2 q'_2 - p'_2 q_2) \|\bar{Y}\| \overline{\langle \bar{X}, \bar{Y} \rangle} \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_1 q'_1 - p'_1 q_1) \|\bar{X}\| \overline{\langle \bar{X}, J\bar{Y} \rangle} \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_2 q'_2 - p'_2 q_2) \|\bar{Y}\| \overline{\langle \bar{X}, J\bar{Y} \rangle} \\
& + \{4(p_1 p_2 + p'_1 p'_2)(q_1 q_2 + q'_1 q'_2) - (p_1 q_2 + p_2 q_1 + p'_1 q'_2 + p'_2 q'_1)^2\} \overline{\langle \bar{X}, \bar{Y} \rangle}^2 \\
& + \{4(p_1 p'_2 - p'_1 p_2)(q_1 q'_2 - q'_1 q_2) - (p_1 q'_2 - p'_1 q_2 + p'_2 q_1 - p_2 q'_1)^2\} \overline{\langle \bar{X}, J\bar{Y} \rangle}^2 \\
& + 2(p_2 q_1 - p_1 q_2 + p'_2 q'_1 - p'_1 q'_2)(p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1) \overline{\langle \bar{X}, \bar{Y} \rangle} \overline{\langle \bar{X}, J\bar{Y} \rangle},
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad & \overline{\langle \bar{R}(\bar{X}' J\bar{X}') J\bar{Y}', \bar{Y}' \rangle} = (p_1^2 + p_1'^2)(q_1^2 + q_1'^2) \overline{\langle \bar{R}(\bar{X}, J\bar{X}) J\bar{X}, \bar{X} \rangle} \\
& + (p_2^2 + p_2'^2)(q_2^2 + q_2'^2) \overline{\langle \bar{R}(\bar{Y}, J\bar{Y}) J\bar{Y}, \bar{Y} \rangle} + 4(p_1 p_2 + p'_1 p'_2)(q_1 q_2 \\
& + q'_1 q'_2) \overline{\langle \bar{R}(\bar{X}, J\bar{Y}) J\bar{Y}, \bar{X} \rangle} + 4(p_1 p'_2 - p_2 p'_1)(q_1 q'_2 - q_2 q'_1) \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{Y}, \bar{X} \rangle} \\
& + 2\{p_1^2 + p_1'^2\}(q_1 q_2 + q'_1 q'_2) + (q_1^2 + q_1'^2)(p_1 p_2 + p'_1 p'_2) \overline{\langle \bar{R}(\bar{X}, J\bar{X}) J\bar{X}, \bar{Y} \rangle} \\
& + 2\{p_2^2 + p_2'^2\}(q_1 q_2 + q'_1 q'_2) + (q_2^2 + q_2'^2)(p_1 p_2 + p'_1 p'_2) \overline{\langle \bar{R}(\bar{Y}, J\bar{Y}) J\bar{Y}, \bar{X} \rangle} \\
& + 2\{p_1^2 + p_1'^2\}(q_1 q'_2 - q_2 q'_1) + (q_1^2 + q_1'^2)(p_1 p'_2 - p_2 p'_1) \overline{\langle \bar{R}(\bar{X}, \bar{Y}) \bar{X}, J\bar{X} \rangle}
\end{aligned}$$

$$\begin{aligned}
& + 2\{(p_2^2 + p_2'^2)(q_1q_2' - q_2q_1') + (q_2^2 + q_2'^2)(p_1p_2' - p_2p_1')\} \langle \overline{R(\overline{X}, \overline{Y})\overline{Y}, J\overline{Y}} \rangle \\
& + \{(p_1^2 + p_1'^2)(q_2^2 + q_2'^2) + (p_2^2 + p_2'^2)(q_1^2 + q_1'^2)\} \langle \overline{R(\overline{X}, J\overline{X}), J\overline{Y}, \overline{Y}} \rangle \\
& + 4\{(p_1p_2 + p_1'p_2')(q_1q_2' - q_2q_1') + (q_1q_2 + q_1'q_2')(p_1p_2' - p_2p_1')\} \langle \overline{R(\overline{X}, \overline{Y})\overline{X}, J\overline{Y}} \rangle \\
(3.6) \quad & \|\overline{X}'\|^2 \|\overline{Y}'\|^2 = (p_1^2 + p_1'^2)(q_1^2 + q_1'^2) \|\overline{X}\|^4 + (p_2^2 + p_2'^2)(q_2^2 + q_2'^2) \|\overline{Y}\|^4 \\
& + 4(p_1p_2 + p_1'p_2')(q_1q_2 + q_1'q_2') \langle \overline{X}, \overline{Y} \rangle^2 + 4(p_1p_2' - p_2p_1')(q_1q_2' \\
& - q_2q_1') \langle \overline{X}, J\overline{Y} \rangle^2 + 2\{(p_1^2 + p_1'^2)(q_1q_2 + q_1'q_2') + (q_1^2 + q_1'^2)(p_1p_2 \\
& + p_1'p_2')\} \|\overline{X}\|^2 \langle \overline{X}, \overline{Y} \rangle + 2\{(p_2^2 + p_2'^2)(q_1q_2 + q_1'q_2') + (q_2^2 + q_2'^2)(p_1p_2 \\
& + p_1'p_2')\} \|\overline{Y}\|^2 \langle \overline{X}, \overline{Y} \rangle + 2\{(p_1^2 + p_1'^2)(q_1q_2' - q_2q_1') + (q_1^2 + q_1'^2)(p_1p_2' \\
& - p_2p_1')\} \|\overline{X}\|^2 \langle \overline{X}, J\overline{Y} \rangle + 2\{(p_2^2 + p_2'^2)(q_1q_2' - q_2q_1') + (q_2^2 + q_2'^2)(p_1p_2' \\
& - p_2p_1')\} \|\overline{Y}\|^2 \langle \overline{X}, J\overline{Y} \rangle + \{(p_1^2 + p_1'^2)(q_2^2 + q_2'^2) + (p_2^2 + p_2'^2)(q_1^2 \\
& + q_1'^2)\} \|\overline{X}\|^2 \|\overline{Y}\|^2 + 4\{(p_1p_2 + p_1'p_2')(q_1q_2' - q_2q_1') + (q_1q_2 + q_1'q_2') \\
& (p_1p_2' - p_2p_1')\} \langle \overline{X}, \overline{Y} \rangle \langle \overline{X}, J\overline{Y} \rangle.
\end{aligned}$$

The equations (3.3) and (3.5) will be established if  $\langle \overline{R(\overline{X}', \overline{Y}')\overline{Y}', \overline{X}'} \rangle$ ,  $\langle \overline{R(\overline{X}', J\overline{X}') J\overline{Y}', \overline{Y}'} \rangle$ ,  $\langle \overline{R(\overline{X}, \overline{Y})\overline{Y}, \overline{X}} \rangle$ , ...and etc. are substituted by the corresponding  $\langle R(X', Y')Y', X' \rangle$ ,  $\langle R(X', JX')JY', Y' \rangle$ ,  $\langle R(X, Y)Y, X \rangle$ ...and etc.

LEMMA 3.2. *If the equation*

$$\begin{aligned}
(3.7) \quad & \overline{D} + \overline{E} \sin 2\varphi + \overline{F} \cos 2\varphi + \overline{G} \sin 4\varphi + \overline{H} \cos 4\varphi \\
& = (D + E \sin 2\varphi + F \cos 2\varphi + G \sin 4\varphi + H \cos 4\varphi)(a + b \sin 2\varphi \\
& \quad + c \cos 2\varphi + d \sin 4\varphi + e \cos 4\varphi)
\end{aligned}$$

holds for any  $\left(0 \leq \varphi \leq \frac{\pi}{2}\right)$ , then

$$\begin{aligned}
(3.7) \quad & (a) \quad Ee + Hb + Fd + Gc = 0, \quad (b) \quad Fe + Hc - Gb - Ed = 0, \\
& (c) \quad Ge + Hd = 0, \quad (d) \quad He - Gd = 0.
\end{aligned}$$

*Proof.* Expanding the right hand side of (C) and putting the coefficients of  $\sin 6\varphi$ ,  $\cos 6\varphi$ ,  $\sin 8\varphi$  and  $\cos 8\varphi$  are equal to zero, we have (3.7, a)~(3.7, d) respectively.

#### 4. Proof of Theorems.

*Proof of Theorem I).* Putting  $\alpha_{11} = -\frac{\pi}{4}$ ,  $\alpha_{12} = \frac{\pi}{4}$  and  $\alpha_{22} = -\alpha$  in  $\alpha_{11} - \alpha_{12} - \alpha_{21} + \alpha_{22} = (2n+1)\pi$ , we have  $\cos \alpha_{21} = \sin \alpha$  and  $\sin \alpha_{21} = \cos \alpha$ . Substituting these into (3.2) we have

$$\begin{aligned}
 p_1 &= \frac{1}{\sqrt{2}} \cos \varphi + \frac{a}{b} \cos \alpha \sin \varphi, & p'_1 &= -\frac{1}{\sqrt{2}} \cos \varphi - \frac{a}{b} \sin \alpha \sin \varphi, \\
 p_2 &= \frac{1}{b} \sin \alpha \sin \varphi, & p'_2 &= \frac{1}{b} \cos \alpha \sin \varphi, \\
 q_1 &= \left( \frac{b}{\sqrt{2}} + \frac{a^2}{b} \sin \alpha \right) \sin \varphi + a \left( \frac{1}{\sqrt{2}} - \sin \alpha \right) \cos \varphi, \\
 q'_1 &= \left( \frac{b}{\sqrt{2}} + \frac{a^2}{b} \cos \alpha \right) \sin \varphi + a \left( \frac{1}{\sqrt{2}} - \cos \alpha \right) \cos \varphi, \\
 q_2 &= \cos \alpha \cos \varphi - \frac{a}{b} \cos \alpha \sin \varphi, \\
 q'_2 &= -\sin \alpha \cos \varphi + \frac{a}{b} \sin \alpha \sin \varphi, \quad \text{with } b = \sqrt{1-a^2}.
 \end{aligned}
 \tag{4.1}$$

Substituting (4.1) into (3.3) and (3.4) and using the equations thus obtained, we have from (A) an equation of the form (C) with coefficients  $E, G, H$  and  $d$  given respectively by the following equations:

$$\begin{aligned}
 E &= \frac{a}{2b^3} \{ -4a^2 K(X, Y) + (b^2 - a^2)H(X) - H(Y) - 2a^2 \langle R(X, JX)JY, Y \rangle \\
 &\quad - 4a^3 \langle R(X, Y)X, JX \rangle - 4a \langle R(X, Y)Y, JY \rangle \} + \frac{a}{\sqrt{2}b^3} \{ 2aK(X, Y) + aH(X) \\
 &\quad + a \langle R(X, JX)JY, Y \rangle + \langle R(X, JX)JX, Y \rangle + \langle R(Y, JY)JY, X \rangle \\
 &\quad + (1 + 2a^2) \langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle + 2a \langle R(X, Y)X, JY \rangle \} \sin \alpha \\
 &\quad + \frac{a^2}{\sqrt{2}b^3} \{ 2aK(X, Y) + aH(X) + a \langle R(X, JX)JY, Y \rangle - \langle R(X, JX)JX, Y \rangle \\
 &\quad - \langle R(Y, JY)JY, X \rangle + (2a^2 + 1) \langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle \\
 &\quad - 2a \langle R(X, Y)X, JY \rangle \} \cos \alpha,
 \end{aligned}
 \tag{4.2}$$

$$\begin{aligned}
 G &= \frac{a}{4b^3} \{ 2(2a^2 - b^2)K(X, Y) + (1 - 6a^2b^2)H(X) + H(Y) - 2(b^2 \\
 &\quad - a^2) \langle R(X, JX)JY, Y \rangle - 4a(2b^2 - a^2) \langle R(X, Y)X, JX \rangle + 4a \langle R(X, Y)Y, JY \rangle \\
 &\quad - 2b^2 \langle R(X, JY)JY, X \rangle \} + \frac{b^2 - a^2}{2\sqrt{2}b^3} \{ 2aK(X, Y) - a(b^2 - a^2)H(X) \\
 &\quad + a \langle R(X, JX)JY, Y \rangle + (a^2 - b^2) \langle R(X, JX)JX, Y \rangle + \langle R(Y, JY)JY, X \rangle \\
 &\quad + (3a^2 - b^2) \langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle + 2a \langle R(X, Y)X, JY \rangle \} \sin \alpha \\
 &\quad + \frac{b^2 - a^2}{2\sqrt{2}b^3} \{ 2aK(X, Y) + a(a^2 - b^2)H(X) + a \langle R(X, JX)JY, Y \rangle
 \end{aligned}
 \tag{4.3}$$

$$\begin{aligned}
& -(a^2-b^2)\langle R(X, JX)JX, Y \rangle - \langle R(Y, JY)JY, X \rangle + (3a^2-b^2)\langle R(X, Y)X, JX \rangle \\
& + \langle R(X, Y)Y, JY \rangle - 2a\langle R(X, Y)X, JY \rangle \} \cos \alpha + \frac{a}{2b} \{-K(X, Y) - a^2H(X) \\
& - 2a\langle R(X, Y)X, JX \rangle + \langle R(X, JY)JY, X \rangle \} \sin 2\alpha + \frac{a}{b} \{a\langle R(X, JX)JX, Y \rangle \\
& + \langle R(X, Y)X, JY \rangle \} \cos 2\alpha,
\end{aligned}$$

$$\begin{aligned}
(4.4) \quad H = & \frac{a^2-b^2}{4b^4} \{(2a^2-b^2)K(X, Y) + \frac{1}{2}(1-6a^2b^3)H(X) + \frac{1}{2}H(Y) \\
& + (a^2-b^2)\langle R(X, JX)JY, Y \rangle + 2a(a^2-2b^2)\langle R(X, Y)X, JX \rangle \\
& + 2a\langle R(X, Y)Y, JY \rangle - b^2\langle R(X, JY)JY, X \rangle \} + \frac{a}{\sqrt{2}b^2} \{2aK(X, Y) \\
& + a(a^2-b^2)H(X) + a\langle R(X, JX)JY, Y \rangle + (a^2-b^2)\langle R(X, JX)JX, Y \rangle \\
& + \langle R(Y, JY)JY, X \rangle + (3a^2-b^2)\langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle \\
& + 2a\langle R(X, Y)X, JY \rangle \} \sin \alpha + \frac{a}{\sqrt{2}b^2} \{2aK(X, Y) + a(a^2-b^2)H(X) \\
& + a\langle R(X, JX)JY, Y \rangle - (a^2-b^2)\langle R(X, JX)JX, Y \rangle - \langle R(Y, JY)JY, X \rangle \\
& + (3a^2-b^2)\langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle - 2a\langle R(X, Y)X, JY \rangle \} \cos \alpha \\
& + \frac{b^2-a^2}{4b^2} \{K(X, Y) + a^2H(X) + 2a\langle R(X, Y)X, JX \rangle \\
& - \langle R(X, JY)JY, X \rangle \} \sin 2\alpha + \frac{a^2-b^2}{2b^2} \{a\langle R(X, JX)JX, Y \rangle \\
& + \langle R(X, Y)X, JY \rangle \} \cos 2\alpha,
\end{aligned}$$

$$\begin{aligned}
(4.5) \quad d = & \frac{a}{4b^3} \{(1-6a^2b^2)\|\bar{X}\|^4 + \|\bar{Y}\|^4 + 2(a^2-b^2)\|\bar{X}\|^2\|\bar{Y}\|^2 - 4a(2b^2-a^2)\langle \bar{X}, \bar{J}\bar{Y} \rangle\|\bar{X}\|^2 \\
& + 4a\langle \bar{X}, \bar{J}\bar{Y} \rangle\|\bar{Y}\|^2 - 2b^2\langle \bar{X}, \bar{Y} \rangle^2 + 2(2a^2-b^2)\langle \bar{X}, \bar{J}\bar{Y} \rangle^2 \} + \langle \sin \alpha \rangle \sin \alpha \\
& + \langle \cos \alpha \rangle \cos \alpha + \frac{a}{2b} \{-a^2\|\bar{X}\|^4 - 2a\|\bar{X}\|^2\langle \bar{X}, \bar{J}\bar{Y} \rangle + \langle \bar{X}, \bar{Y} \rangle^2 - \langle \bar{X}, \bar{J}\bar{Y} \rangle^2 \} \sin 2\alpha \\
& + \frac{a}{b} \{a\|\bar{X}\|^2\langle \bar{X}, \bar{Y} \rangle + \langle \bar{X}, \bar{Y} \rangle\langle \bar{X}, \bar{J}\bar{Y} \rangle \} \cos 2\alpha,
\end{aligned}$$

Where we denote by  $\langle \sin \alpha \rangle$  the coefficient of  $\sin \alpha$  and by  $\langle \cos \alpha \rangle$  the coefficient of  $\cos \alpha$ .

However because of Lemma 3.2, we have the equations (3.7, a)~(3.7, d) con-

taining these coefficients. These equations imply that there occur only two cases, that is, Case 1, where  $d=e=0$ , and Case 2, where  $d^2+e^2\neq 0$  (i.e  $d\neq 0$  or  $e\neq 0$ ).

Case 1). In this case, using (4.5), we have

$$(4.6) \quad (1-6a^2b^2)\|\bar{X}\|^4+\|\bar{Y}\|^4+2(a^2-b^2)\|\bar{X}\|^2\|\bar{Y}\|^2+4a\|\bar{Y}\|^2\langle\bar{X}, J\bar{Y}\rangle \\ -4a(2b^2-a^2)\|\bar{X}\|\langle\bar{X}, J\bar{Y}\rangle-2b^2\langle\bar{X}, \bar{Y}\rangle^2+(4a^2-2b^2)\langle\bar{X}, J\bar{Y}\rangle^2=0,$$

$$(4.7) \quad a\|\bar{X}\|^2+\langle\bar{X}, J\bar{Y}\rangle^2-\langle\bar{X}, \bar{Y}\rangle^2=0,$$

$$(4.8) \quad \langle\bar{X}, \bar{Y}\rangle(a\|\bar{X}\|^2+\langle\bar{X}, J\bar{Y}\rangle)=0.$$

Consequently we have  $\langle\bar{X}, \bar{Y}\rangle=0$ ,  $\|\bar{X}\|=\|\bar{Y}\|$  and  $\langle J\bar{X}, \bar{Y}\rangle=a\|\bar{X}\|^2$ , from which  $\frac{\langle J\bar{X}, \bar{Y}\rangle}{\|\bar{X}\|\|\bar{Y}\|}=a$ . Therefore in Case 1  $f_*$  is conformal on the subspace spanned by  $X, Y, JX$  and  $JY$ .

Case 2). In this case, (3.7, c) and (3.7, d) imply  $G=H=0$ , and this equation implies together with (3.7, a) and (3.7, b)  $E=F=0$ . Taking account of (4.2), we have from  $E=0$ ,

$$(4.9) \quad 4a^2K(X, Y)+(a^2-b^2)H(X)+H(Y)+2a^2\langle R(X, JX)JY, Y\rangle \\ +4a^3\langle R(X, Y)X, JX\rangle+4a\langle R(X, Y)Y, JY\rangle=0,$$

$$(4.10) \quad 2aK(X, Y)+aH(X)+a\langle R(X, JX)JY, Y\rangle+\langle R(X, JX)JX, Y\rangle \\ +\langle R(Y, JY)JY, X\rangle+(1+2a^2)\langle R(X, Y)X, JX\rangle \\ +\langle R(X, Y)Y, JY\rangle+2a\langle R(X, Y)X, JY\rangle=0$$

and

$$(4.11) \quad 2aK(X, Y)+aH(X)+a\langle R(X, JX)JY, Y\rangle-\langle R(X, JX)JX, Y\rangle \\ +\langle R(Y, JY)JY, X\rangle+(1+2a^2)\langle R(X, Y)X, JX\rangle+\langle R(X, Y)Y, JY\rangle \\ -2a\langle R(X, Y)X, JY\rangle=0.$$

The equations (4.10) and (4.11) are equivalent respectively to

$$(4.10)' \quad 2aK(X, Y)+aH(X)+a\langle R(X, JX)JY, Y\rangle+(1+2a^2)\langle R(X, Y)X, JX\rangle \\ +\langle R(X, Y)Y, JY\rangle=0,$$

$$(4.11)' \quad \langle R(X, JX)JX, Y\rangle+\langle R(Y, JY)JY, X\rangle+2a\langle R(X, Y)X, JY\rangle=0.$$

Similarly, (4.3) and  $G=0$  imply

$$(4.12) \quad 4a^2K(X, Y)+(1-6a^2b^2)H(X)+H(Y)-2(2b^2-a^2)\langle R(X, JX)JY, Y\rangle \\ -4a(2b^2-a^2)\langle R(X, Y)X, JX\rangle+4a\langle R(X, Y)Y, JY\rangle=0,$$

$$(4.13) \quad K(X, Y) + a^2 H(X) + 2a \langle R(X, Y)X, JX \rangle - \langle R(X, JY)JY, X \rangle = 0,$$

$$(4.14) \quad a \langle R(X, JX)JX, Y \rangle + \langle R(X, Y)X, JY \rangle = 0.$$

Finally, (4.4) and  $H=0$  imply

$$(4.15) \quad 2a K(X, Y) - a(b^2 - a^2)H(X) + a \langle R(X, JX)JY, Y \rangle \\ + (3a^2 - b^2) \langle R(X, Y)X, JX \rangle + \langle R(X, Y)Y, JY \rangle = 0,$$

$$(4.16) \quad (a^2 - b^2) \langle R(X, JX)JX, Y \rangle + \langle R(Y, JY)JY, X \rangle + 2a \langle R(X, Y)X, JY \rangle = 0.$$

Using the equalities (4.10)'~(4.16) obtained above, we are now going to derive some equations containing curvatures. First from (4.11), (4.14) and (4.16) we have

$$(4.17) \quad \langle R(X, JX)JX, Y \rangle = \langle R(Y, JY)JY, X \rangle = \langle R(X, Y)X, JY \rangle = 0.$$

From (4.10)' and (4.15) we obtain

$$(4.18) \quad aH(X) = -\langle R(X, Y)X, JX \rangle.$$

From (4.9) and (4.12) we get

$$(1 - 3a^2)H(X) - 4a \langle R(X, Y)X, JX \rangle - 2 \langle R(X, JX)JY, Y \rangle = 0.$$

Substituting (4.18) into this equation, we have

$$(4.19) \quad \langle R(X, JX)JY, Y \rangle = \frac{1+a^2}{2} H(X)$$

We have also, from (4.10) and (4.7),

$$a(a^2 - 1)H(X) - \langle R(X, Y)X, JX \rangle - 2a \langle R(X, JX)JY, Y \rangle - \langle R(X, Y)Y, JY \rangle = 0.$$

Substituting (4.15) and (4.16) into this equation, we obtain

$$(4.20) \quad aH(X) = -\langle R(X, Y)Y, JY \rangle,$$

which together with (4.15), implies

$$(4.21) \quad \langle R(X, Y)Y, JY \rangle = \langle R(X, Y)X, JX \rangle.$$

Substituting (4.18) and (4.19) into (4.13) we have

$$(4.22) \quad K(X, Y) = \frac{1+3a^2}{4} H(X).$$

Furthermore, substituting (4.18), (4.19), (4.20) and (4.22) into (4.9) we get

$$(4.23) \quad H(X) = H(Y).$$

Finally, from (4.19), (4.22) and the first Bianchi's identity, we have

$$(4.24) \quad \langle R(X, JY)JY, X \rangle = \frac{1-a^2}{4} H(X).$$

Therefore, using (4.17), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23) and (4.24) we shall show that the holomorphic sectional curvatures are constant for any holomorphic section belonging to the subspace spanned by  $X, Y, JX$  and  $JY$ . To prove this fact, we take an arbitrary unit vector  $Z$  such as  $Z = xX + yY + uJX + vJY$ . Then, by direct calculations and taking account of (4.17), we have

$$\begin{aligned} H(Z) &= (x^2 + u^2)H(X) + (y^2 + v^2)H(Y) + 4(vx - uy)^2K(X, Y) \\ &\quad + 4(vx^3 - x^2yu + xvu^2 - u^3y)\langle R(X, Y)X, JX \rangle \\ &\quad + 4(xvy^2 + xv^3 - y^3u - uv^2y)\langle R(X, Y)Y, JY \rangle \\ &\quad + 2(x^2 + u^2)(y^2 + v^2)\langle R(X, JX)JY, Y \rangle \\ &\quad + 4(xy + uv)^2\langle R(X, JY)JY, X \rangle. \end{aligned}$$

Therefore, since  $\|Z\|^2 = x^2 + y^2 + u^2 + v^2 + 2(yu - xv)a = 1$ ,

$$\begin{aligned} H(Z) &= H(X)\{(x^2 + y^2 + u^2 + v^2)^2 - 2(x^2 + u^2)(y^2 + v^2) \\ &\quad + (1 + 3a^2)(xv - yu)^2 - 4a(x^2 + y^2 + u^2 + v^2)(vx - yu) \\ &\quad + (1 + a^2)(x^2 + u^2)(y^2 + v^2) + (1 - a^2)(xy + uv)^2\} \\ &= H(X)\{(1 - 2(yu - vx)a)^2 + (1 + 3a^2)(xv - yu)^2 \\ &\quad - 4a(1 - 2(yu - vx)a)(vx - yu) + (a^2 - 1)(xv - yu)^2\} \\ &= H(X). \end{aligned}$$

The equation  $H(Z) = H(X)$  shows that the holomorphic sectional curvatures are constant for any holomorphic section belonging to the subspace spanned by  $X, Y, JX$  and  $JY$ . Here we recall that if  $p$  is a non-isotropic point of  $M$ , then there exists an orthogonal vectors  $e_1, \dots, e_n, Je_1, \dots, Je_n$  belonging to the tangent space  $T_p(M)$  such that

$$(*) \quad (R_{iij})^2 + (H(e_i) - H(e_j))^2 \neq 0 \quad (i \neq j)$$

(see R.S. Kulkarni, [4]), where  $R_{iij} = \langle R(e_i, Je_i)e_i, Je_j \rangle$ .

If we put  $X = e_i$  and  $Y = \cos \theta Je_i + \sin \theta e_j$ , then  $X$  and  $Y$  span a  $\theta$ -holomorphic section. Since  $H(Z) = \text{constant}$ , we get  $H(e_i) = H(e_j)$ . On the other hand, (4.17) implies

$$\begin{aligned} R_{iij} &= \frac{1}{\sin \theta} \langle R(X, JX)X, JY + \cos \theta X \rangle \\ &= -\frac{1}{\sin \theta} \langle R(X, JX)JX, Y \rangle = 0 \end{aligned}$$

because of  $e_i = X, e_j = \frac{1}{\sin \theta}(Y - \cos \theta JX)$ . This contradicts (\*). Thus at any non-isotropic points Case 2 does not occur.

Since the set of non-isotropic points is dense, the given mapping  $f$  is necessary conformal because of the conclusions derived in Case 1 and Case 2. Thus by the same argument developed in [4] we can show that  $f$  is isometric. Therefore Theorem 1 is completely proved.

*Proof of Theorem II.* As in the proof of Theorem I, substituting (4.1) into (3.5) and (3.6), and using the equations obtained, we have from (B) an equation of the form (C). Because of Lemma 3.2, we have the equations of the forms (3.7, a) (3.7, d). From (3.7, a) and (3.7, b) we have again only two cases, that is, Case 1, where  $d=e=0$  and Case 2, where  $d^2+e^2 \neq 0$  (i.e.  $d \neq 0$  or  $e \neq 0$ ).

(Case 1). In this case, we have

$$\begin{aligned} 0=d &= \left\{ \frac{a(1-6a^2b^2)}{4b^3} \|\bar{X}\|^4 + \frac{a}{4b^3} \|\bar{Y}\|^4 - \frac{a}{2b} \langle \bar{X}, \bar{Y} \rangle^2 + \frac{a(2a^2-b^2)}{2b^3} \langle \bar{X}, J\bar{Y} \rangle^2 \right. \\ &\quad \left. - \frac{a^2(2b^2-a^2)}{b^3} \|\bar{X}\|^2 \langle \bar{X}, J\bar{Y} \rangle + \frac{a^2}{b^3} \|\bar{Y}\|^2 \langle \bar{X}, J\bar{Y} \rangle - \frac{a(b^2-a^2)}{2b^3} \|\bar{X}\|^2 \|\bar{Y}\|^2 \right\} \\ &\quad + \langle \sin \alpha \rangle \sin \alpha + \langle \cos \alpha \rangle \cos \alpha + \left\{ -\frac{a^3}{2b} \|\bar{X}\|^4 + \frac{a}{2b} \langle \bar{X}, \bar{Y} \rangle - \frac{a}{2b} \langle \bar{X}, J\bar{Y} \rangle \right. \\ &\quad \left. - \frac{a^2}{b} \|\bar{X}\|^2 \langle \bar{X}, J\bar{Y} \rangle \right\} \sin 2\alpha + \left\{ \frac{a^2}{b} \|\bar{X}\|^2 \langle \bar{X}, \bar{Y} \rangle + \frac{a}{b} \langle \bar{X}, \bar{Y} \rangle \langle \bar{X}, J\bar{Y} \rangle \right\} \cos 2\alpha. \end{aligned}$$

From which we get (4.6), (4.7) and (4.8). Therefore we have  $\langle \bar{X}, \bar{Y} \rangle = 0$ ,  $\|\bar{X}\| = \|\bar{Y}\|$  and  $\langle J\bar{X}, \bar{Y} \rangle = a\|\bar{X}\|^2$  and consequently  $\frac{\langle \bar{X}, J\bar{Y} \rangle}{\|\bar{X}\| \|\bar{Y}\|} = a$ .

Case 2). In this case we have again  $G=H=E=F=0$ . Now by similar calculations as in the proof of the Theorem I, we obtain equations (4.12), (4.13), (4.14), (4.9), (4.10), (4.11), (4.15) and (4.16). Therefore by the same argument as in the Theorem I we can prove Theorem II.

*Appendix.* The following formula have been used in obtaining  $E, G, H$  and  $d$  appearing in Theorem I.

$$\begin{aligned} p_1q'_1 - p'_1q_1 &= \frac{a}{2b^2} + \left( \frac{b^2-a^2}{2b} + \frac{a^2}{\sqrt{2}b} \sin \alpha + \frac{a^2}{\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\ &\quad + \left( \frac{a(b^2-a^2)}{2b^2} - \frac{a}{\sqrt{2}} \sin \alpha - \frac{a}{\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\ p_2q'_2 - p'_2q_2 &= \frac{a}{2b^2} - \frac{1}{2b} \sin 2\varphi - \frac{a}{2b^2} \cos 2\varphi, \\ p_1q_2 - p_2q_1 &= \left( -\frac{a^2}{2b^2} - \frac{1}{2\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha \right) + \left( \frac{a}{2b} - \frac{a}{2\sqrt{2}b} \sin \alpha \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{a}{2\sqrt{2}b} \cos \alpha) \sin 2\varphi + \left( \frac{a^2}{2b^2} + \frac{1}{2\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p_1 q'_2 - p'_2 q_1 &= \left( -\frac{1}{2\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \cos \alpha \right) + \left( \frac{a}{2\sqrt{2}b} \sin \alpha - \frac{a}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\
& + \left( -\frac{1}{2\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p'_1 q_2 - p_2 q'_1 &= \left( -\frac{1}{2\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \cos \alpha \right) + \left( -\frac{a}{2\sqrt{2}b} \sin \alpha + \frac{a}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\
& + \left( \frac{1}{2\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p'_2 q'_1 - p_1 q'_2 &= \left( \frac{a^2}{2b^2} - \frac{1}{2\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha \right) + \left( -\frac{a}{2b} + \frac{a}{2\sqrt{2}b} \sin \alpha \right. \\
& \left. + \frac{a}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi + \left( -\frac{a^2}{2b^2} - \frac{1}{2\sqrt{2}} \sin \alpha - \frac{1}{2\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p_1 q'_2 - p'_1 q_2 + p_2 q'_1 - p'_2 q_1 &= \left( \frac{a}{\sqrt{2}b} \sin \alpha - \frac{a}{\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\
& + \left( -\frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p_1 p_2 + p'_1 p'_2 &= \left( \frac{1}{2\sqrt{2}b} \sin \alpha - \frac{1}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi, \\
q_1 q_2 + q'_1 q'_2 &= \left( \frac{a^2 - b^2}{2\sqrt{2}b} \sin \alpha + \frac{b^2 - a^2}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\
& + \left( -\frac{a}{\sqrt{2}} \sin \alpha + \frac{a}{\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p_1 q_2 + p_2 q_1 + p'_1 q'_2 + p'_2 q'_1 &= \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha, \\
p_1 p'_2 - p'_1 p_2 &= \frac{a}{2b^2} + \left( \frac{1}{2\sqrt{2}b} \sin \alpha + \frac{1}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi - \frac{a}{2b^2} \cos 2\varphi, \\
q_1 q'_2 - q'_1 q_2 &= \frac{a}{2b^2} + \left( -\frac{a^2}{b} + \frac{a^2 - b^2}{2\sqrt{2}b} \sin \alpha + \frac{a^2 - b^2}{2\sqrt{2}b} \cos \alpha \right) \sin 2\varphi \\
& + \left( \frac{a(b^2 - a^2)}{2b^2} - \frac{a}{\sqrt{2}} \sin \alpha - \frac{a}{\sqrt{2}} \cos \alpha \right) \cos 2\varphi, \\
p_1 q'_2 - p'_1 q_2 + p'_2 q_1 - p_2 q'_1 &= -\frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha,
\end{aligned}$$

$$p_2q_1 - p_1q_2 + p'_2q'_1 - p'_1q'_2 = \frac{a^2}{b^2} + \left(-\frac{a}{b} + \frac{a}{\sqrt{2}b} \sin \alpha + \frac{a}{\sqrt{2}b} \cos \alpha\right) \sin 2\varphi \\ - \left(\frac{a^2}{b^2} + \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{\sqrt{2}} \cos \alpha\right) \cos 2\varphi.$$

The following ones have been used in obtaining  $E, G, H$  and  $d$  appearing in Theorem II.

$$p_1^2 + q_1'^2 = \frac{1}{2b^2} + \frac{a}{\sqrt{2}b} \left(\cos \alpha + \sin \alpha\right) \sin 2\varphi + \frac{b^2 - a^2}{2b^2} \cos 2\varphi,$$

$$p_2^2 + p_2'^2 = \frac{1}{2b^2} - \frac{1}{2b^2} \cos 2\varphi,$$

$$q_2^2 + q_1'^2 = \frac{1}{2b^2} + \left(\frac{a(b^2 - a^2)}{b^2} + \frac{a(a^2 - b^2)}{\sqrt{2}b} \sin \alpha + \frac{a(a^2 - b^2)}{\sqrt{2}b} \cos \alpha\right) \sin 2\varphi \\ + \left(-\frac{(a^2 - b^2)^2}{2b^2} - \sqrt{2}a^2 \sin \alpha - \sqrt{2}a^2 \cos \alpha\right) \cos 2\varphi,$$

$$q_2^2 + q_2'^2 = \frac{1}{2b^2} - \frac{a}{b} \sin 2\varphi + \frac{b^2 - a^2}{2b^2} \cos 2\varphi,$$

$$p_1p_2 + p'_1p'_2 = \left(\frac{1}{2\sqrt{2}b} \sin \alpha - \frac{1}{2\sqrt{2}b} \cos \alpha\right) \sin 2\varphi,$$

$$p_1p_2' - p_2p_1' = \frac{a}{2b^2} + \left(\frac{1}{2\sqrt{2}b} \sin \alpha + \frac{1}{2\sqrt{2}b} \cos \alpha\right) \sin 2\varphi - \frac{a}{2b^2} \cos 2\varphi,$$

$$q_1q_2 + q_1'q_2' = \left(\frac{a^2 - b^2}{2\sqrt{2}b} \sin \alpha - \frac{a^2 - b^2}{2\sqrt{2}b} \cos \alpha\right) \sin 2\varphi \\ + \left(-\frac{a}{\sqrt{2}} \sin \alpha + \frac{a}{\sqrt{2}} \cos \alpha\right) \cos 2\varphi,$$

$$q_1'q_2' - q_1q_2 = \frac{a}{2b^2} + \left(-\frac{a^2}{b} + \frac{a^2 - b^2}{2\sqrt{2}b} \sin \alpha + \frac{a^2 - b^2}{2\sqrt{2}b} \cos \alpha\right) \sin 2\varphi \\ + \left(\frac{a(b^2 - a^2)}{2b^2} - \frac{a}{\sqrt{2}} \sin \alpha - \frac{a}{\sqrt{2}} \cos \alpha\right) \cos 2\varphi.$$

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DEPARTMENT OF MATHEMATICS,  
JOSAI UNIVERSITY,  
SAKADO SAITAMA JAPAN