THE HARMONIC FUNCTIONS IN A HALF-PLANE AND FOURIER TRANSFORMS

By Tatsuo KAWATA.

1. Introduction. Let $f(x+iy)=f(z)$ be regular in the upper half-plane $y>0$ satisfying the condition

$$(1.1)\qquad \int_{-\infty}^{\infty}|f(x+iy)|^{p}dx \le C, \quad p>0,$$

where C is a constant independent of $y>0$. We say the class of such functions H^p. The following theorem concerning a function of H^p is well known.

Theorem A. Let $f(z)\in H^p$, $p>0$. Then

(i) $f(x+iy)$ converges to a function $f(x)$ as $y\to+0$ for almost all x, which is said the boundary function of $f(z)$,

(ii) $f(x+iy)$ converges to $f(x)$ in mean with index p, or

$$(1.2)\qquad \|f(x+iy)-f(x)\|^{p}=\int_{-\infty}^{\infty}|f(x+iy)-f(x)|^{p}dx \to 0 \quad \text{as} \quad y\to+0,$$

(iii) $\|f(x+iy)\|$ is a non-increasing function of y and (as a consequence of (ii)) $\|f(x+iy)\|$ tends to $\|f(x)\|$, and

(iv) $f(z)$ can be represented as a Poisson integral of the boundary function $f(x)$ or in other words

$$(1.3)\qquad f(x+iy)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,y}{(\xi-x)^2+y^2}\,d\xi.$$

The case $p\geq 1$ is due to E. Hille and J. D. Tamarkin(1) and the case $0<p<1$ was proved by the author(2). Analogous results for functions regular in the unit circle are well known(3).

Consider the function $f(x,y)$ harmonic in $y>0$, such that

$$(1.4)\qquad \int_{-\infty}^{\infty}|f(x+iy)|^{p}\leqq C, \quad y>0, \quad p\geqq 1,$$

C being a constant independent of $y>0$. When $p=1$, we consider the harmonic function satisfying (1.4) with $p=1$ and in addition satisfying the condition that

$$(1.5)\qquad \int_{e}|f(x,y)|\,dx<\varepsilon,$$

where ε is a given arbitrary positive number and e is any set such that $m(e)\le\delta$, ($\delta=\delta(\varepsilon)$). We shall denote the class of functions having these properties, H_a^p. In § 2, we shall prove the analogous theorem for a function of H_a^p as Theorem A.

The main arguments of Hille and Tamarkin for proving Theorem A is to transform the theorem into the one for functions regular in the unit circle, using a fact due to Gabriel(4) concerning subharmonic functions. The proof of Theorem 1 in §2 consists, on the contrary, of showing, first, the fact (iv) and then of deducing (i), (ii) and (iii) and we do not use the transformation of a half-plane into a circle. Thus it gives incidentally an another proof for analytic case $p>1$.

We can also prove the theorem by reducing to Theorem A. Indeed originally I have proved Theorem 1 in this way. Afterwards Mr. T. Ugaeri has given the proof written in this paper. With his permission I have given his proof.

S. Verblunsky(5) has proved the

Theorem B. Let $g(u)$ be a function such that

$$(1.6)\qquad \int_{-\infty}^{\infty}e^{-y|u|}|g(u)|\,du<\infty$$

for every $y>0$ and

$$(1.7)\qquad f(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y|u|}e^{ixu}g(u)\,du$$

is bounded for x and $y>0$. Then $f(x,y)$ converges to a function $f(x)$ for almost every x as $y\to+0$, and

$$(1.8)\qquad g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixt}f(x)\,dx \quad (C,1).$$

A. C. Offord(6) generalized this theorem assuming only the fact that (1.7) exists in some Cesàro sense for $y>0$ and $f(x,y)$ is bound in x and $y(>0)$.

In Verblunsky's theorem $f(x,y)$ evidently defines a harmonic function in the upper half-plane and is a consequence of the well known theorem of Fatou (transforming the half-plane into unit circle).

A. C. Offord, on the other hand, has also considered a general class of functions and treated the analogous Fourier transform problems (7).

Write

$$(1.9)\qquad f_\omega(x)=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)g(u)e^{-ixu}du$$

and suppose that if $p>1$

$$(1.10)\qquad \int_{-\infty}^{\infty}|f_\omega(x)|^{p}dx\le C,$$

C being a constant independent of ω and if $p \geq 1$, $f_\omega(x)$ satisfies (1.10) with $p=1$ and in addition

$$(1.11) \quad \int_e |f_\omega(x)|\,dx < \varepsilon,$$

for every set e such that $m(e) \leq \delta$, $\delta = \delta(\varepsilon)$. We call the class of such functions H_0^p. Offord proved that if $g(u) \in H_0^p$, $p \geq 1$, then

$$(1.12) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u)\, e^{-ixu}\,du \quad (C,1)$$

exists for almost all x and further $g(u)$ has Fourier transform $f(x)$ in L_p or

$$(1.13) \quad \lim_{T\to\infty} \int_{-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} g(u)\, e^{-ixu}\,du - f(x) \right|^p dx = 0.$$

In proving these facts, he avoids the use of harmonic functions and mainly uses weak convergence.

In § 4, we consider the more general class H_{0a}^p of functions

$$(1.14) \quad f_\omega(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) e^{-y|u|} g(u)\, e^{-ixu}\,du$$

with the condition

$$(1.15) \quad \int_{-\infty}^{\infty} |f_\omega(x,y)|^p dx \leq C, \quad p > 1,$$

C being a constant independent of $\omega (>0)$ and $y > 0$, and when $p = 1$, in addition

$$(1.16) \quad \int_e |f_\omega(x,y)|\,dx \leq \varepsilon, \quad y > 0,$$

e being any set $m(e) \leq \delta$, $\delta = \delta(\varepsilon)$.

We shall prove the Fourier transform theorem concerning H_{0a}^p.

2. The harmonic function in a half-plane. We shall prove the following theorem.

Theorem 1. If $f(x,y) \in H_a^p \ (p \geq 1)$, then

(i) $f(x,y)$ converges to a function $f(x)$ as $y \to +0$ for almost all x,

(ii) $f(x,y)$ converges to $f(x)$ in mean with index p, and hence $f(x) \in L_p$,

(iii) $\|f(x,y)\|$ tends increasingly to $\|f(x)\|$ as $y \to +0$, and

(iv) $f(x,y)$ can be represented as the Poisson integral of $f(x)$, or

$$(2.1) \quad f(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(\xi - x)^2 + y^2} f(\xi)\,d\xi.$$

Proof. We shall first that if $f(x,y) \in H_a^p \ (p \geq 1)$, then $f(x,y)$ is bounded for $y \geq y_0 \ (>0)$, $-\infty < x < \infty$, y_0 being an arbitrary but fixed positive number.

Let r be a positive fixed constant less than y_0. Since $f(x,y)$ is harmonic, we have, for $0 < \rho < r$

$$(2.2) \quad f(x,y) = \frac{1}{2\pi} \int_0^{2\pi} f(x + \rho\cos\theta,\, y + \rho\sin\theta)\,d\theta$$

from which it results

$$\frac{r^2}{2} |f(x,y)| = \int_0^r \rho\, |f(x,y)|\,d\rho$$
$$\leq \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^r |f(x+\rho\cos\theta,\, y+\rho\sin\theta)|\,\rho\,d\rho$$
$$\leq \left(\frac{1}{2\pi}\right)^{1/p} \frac{r^{2(p-1)/p}}{2^{(p-1)/p}} \left\{ \int_0^{2\pi} d\theta \cdot \int_0^r |f(x+\rho\cos\theta,\, y+\rho\sin\theta)|^p \rho\,d\rho \right\}^{1/p}.$$

Thus we have

$$|f(x,y)|^p \leq C_r \int_0^{2\pi} d\theta \int_0^r |f(x+\rho\cos\theta,\, y+\rho\sin\theta)|^p \rho\,d\rho$$
$$\leq C_r \iint_{(\xi-x)^2 + (\eta-y)^2 \leq r} |f(\xi,\eta)|^p d\xi\,d\eta$$
$$\leq C_r \int_{y-r}^{y+r} d\eta \int_{x-r}^{x+r} |f(\xi,\eta)|^p d\xi$$
$$\leq C_r \int_{y-r}^{y+r} d\eta \int_{-\infty}^{\infty} |f(\xi,\eta)|^p d\xi \leq C_r \cdot C \cdot 2r,$$

where C_r is dependent only of r. Hence $f(x,y)$ is bounded for $y \geq y_0$.

Now for fixed y_0, we consider the function

$$f^*(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi, y_0) \frac{y}{(\xi-x)^2 + y^2}\,d\xi,$$

which is evidently a harmonic function for $-\infty < x < \infty$, $y > 0$. Since $f(x, y_0)$ is bounded, setting $|f(x,y_0)| \leq M$

$$|f^*(x,y)| \leq M \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(\xi-x)^2 + y^2}\,d\xi = M.$$

And by the known fact, $f^*(x,y)$ converges $f(x,y_0)$ for almost all x. If we consider the function

$$F(x,y) = f^*(x,y) - f(x, y+y_0)$$

which is clearly harmonic for $y > y_0$ and bounded in $-\infty < x < \infty$, $y > 0$, then since $f(x, y+y_0) \to f(x,y_0)$, $F(x,y) \to 0$ as $y \to +0$. Thus by Fatou's theorem $F(x,y) = 0$ for $y > 0$,
Then we get

$$(2.3) \quad f(x, y+y_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi, y_0) \frac{y}{(\xi-x)^2 + y^2}\,d\xi$$ [8]

Now we vary y_0, and fix y. By (1.4) and (1.5) and known theorem on weak convergence, there exists a function $f(x) \in L_p$ such that

$$(2.4) \quad \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi, y_k) \frac{y}{(\xi-x)^2 + y^2}\,d\xi \to \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \frac{y}{(\xi-x)^2 + y^2}\,d\xi$$

for suitable sequence $y_k \ (y_k \to 0)$. Since the left side of (2.3) tends to

$f(x,y)$ as $y_0 \to 0$, we have

$$(2.5)\quad f(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\frac{y}{(\xi-x)^2+y^2}\,d\xi .$$

From this, (i) is trivial and (ii) is also well known [9]. (iii) is an immediate consequence of Jensen's inequality and of (1.3) for

$$\int_{-\infty}^{\infty}|f(x,y+y_0)|^p dx=\int_{-\infty}^{\infty}dx\left|\frac{1}{\pi}\int_{-\infty}^{\infty} f(x+\xi,y_0)\cdot\frac{y}{\xi^2+y^2}\,d\xi\right|^p\le\int_{-\infty}^{\infty}dx\frac{1}{\pi}\int_{-\infty}^{\infty}|f(\xi+x,y_0)|^p\frac{y}{\xi^2+y^2}\,d\xi$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{\xi^2+y^2}\,d\xi\int_{-\infty}^{\infty}|f(\xi+x,y_0)|^p dx$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{\xi^2+y^2}\,d\xi\int_{-\infty}^{\infty}|f(x,y_0)|^p dx$$
$$=\int_{-\infty}^{\infty}|f(x,y_0)|^p dx .$$

Thus the theorem is proved.

We remark that $f(x,y)$ tends to zero as $x\to\infty$ uniformly in $y\ge\delta>0$, δ being an arbitrary but fixed positive number. This is a consequence of (2.3). For if $p=1$, this is evident since Poisson kernel $y/((y-x)^2+y^2)$ is boundedly convergent to zero as $x^2+y^2\to\infty$. When $p>1$, by Jensen's inequality

$$|f(x,y+y_0)|^p\le\frac{1}{\pi}\int_{-\infty}^{\infty}|f(\xi,y_0)|^p\frac{y}{(\xi-x)^2+y^2}\,d\xi,$$

from which our assertion follows.

3. Analogue of Theorem B. First we shall prove the following theorem which is an immediate consequence of Theorem 1 and is an L_p-analogue of Verblunsky's theorem B.

Theorem 2. *Let*

$$(3.1)\quad \int_{-\infty}^{\infty}e^{-y|u|}|g(u)|\,du<\infty \text{ for every } y>0$$

and write

$$(3.2)\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y|u|}g(u)\,e^{-ixu}\,du=f(x,y).$$

If, $p>1$

$$(3.3)\quad \int_{-\infty}^{\infty}|f(x,y)|^p dx\le C,\quad y>0,$$

C *being independent of* $y>0$ *and when* $p=1$, *if in addition, for given* $\varepsilon>0$

$$\int_e|f(x,y)|\,dx\le\varepsilon$$

e *being an arbitrary set* $m(e)\le\delta$, $\delta=\delta(\varepsilon)$, *then there exists a function* $f(x)\in L_p$, *such that*

(i) $\lim_{y\to+0}f(x,y)=f(x)$ *for almost all values of* x,

(ii) $f(x,y)$ *converges in mean with index* p *to* $f(x)$,

(iii) $\|f(x,y)\|\uparrow\|f(x)\|$ *as* $y\to+0$

and

(iv) $f(x,y)$ *can be represented as Poisson integral of* $f(x)$.

This is trivial since $f(x,y)$ in (2.2) defines a harmonic function in $y>0$.

We can further show that thus gotten $f(x)$ and $g(x)$ are Fourier transform in each other.

Theorem 3. *Let* $p\ge1$. *Under the hypotheses of Theorem 2, we have*

$$(3.4)\quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(u)e^{-ixu}du\quad(C,1)$$

almost everywhere,

$$(3.5)\quad g(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{ixu}dx\quad(C,1)$$

almost everywhere.

Theorem 4. *Under the hypotheses of Theorem 2,* $f(x)$ *is the Fourier transform in* L_p *of* $g(u)$ *if* $p>1$.

The following argument is completely analogous as the one used by A. C. Offord.

Let $r(x)$ bounded and of $L_1(-\infty,\infty)$ and its Fourier transform be $s(x)$. Then we have, $f(x)$ being the one in Theorem 2,

$$\int_{-\infty}^{\infty}f(x)\,r(x)\,dx=\int_{-\infty}^{\infty}\underset{(L_p)}{\mathrm{l.i.m}}\,f(x,y)\,r(x)\,dx$$
$$=\lim_{y\to0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}r(x)\,dx\int_{-\infty}^{\infty}e^{-y|t|}e^{-ixt}g(t)\,dt$$
$$=\lim_{y\to0}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y|t|}g(t)\,dt\int_{-\infty}^{\infty}r(x)e^{-ixt}dx$$
$$(3.6)\quad =\lim_{y\to0}\int_{-\infty}^{\infty}e^{-y|t|}g(t)s(t)\,dt.$$

Now put, x being fixed,

$$r(t)=\frac{2}{\pi\omega}\,\frac{\sin^2\frac{1}{2}\omega(x-t)}{(x-t)^2},$$

$$s(u)=\frac{1}{\sqrt{2\pi}}\left(1-\frac{|u|}{\omega}\right)e^{-ixu},\quad |u|\le\omega$$
$$=0,\quad |u|>\omega.$$

Then we have by (3.6)

$$\frac{2}{\pi\omega}\int_{-\infty}^{\infty} f(u)\frac{\sin^2\frac{1}{2}\omega(x-u)}{(x-u)^2}du$$

$$=\lim_{y\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\left(1-\frac{|t|}{\omega}\right)e^{-y|t|}g(t)e^{-ixt}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\left(1-\frac{|t|}{\omega}\right)g(t)e^{-ixt}dt$$

As is well known, the left side tends to $f(x)$ almost everywhere, we get (2.4).

Before proving (3.5), we shall prove Theorem 4. If we put in (2.6)

$$s(u)=\frac{1}{\sqrt{2\pi}}e^{-ixu}, \quad |u|\le\omega$$
$$=0, \quad |u|>\omega$$

$$r(t)=\frac{1}{\pi}\frac{\sin\omega(x-t)}{x-t},$$

then we have

$$I(x,\omega)=\lim_{y\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}g(t)e^{-ixt}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}g(t)e^{-ixt}dt$$

$$(3.7)\qquad =\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\omega(x-t)}{x-t}dt.$$

($r(x)$ does not belong to L_1 but has the Fourier transform $s(x)$ in B sense [11] and (3.6) also holds). A result due to E. Hille and J. D. Tamarkin shows that (3.7) tends to $f(x)$ in mean with index $p(>1)$.

Now we shall prove (3.5). Similarly as (3.7) we have

$$(3.8)\qquad \frac{1}{\sqrt{2\pi}}\int_0^{\omega}g(t)e^{-ixt}dt=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\frac{\sin\omega(x-t)}{x-t}e^{\frac{i\omega t}{2}}dt$$

If (3.3) is satisfied, then $f(x)\in L_p$. Thus for $p>1$, putting $x=0$ in (3.8) and using Holder s inequality to the right side, we have

$$\left|\frac{1}{\sqrt{2\pi}}\int_0^{\omega}g(t)dt\right|\le\frac{1}{\pi}\left(\int_{-\infty}^{\infty}|f(t)|^p dt\right)^{1/p}\cdot\left(\int_{-\infty}^{\infty}\left|\frac{\sin\omega t}{t}\right|^q dt\right)^{1/q}$$

$$\le C\,\omega^{1/p},$$

where C is an absolute constant. Hence

$$(3.9)\qquad \left|\int_0^t g(u)du\right|\le C\,t^{1/p},\quad p>1.$$

Now since by Theorem 4, $\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}g(t)e^{-ixt}dt$ converges in mean with index p to $f(x)$, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}f(u)\left(1-\frac{|u|}{\omega}\right)e^{ixu}du$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)e^{ixu}du\,\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}g(v)e^{-iuv}dv$$

$$=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)e^{ixu}du\int_{-A}^{A}g(v)e^{-iuv}dv$$

$$=\lim_{A\to\infty}\frac{2}{\pi\omega}\int_{-A}^{A}g(v)dv\int_{-\omega}^{\omega}(\omega-|u|)e^{iu(x-v)}du$$

$$=\lim_{A\to\infty}\frac{2}{\pi\omega}\int_{-A}^{A}g(v)\frac{\sin^2\frac{1}{2}\omega(x-v)}{(x-v)^2}dv$$

$$=\frac{2}{\pi\omega}\int_{-\infty}^{\infty}g(v)\frac{\sin^2\frac{1}{2}\omega(x-v)}{(x-v)^2}dv$$

$$=\frac{2}{\pi\omega}\int_{-K}^{K}+\frac{2}{\pi\omega}\int_{-\infty}^{-K}+\frac{2}{\pi\omega}\int_{K}^{\infty}$$

$$(3.10)\qquad = I_1+I_2+I_3,$$

say. Let $p>1$. Then integration by parts shows, denoting $\int_0^t g(u)du=G(t)$

$$I_3=-\frac{2}{\pi\omega}G(K)\frac{\sin^2\frac{1}{2}\omega(x-K)}{(x-K)^2}$$
$$-\frac{2}{\pi\omega}\int_K^{\infty}G(t)\frac{d}{dt}\frac{\sin^2\frac{\omega}{2}(x-t)}{(x-t)^2}dt$$
$$=-\frac{2}{\pi\omega}G(K)\frac{\sin^2\frac{1}{2}\omega(x-K)}{(x-K)^2}$$
$$-\frac{4}{\pi\omega}\int_K^{\infty}G(t)\frac{\sin^2\frac{\omega}{2}(x-t)}{(x-t)^3}dt+\frac{1}{\pi}\int_K^{\infty}G(t)\frac{\sin\omega(x-t)}{(x-t)^2}dt.$$

Hence for $\omega>1$

$$|I_3|\le|G(K)|\frac{1}{(K-x)^2}+\frac{4}{\pi}\int_K^{\infty}t^{1/p}\frac{dt}{(t-x)^3}$$
$$+\frac{1}{\pi}\int_K^{\infty}t^{1/p}\frac{dt}{(t-x)^2}.$$

Thus taking K sufficiently large $(>x)$, we can take

$$(3.11)\qquad |I_3|<\varepsilon.$$

Similarly we may have

$$(3.12)\qquad |I_2|<\varepsilon.$$

Since I_1 tends to $g(x)$ as $\omega\to\infty$ at almost every x less than K in absolute value, and by (3.11), (3.12) and (3.10), ε being arbitrary, we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega} f(u)\left(1-\frac{|u|}{\omega}\right)e^{ixu}du \to g(x)$$

for almost all values of x. Thus (3.5) is proved for $p>1$. For $p=1$, $f(t)\in L_1(-\infty,\infty)$ and (3.8) gives for $x=0$

$$\int^{\omega} g(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_0^{\omega} du\int_{-\infty}^{\infty} e^{iut}f(t)dt$$

Hence

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixt}f(t)\,dt$$

holds almost everywhere which proves (3.5). Thus Theorem 3 is proved.

4. <u>The class H_{0a}^{p}</u>. Let $g(u)$ be a function integrable in every finite interval and put

(4.1)
$$f_\omega(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\left(1-\frac{|u|}{\omega}\right)e^{-y|u|}e^{-ixu}g(u)\,du$$

for $-\infty<x<\infty$, $y>0$ and $\omega>0$. We shall first show the following theorem.

<u>Theorem 5.</u> <u>If</u> $p\ge 1$

(4.2) $\int_{-\infty}^{\infty}|f_\omega(x,y)|^p dx \le C,\ y>0,$

<u>C being a constant independent of y, then $f_\omega(x,y)$ converges in mean with index p to a harmonic function $f(x,y)$ uniformly as $\omega\to\infty$ for every $y>y_0$, y_0 being an arbitrary but fixed constant, or in other words</u>

(4.3) $\lim_{\omega\to\infty}\int_{-\infty}^{\infty}|f_\omega(x,y)-f(x,y)|^p dx = 0$

<u>holds uniformly with respect to $y\ge y_0$. Furthermore $F_\omega(x,y)$ converges uniformly in ordinary sense to $f(x,y)$ with respect to $-\infty<x<\infty$ and $y\ge y_0$.</u>

Let $y=y_1+y_2$, where $y_1=y_0/2$. Then $y_2\ge y_0/2$. If we put

(4.4) $f_2(x,y;\omega) = \frac{1}{\sqrt{2\pi}}\int_0^{\omega}du\int_{-u}^{u}e^{-y|v|}g(v)e^{-ixv}dv$

then evidently

(4.5) $f_\omega(x,y) = \frac{1}{\omega}f_2(x,y;\omega)$

Twice integrations by parts yield that

$$\omega f_\omega(x,y) = e^{-y_1\omega}f_2(x,y_2;\omega) + 2y_1\int_0^{\omega}e^{-y_1u}f_2(x,y_2;u)\,du + y_1^2\int_0^{\omega}(\omega-u)e^{-y_1u}f_2(x,y_2;u)\,du$$
$$= \omega I_1(\omega) + \omega I_2(\omega) + \omega I_3(\omega),$$

say.

By (4.2) and (4.5), we have

$$\int_{-\infty}^{\infty}|I_1(\omega)|^p dx = e^{-y_1p\omega}\int_{-\infty}^{\infty}\left|\frac{1}{\omega}f_2(x,y_2;\omega)\right|^p dx$$

(4.6) $\le Ce^{-y_1p\omega} = Ce^{-y_0p\omega/2}$

Hence, as $\omega\to\infty$, $\int_{-\infty}^{\infty}|I_1(\omega)|^p dx$ tends to zero uniformly for $y\ge y_0$. If $p>1$, then

$$\int_{-\infty}^{\infty}|I_2(\omega)|^p dx$$
$$= \int_{-\infty}^{\infty}dx\,2y_1\frac{1}{\omega^p}\left|\int_0^{\omega}e^{-y_1u}u\frac{1}{u}f_2(x,y_2;u)\,du\right|^p$$
$$\le \int_{-\infty}^{\infty}\frac{2y_1}{\omega^p}dx\left(\int_0^{\omega}e^{-qy_1u}u^q du\right)^{p/q}\cdot\int_0^{\omega}\left|\frac{1}{u}f_2(x,y_2;u)\right|^p du,$$

where $1/p+1/q=1$. Putting

$$A_p = \left(\int_0^{\infty}e^{-qy_0u/2}u^q\,du\right)^{p/q}$$

we get

$$\int_{-\infty}^{\infty}|I_2(\omega)|^p dx$$
$$\le A_p\frac{2y_1}{\omega^p}\int_{-\infty}^{\infty}dx\int_0^{\omega}\left|\frac{1}{u}f_2(x,y_2;u)\right|^p du$$
$$= A_p\frac{2y_1}{\omega^p}\int_0^{\omega}du\int_{-\infty}^{\infty}\left|\frac{1}{u}f_2(x,y_2;u)\right|^p du$$
$$\le A_p y_0\omega^{1-p}C,$$

which converges to zero uniformly for $y\ge y_0$ as $\omega\to\infty$.

If $p=1$,

$$\int_{-\infty}^{\infty}|I_2(\omega)|\,dx \le \frac{2y_1}{\omega}\int_{-\infty}^{\infty}dx\int_0^{\omega}e^{-y_1u}u\left|\frac{1}{u}f_2(x,y_2,u)\right|\cdot$$
$$\cdot du = \frac{2y_1}{\omega}\int_0^{\omega}e^{-y_1u}u\,du\int_{-\infty}^{\infty}\left|\frac{1}{u}f_2(x,y_2;u)\right|dx$$
$$\le \frac{Cy_0}{\omega}\int_0^{\infty}e^{-y_1u}u\,du = \frac{A_1Cy_0}{\omega}.$$

Hence

(4.7) $\lim_{\omega\to\infty}\int_{-\infty}^{\infty}|I_2(\omega)|^p dx = 0$

holds uniformly for $y\ge y_0$.

Lastly, we have, for $\omega'>\omega$.

$$I_3(\omega')-I_3(\omega)$$
$$= y_1^2\int_{\omega}^{\omega'}e^{-y_1u}f_2(x,y_2,u)\,du$$
$$- y_1^2\frac{1}{\omega'}\int_0^{\omega'}ue^{-y_1u}f_2(x,y_2;u)\,du$$
$$+ y_1^2\frac{1}{\omega}\int_0^{\omega}ue^{-y_1u}f_2(x,y_2;u)\,du$$
$$= I_4(\omega,\omega') - I_5(\omega') + I_5(\omega).$$

say.

We can prove that $\int_{-\infty}^{\infty} |I_5(\omega')|^p dx = o(1)$, $\int_{-\infty}^{\infty} |I_6(\omega)|^p dx = o(1)$ uniformly for $y \ge y_0$, as $\omega' \to \infty$, $\omega \to \infty$, quite analogously as in the case of $I_2(\omega)$. We shall show here that

(4.8) $$\lim_{\omega\,\omega' \to \infty} \int_{-\infty}^{\infty} |I_4(\omega, \omega')|^p dx = 0$$

uniformly for $y \ge y_0$

$$J = \int_{-\infty}^{\infty} |I_4(\omega, \omega')|^p dx$$
$$= y_1^2 \int_{-\infty}^{\infty} dx \left| \int_{\omega}^{\omega'} e^{-y_1 u} u f_2(x, y_2; u)\, du \right|^p$$
$$= y_1^2 \int_{-\infty}^{\infty} dx \left| \int_{\omega}^{\omega'} e^{-y_1 u} u^2 \frac{1}{u} f_2(x, y_2; u)\, du \right|^p.$$

Putting $A_p(\omega, \omega') = \int_{\omega}^{\omega'} e^{-y_1 u} u^2\, du$,

we have by Jensen's inequality, $p \ge 1$,

$$J = y_1^2 \int_{-\infty}^{\infty} dx\, A_p^{1/p}(\omega,\omega') \left| \int_{\omega}^{\omega'} \frac{e^{-y_1 u} u^2}{A_p(\omega,\omega')} \frac{1}{u} f_2(x, y_2; u)\, du \right|^p$$
$$\le y_1^2 A_p^{1/p}(\omega,\omega') \int_{-\infty}^{\infty} dx \int_{\omega}^{\omega'} \frac{e^{-y_1 u} u^2}{A_p(\omega,\omega')} \left| \frac{1}{u} f_2(x, y_2; u) \right|^p du$$
$$= \frac{y_0^2}{4} A_p^{1/p}(\omega,\omega') \int_{\omega}^{\omega'} \frac{e^{-y_1 u} u^2}{A_p(\omega,\omega')} du \int_{-\infty}^{\infty} \left| \frac{1}{u} f_2(x, y_2; u) \right|^p dx$$
$$\le C\, \frac{y_0^2}{4} A_p^{1/p}(\omega, \omega')$$

which tends to zero uniformly for $y > y_0$ as $\omega, \omega' \to \infty$. Thus we have that

$$\left\{ \int_{-\infty}^{\infty} |f_\omega(x,y) - f_{\omega'}(x,y)|^p dx \right\}^{1/p}$$
$$= \left\{ \int_{-\infty}^{\infty} | I_1(\omega) + I_2(\omega) + I_3(\omega) - I_1(\omega') - I_2(\omega') - I_3(\omega')|^p dx \right\}^{1/p}$$
$$\le \left\{ \int_{-\infty}^{\infty} |I_1(\omega)|^p dx \right\}^{1/p} + \left\{ \int_{-\infty}^{\infty} |I_2(\omega)|^p dx \right\}^{1/p}$$
$$+ \left\{ \int_{-\infty}^{\infty} |I_1(\omega')|^p dx \right\}^{1/p} + \left\{ \int_{-\infty}^{\infty} |I_2(\omega')|^p dx \right\}^{1/p}$$
$$+ \left\{ \int_{-\infty}^{\infty} |I_4(\omega,\omega')|^p dx \right\}^{1/p} + \left\{ \int_{-\infty}^{\infty} |I_5(\omega')|^p dx \right\}^{1/p}$$
$$+ \left\{ \int_{-\infty}^{\infty} |I_6(\omega)|^p dx \right\}^{1/p}$$

which converges to zero uniformly for $y \ge y_0$ as $\omega, \omega' \to \infty$. Thus

(4.9) $$\int_{-\infty}^{\infty} |f_\omega(x,y) - f_{\omega'}(x,y)|^p dx = o(1)$$

as $\omega, \omega' \to \infty$, uniformly for $y \ge y_0$ (>0), from which it results that there exists a function $f(x,y)$ such that (4.3) holds.

Now since $f_\omega(x,y)$ is harmonic for $-\infty < x < \infty$, $y > 0$, $\Phi_{\omega\omega'}(x,y) = f_\omega(x,y) - f_{\omega'}(x,y)$ is also harmonic for every $\omega, \omega' > 0$ and hence we have

$$\Phi_{\omega\omega'}(x,y)$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Phi_{\omega\omega'}(x + r\cos\theta, y + r\sin\theta)\, d\theta$$

where $r \le y_0/2$, $y \ge y_0$, y_0 being an arbitrary but fixed positive number. Then

$$\frac{r^2}{2} |\Phi_{\omega\omega'}(x,y)| \le \int_0^r \rho\, |\Phi_{\omega\omega'}(x,y)|\, d\rho$$
$$\le \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^r |\Phi_{\omega\omega'}(x + \rho\cos\theta, y + \rho\sin\theta)| \rho\, d\rho$$
$$\le \left(\frac{1}{2\pi}\right)^{1/p} \frac{r^{2(p-1)/p}}{2^{(p-1)/p}} \left\{ \int_0^{2\pi} d\theta \int_0^r |\Phi_{\omega\omega'}(x + \rho\cos\theta, y + \rho\sin\theta)|^p \rho\, d\rho \right\}^{1/p}$$
$$\le C_r \left\{ \iint_{(\xi-x)^2 + (\eta-y)^2 \le r^2} |\Phi_{\omega\omega'}(\xi,\eta)|^p d\xi\, d\eta \right\}^{1/p}$$
$$\le C_r \left\{ \int_{y-r}^{y+r} d\eta \int_{x-r}^{x+r} |\Phi_{\omega,\omega'}(\xi,\eta)|^p d\xi \right\}^{1/p}$$
$$\le C_r \left\{ \int_{y-r}^{y+r} d\eta \int_{-\infty}^{\infty} |\Phi_{\omega\omega'}(\xi,\eta)|^p d\xi \right\}^{1/p}$$

Since by (4.9)

$$\int_{-\infty}^{\infty} |\Phi_{\omega,\omega'}(\xi,\eta)|^p d\xi = o(1)$$

as $\omega, \omega' \to \infty$ uniformly for $y > y_0/2$, noticing $y - r > y_0/2$, if $y > y_0$, we have

$$\lim_{\omega,\omega' \to \infty} \Phi_{\omega\omega'}(x,y) = 0$$

uniformly for $-\infty < x < \infty$ and $y \ge y_0$. Hence $F_\omega(x,y)$ converges to a function $F^*(x,y)$ uniformly as $\omega \to \infty$. And $F^*(x,y)$ is harmonic and concides with $F(x,y)$ for almost all x for every $y\ (>0)$. Thus $F^*(x,y) = F(x,y)$ for almost all x, $y\ (>0)$. Hence we may consider $F(x,y)$ to be harmonic in $y > 0$. We have completely proved the theorem.

5. <u>The Fourier transforms of a function of H_{0a}^p.</u> Let $g(u)$ be a function integrable in every finite interval and be of the class H_{0a}^p $(p \ge 1)$, or let

(5.1) $$f_\omega(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\omega} \left(1 - \frac{|u|}{\omega}\right) e^{-y|u|} g(u) e^{-ixu} du$$

satisfy the condition (1.10) for $p > 1$, and for $p = 1$ let (5.1) satisfy (1.10) with (1.11). Then by Theorem 5, $f_\omega(x,y)$ converges uniformly for $y \ge y_0$, to a harmonic function $F(x,y)$ and moreover $f_\omega(x,y)$ converges in mean with index p to $f(x,y)$ for fixed $y>0$. It is almost trivial that $f(x,y)$ belongs to H^p_a. For by Fatou's lemma,

$$\int_{-\infty}^{\infty}|f(x,y)|^p dx \le \lim_{\omega\to\infty}\int_{-\infty}^{\infty}|f_\omega(x,y)|^p dx$$
$$\le C, \quad y>0$$

and

$$\int_e |f(x,y)|\,dx \le \lim_{\omega\to\infty}\int_e |f_\omega(x,y)|\,dx \le \varepsilon.$$

Hence by Theorem 1, $f(x,y)$ converges for almost all x to a function $f(x)$ as $y\downarrow 0$, and further $f(x,y)$ converges to $f(x)$ in mean with index p. With these notations we have the following theorems.

<u>Theorem 6.</u> <u>Let</u> $g(u)\in H^p_{oa}\ (p\ge 1)$. <u>Then</u>

$$(5.3)\quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)e^{-ixu}du \quad (C,1)$$

<u>almost everywhere</u>

and

$$(5.4)\quad g(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{ixu}dx \quad (C,1)$$

<u>almost everywhere</u>.

<u>Theorem 7.</u> <u>Let</u> $g(u)\in H^p_{oa}\ (p>1)$. <u>Then</u> $f(x)$ <u>in Theorem</u> 6 <u>is the Fourier transform in</u> L_p <u>or</u>

$$\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} g(u)e^{-ixu}du$$

<u>converges to</u> $f(x)$ <u>in mean with index</u> $p\,(>1)$.

The proofs of these Theorems run similarly as in the proofs of Theorems 3 and 4.

Let $r(x)$ be bounded and of $L_1(-\infty\,\infty)$ and its Fourier transform be $s(x)$. Then we have

$$\int_{-\infty}^{\infty} f(x)r(x)\,dx = \int_{-\infty}^{\infty} \underset{y\to 0}{\mathrm{l.i.m.}}\, f(x,y)\,r(x)\,dx$$
$$= \lim_{y\to 0}\int_{-\infty}^{\infty} f(x,y)\,r(x)\,dx$$
$$= \lim_{y\to 0}\int_{-\infty}^{\infty} \underset{\omega\to\infty}{\mathrm{l.i.m.}}\, f_\omega(x,y)\,r(x)\,dx$$
$$= \lim_{y\to 0}\lim_{\omega\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r(x)\,dx\int_{-\omega}^{\omega}\Bigl(1-\frac{|u|}{\omega}\Bigr)e^{-y|u|-ixu}g(u)\,du$$
$$= \lim_{y\to 0}\lim_{\omega\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^{\omega}\Bigl(1-\frac{|u|}{\omega}\Bigr)e^{-y|u|}g(u)\,du\int_{-\infty}^{\infty} r(x)\cdot e^{-ixu}dx$$
$$(5.5)\quad = \lim_{y\to 0}\lim_{\omega\to\infty}\int_{-\omega}^{\omega}\Bigl(1-\frac{|u|}{\omega}\Bigr)e^{-y|u|}g(u)s(u)\,du.$$

If we put,

$$r(t)=\frac{2}{\pi A}\,\frac{\sin^2\frac{1}{2}A(x-t)}{(x-t)^2},$$

and

$$s(u)=\frac{1}{\sqrt{2\pi}}\Bigl(1-\frac{|u|}{A}\Bigr)e^{-ixu}, \quad -A\le u\le A,$$
$$=0,$$

then by (5.5) we get

$$\frac{2}{\pi A}\int_{-\infty}^{\infty} f(t)\,\frac{\sin^2\frac{1}{2}A(x-t)}{(x-t)^2}\,dt$$
$$=\lim_{y\to 0}\lim_{\omega\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\Bigl(1-\frac{|u|}{\omega}\Bigr)e^{-y|u|}g(u)\Bigl(1-\frac{|u|}{A}\Bigr)e^{-ixu}du$$
$$=\lim_{y\to 0}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} e^{-y|u|}g(u)\Bigl(1-\frac{|u|}{A}\Bigr)e^{-ixu}du$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\Bigl(1-\frac{|u|}{A}\Bigr)g(u)\,e^{-ixu}du$$

from which (5.3) follows.

The proofs of (5.4) and Theorem 7 can be performed quite similarly as those of (3.5) and Theorem 4, using the argument in proving (5.3) above and we shall omit the detail proofs here.

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(10) $\frac{1}{\sqrt{2\pi}}\int_{-T}^{T} r(x)\,e^{-ixt}dx$ converges boundedly to a function $S(x)$ as $T\to\infty$.

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