SOME PROPERTIES OF ASYMPTOTIC DISTRIBUTIONS

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This note contains two different problems. In §1, §2 we shall give some results similar as the one which were obtained by Kac and Steinhaus.[1] The definitions used here of asymptotic distributions are different from them, and the hypothesis in the theorem are less restrictive. In §3, we are concerned with some limit theorems.

§ 1.

Definition 1. Let $x(t)$ be a measurable function of a real variable t taking values in R^n.
For every $T>0$, we define

$$\varphi_T(E)=\frac{1}{2T}mE_t[-T\le t\le T,\ x(t)\in E],$$

E being an arbitrary Borel set in R^n.
Then $\varphi_T(E)$ is a distribution function for every fixed T.
If the distribution function φ_T tends to a distribution function φ[2] for $T\to\infty$, we say that $x(t)$ has an asymptotic distribution function φ. This definition is due to Harteman and Wintner.
Now we shall prove the theorem.

Theorem 1. If $x(t)$ has an asymptotic distribution function, then for any continuous function $f(x)$ in R^n, $f(x(t))$ has an asymptotic distribution function.

To prove the theorem we need a following lemma.

Lemma 1. A measurable function has an asymptotic distribution function φ, if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{iu\cdot x(t)}dt$$

exists, uniformly in $|u|\le C$ for any $C>0$.

This lemma is known.[3]

Proof of Theorem. For simplicity, we restrict ourselves for the case where $x(t)$ is a real function. We write

$$f(x(t))=y(t)$$

By Lemma 1, it is sufficient to prove that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{iu\cdot y(t)}dt$$

exists, uniformly in $|u|\le C$ for every $C>0$. Obviously

$$\frac{1}{2T}\int_{-T}^{T}e^{iu\cdot y(t)}dt=\int_{-\infty}^{\infty}e^{iu\cdot f(x)}d\varphi_T(x),$$

so that it suffices to show that

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}e^{iu\cdot f(x)}d\varphi_T(x)$$

exists, uniformly in $|u|\le C$.

Since $\varphi_T\to\varphi$, for arbitrarily small $\varepsilon>0$, we can choose x_0 and T_0 which satisfy following conditions:

i) $-x_0,\ x_0$ are continuity points of φ.

ii) $\{1-\varphi_T(x_0)\}+\varphi_T(-x_0)<\varepsilon$ for all $T>T_0$.

Then we have

$$\left|\left(\int_{-\infty}^{-x_0}+\int_{x_0}^{\infty}\right)e^{iu\cdot f(x)}d\varphi_T(x)\right|<\varepsilon,\quad(T>T_0)$$

$$\left|\left(\int_{-\infty}^{-x_0}+\int_{x_0}^{\infty}\right)e^{iu\cdot f(x)}d\varphi(x)\right|<\varepsilon.$$

Now, we can choose $F(x)$ which is absolutely continuous in $(-x_0,x_0)$ such that

$$|f(x)-F(x)|<\frac{\varepsilon}{C},\quad\text{for } -x_0\le x\le x_0.$$

Then

$$\left|\int_{-x_0}^{x_0}\{e^{iu\cdot f(x)}-e^{iu\cdot F(x)}\}d\varphi_T(x)\right|\le\int_{-x_0}^{x_0}|u||f(x)-F(x)|\cdot d\varphi_T(x)<\varepsilon.$$

Similarily

$$\left|\int_{-x_0}^{x_0}\{e^{iuf(x)}-e^{iu\cdot F(x)}\}d\varphi(x)\right|<\varepsilon.$$

Therefore we have

$$\left|\int_{-x_0}^{x_0}e^{iu\cdot f(x)}d\{\varphi_T(x)-\varphi(x)\}\right|\le\left|\int_{-x_0}^{x_0}\{e^{iuf(x)}-e^{iuF(x)}\}d\varphi_T(x)\right|$$
$$+\left|\int_{-x_0}^{x_0}\{e^{iu\cdot f(x)}-e^{iuF(x)}\}d\varphi(x)\right|+\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}d\{\varphi(x)-\varphi_T(x)\}\right|$$
$$\le 2\varepsilon+\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}d\{\varphi(x)-\varphi_T(x)\}\right|.$$

But the integration by parts shows that

$$\left|\int_{-x_0}^{x_0}e^{iu\cdot F(x)}d\{\varphi(x)-\varphi_T(x)\}\right|\le\left|\Big[e^{iuF(x)}\{\varphi(x)-\varphi_T(x)\}\Big]_{-x_0}^{x_0}\right|$$
$$+\left|\int_{-x_0}^{x_0}\{\varphi(x)-\varphi_T(x)\}d(e^{iuF(x)})\right|$$
$$\le 2\varepsilon+\left|\int_{-x_0}^{x_0}uF'(x)\{\varphi(x)-\varphi_T(x)\}e^{iuF(x)}dx\right|.$$

Since $\varphi_T(x)$ tends to $\varphi(x)$ boundedly, there exists a T_1 such that the last term is less than ε for $T > T_1$ and thus we get

$$\left| \int_{-x_0}^{x_0} e^{iu \cdot F(x)} d\{\varphi(x) - \varphi_T(x)\} \right| < 3\varepsilon, \quad \text{for } T > T_1.$$

Hence it results:

$$\left| \int_{-x_0}^{x_0} e^{iu \cdot f(x)} d\{\varphi_T(x) - \varphi(x)\} \right| < 5\varepsilon, \ (T > \max(T_0, T_1))$$

which proves our theorem.

§ 2.

Definition 2. Let $x(t)$, $y(t)$ be measurable, real valued functions in $(-\infty, \infty)$. If these functions satisfy the following conditions, they are called to be statistically independent.

i) A vector function $z(t) = (x(t), y(t))$ has an asymptotic distribution function, (so that each of $x(t)$ and $y(t)$ have also asymptotic distribution functions.)

ii) Let $Q = [a_1, b_1; a_2, b_2]$ be an interval in R^2, and $Q^1 = [a_1, b_1]$, $Q^2 = [a_2, b_2]$ be intervals in R^1. Whenever Q, Q^1, Q^2 are continuity intervals of distribution functions ϕ, ϕ^1, ϕ^2 respectively, ϕ, ϕ^1, ϕ^2 being asymptotic distribution functions of $z(t), x(t), y(t)$ respectively, it holds

$$\phi(Q) = \phi^1(Q^1) \cdot \phi^2(Q^2).$$

We shall prove following result which is, in some sense, a generalization of the result of Kac and Steinhaus.

Theorem 2. Let $x(t)$ and $y(t)$ be statistically independent and both bounded, that is, there exists a constant M, such that

$$|x(t)| \le M, \quad |y(t)| \le M, \qquad \text{for } -\infty < t < \infty,$$

If $f(x)$, $g(x)$ are real and continuous in the interval $[-M, M]$, then $f(x(t))$, $g(y(t))$ are also statistically independent.

To prove the theorem, we shall state a number of lemmas.

Lemma 2. If $x(t)$ and $y(t)$ are bounded real measurable functions, then in order that $x(t)$, $y(t)$ are statistically independent, it is necessarily and sufficient that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^k(t)\, y^l(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^k(t)\, dt \cdot \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} y^l(t)\, dt,$$

for any positive integers k, l.

This is known.(b)

Lemma 3. Under the hypothesis of Lemma 2,

$$f_1(t) = a_0 x^m(t) + a_1 x^{m-1}(t) + \cdots + a_m,$$

$$f_2(t) = b_0 y^n(t) + b_1 y^{n-1}(t) + \cdots + b_n$$

are statistically independent.

This is readily derived from Lemma 2, that is

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T_1}^{T_1} f_1^k(t) f_2^l(t)\, dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (a_0 x^m(t) + \cdots + a_m)^k (b_0 y^n(t) + \cdots + b_n)^l\, dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1^k(t)\, dt \cdot \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_2^l(t)\, dt$$

Lemma 4. Let $x_n(t)$, $y_n(t)$ be statistically independent and be bounded. If $x_n(t) \to x(t)$, $y_n(t) \to y(t)$ $(n \to \infty)$ uniformly in $(-\infty, \infty)$, then $x(t)$, $y(t)$ are also statistically independent.

By uniform convergence, we can prove easily the validity of conditions of Lemma 1.

Proof of Theorem 2. If we choose polynomials sequence $\{f_n(x)\}$, $\{g_n(x)\}$ such that $f_n(x) \to f(x)$, $g_n(x) \to g(x)$ holds uniformly in $[-M, M]$ then $f_n(x(t)) \to f(x(t))$, $g_n(y(t)) \to g(y(t))$ uniformly in $(-\infty, \infty)$. By Lemma 3, $f_n(x(t))$, $g_n(y(t))$ are statistically independent, and hence by Lemma 4, $f(x(t))$, $g(y(t))$ are statistically independent.

§ 3.

Definition 3. Let $x_n(t)$ and $x(t)$ be measurable in $(-\infty, \infty)$, and let

$$m E_t[-T \le t \le T, \ |x_n(t) - x(t)| > \varepsilon] \equiv D_\varepsilon^{(-T,T)}(|x_n - x|),$$

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} D_\varepsilon^{(-T,T)}(|x_n - x|) \equiv D_\varepsilon(|x_n - x|),$$

for every positive ε.

If for every positive ε, $\lim_{n\to\infty} D_\varepsilon(|x_n - x_n|) = 0$, then we say after A. Wintner that the sequence $\{x_n\}$ is convergent in relative measure to x and we write $x_n \} \to \{ x$.

Definition 4. If the function $x_n(t)$ and $x(t)$ have asymptotic distribution functions φ_n and φ respectively, and if $\varphi_n \to \varphi$ at the continuity points of the latter function then after A. Wintner, the sequence $\{x_n\}$ is said to converge to x in distribution; and we write $x_n\,(\to)\,x\ (n\to\infty)$

Theorem 3. If $x_n\{\to\}x$, $y_n\{\to\}y$, and x_n, y_n are statistically independent, then x, y are statistically independent.

To prove the theorem, we use the following lemma.

Lemma 5. If $x_n\{\to\}x$, $y_n\{\to\}y$, then $x_n+y_n\{\to\}x+y$.

Since

$$E_t(-T\le t\le T;\ x_n+y_n-x-y>\varepsilon)$$
$$= E_t(-T\le t\le T;\ x_n-x<\tfrac{\varepsilon}{2},\ x_n+y_n-x-y>\varepsilon) + E_t(-T\le t\le T;\ x_n-x\ge\tfrac{\varepsilon}{2},\ x_n+y_n-x-y>\varepsilon)$$
$$\le E_t(-T\le t\le T,\ y_n-y>\tfrac{\varepsilon}{2}) + E_t(-T\le t\le T;\ x_n-x>\tfrac{\varepsilon}{2}),$$

we have

$$\frac{1}{2T}\, m E_t[-T\le t\le T;\ x_n+y_n-x-y>\varepsilon] \le \frac{1}{2T} D^{(-T,T)}_{\frac{\varepsilon}{2}}(|x_n-x|) + \frac{1}{2T}D^{(-T,T)}_{\frac{\varepsilon}{2}}(|y_n-y|).$$

Similarily we have

$$\frac{1}{2T}\, m E_t[-T\le t\le T;\ x_n+y_n-x-y<-\varepsilon] \le \frac{1}{2T} D^{(-T,T)}_{\frac{\varepsilon}{2}}(|x_n-x|) + \frac{1}{2T}D^{(-T,T)}_{\frac{\varepsilon}{2}}(|y_n-y|)$$

Therefore we have

$$\frac{1}{2T}D^{(-T,T)}_{\varepsilon}(|x_n+y_n-x-y|)\le \frac{1}{2T}D^{(-T,T)}_{\frac{\varepsilon}{2}}(|x_n-x|)+\frac{1}{T}D^{(-T,T)}_{\frac{\varepsilon}{2}}(|y_n-y|)$$

That is

$$D_\varepsilon(|x_n+y_n-x-y|)\le D_{\frac{\varepsilon}{2}}(|x_n-x|)+D_{\frac{\varepsilon}{2}}(|y_n-y|)$$

which proves

$$\lim_{n\to\infty} D_\varepsilon(|x_n+y_n-x-y|)=0,\quad \text{for every } \varepsilon>0.$$

The proof of Theorem 3. From a theorem of Hartman, van Kampen and A. Wintner(7) for any real constants u_1, u_2, the function $u_1x_n(t)+u_2y_n(t)$ has the asymptotic distribution function $F_n * G_n$, where F_n, G_n are asymptotic distribution functions of $u_1x_n(t)$, $u_2y_n(t)$ respectively, and $F_n * G_n$ signifies the convolution of F_n and G_n.
By the above lemma

$$u_1x_n(t)+u_2y_n(t)\{\to\}u_1x(t)+u_2y(t)$$

So from another known theorem(8) $u_1x(t)+u_2y(t)$ has an asymptotic distribution function and

$$u_1x_n(t)+u_2y_n(t)\,(\to)\,u_1x(t)+u_2y(t).$$

But from our hypothesis, $F_n\to F$, $G_n\to G$, where F and G are asymptotic distribution functions of $u_1x(t)$ & $u_2y(t)$ respectively. So we have $F_n * G_n\to F*G$. Thus $F*G$ is an asymptotic distribution function of $u_1x(t)+u_2y(t)$, which proves from the theorem of Hartman, van Kampen and A. Wintner(7) that $x(t)$ and $y(t)$ are statistically independent.

Theorem 4. If $x_n\{\to\}0$, $y_n\{\to\}y$ and x_n+y_n have asymptotic distribution functions for every n, then $x_n+y_n\{\to\}y$.

Proof. We have

$$E_t[-T\le t\le T,\ x_n(t)+y_n(t)>\alpha]$$
$$=E_t[-T\le t\le T,\ x_n(t)<\varepsilon,\ x_n+y_n>\alpha]+E_t[-T\le t\le T,\ x_n>\varepsilon,\ x_n+y_n>\alpha]$$

and so

$$\frac{1}{2T}\, mE_t[-T\le t\le T,\ x_n+y_n>\alpha]$$
$$\le\frac{1}{2T}\, mE_t[-T\le t\le T,\ y_n>\alpha-\varepsilon]+\frac{1}{2T}\, mE_t[-T\le t\le T,\ x_n>\varepsilon],$$

from which by letting $T\to\infty$, we have

$$1-\varphi_{x_n+y_n}(\alpha)\le 1-\varphi_{y_n}(\alpha-\varepsilon)+1-\varphi_{x_n}(\varepsilon),$$

where φ_x denotes the asymptotic distribution function of $x(t)$. Thus we have, by the assumptions

$$\overline{\lim_{n\to\infty}}\,(1-\varphi_{x_n+y_n}(\alpha))\le 1-\varphi_y(\alpha-\varepsilon).$$

That is

$$\underline{\lim_{n\to\infty}}\,\varphi_{x_n+y_n}(\alpha)\ge\varphi_y(\alpha-\varepsilon).$$

Since $\varepsilon>0$ is arbitrary, if α is the continuous point of φ_y, we have

$$\underline{\lim_{n\to\infty}}\,\varphi_{x_n+y_n}(\alpha)\ge\varphi_y(\alpha)$$

On the other hand, since

$$E_t[-T\le t\le T,\ x_n+y_n<\alpha]$$
$$=E_t[-T\le t\le T,\ x_n>-\varepsilon,\ x_n+y_n<\alpha]+E_t[-T\le t\le T,\ x_n<-\varepsilon,\ x_n+y_n<\alpha]$$
$$\le E_t[-T\le t\le T,\ y_n<\alpha+\varepsilon]+E_t[-T\le t\le T,\ x_n<-\varepsilon],$$

by the same argument as above, we have

$$\overline{\lim_{n\to\infty}} \varphi_{x_n+y_n}(\alpha) \leq \varphi_y(\alpha).$$

Combining this with above result, we get

$$\lim_{n\to\infty} \varphi_{x_n+y_n}(\alpha) = \varphi_y(\alpha).$$

This completes the proof.

<u>Theorem 5.</u> <u>If $x(t)$ has the unit asymptotic distribution function and $y(t)$ has an asymptotic distribution function φ_y, then $x(t)+y(t)$ has also an asymptotic distribution function φ_y.</u>

Proof. By the same way as in the proof of Theorem 4, we have

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) > \alpha]$$

$$\leq 1-\varphi_y(\alpha-\varepsilon)+1-\varphi_x(\varepsilon) = 1-\varphi_y(\alpha-\varepsilon).$$

Since $\varepsilon>0$ is arbitrary, if α is a continuity point of φ_y, then we have

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) > \alpha]$$

$$\leq 1-\varphi_y(\alpha),$$

that is

$$1-\underline{\lim_{T\to\infty}} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) < \alpha]$$

$$\leq 1-\varphi_y(\alpha),$$

or

$$\underline{\lim_{T\to\infty}} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) < \alpha] \geq \varphi_y(\alpha).$$

Analogously we have

$$\overline{\lim_{T\to\infty}} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) < \alpha] \leq \varphi_y(\alpha)$$

That is

$$\lim_{T\to\infty} \frac{1}{2T} m E_t[-T \leq t \leq T, x(t)+y(t) < \alpha] = \varphi_y(\alpha).$$

This completes the proof.

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A NOTE ON GENERATORS OF COMPACT LIE GROUPS

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H. Auerbach has obtained the following theorem [1] :

THEOREM: Let $\mathfrak{G}$ be a (connected) compact Lie group, and for any integer k let

$$M(x,y,k) = \{ p ; p = \prod_{i=1}^{k} v_i, \quad v_i = x^{n_i} \text{ when } i \text{ is odd}, \quad v_i = y^{n_i} \text{ when } i \text{ is even} \}$$

$$M(x,y) = \bigcup_{k=1}^{\infty} M(x,y,k)$$

<u>Then there exist x and y such that</u> $\mathfrak{G} = \overline{M(x,y)}$.

Here arises a question: <u>Is there</u> any integer k such that $\mathfrak{G} = \overline{M(x,y,k)}$. The affirmative answer for this question can easily be obtained. Let $f(\mathfrak{G})$ be the minimum of such k. The next problem, to determine $f(\mathfrak{G})$ for each compact Lie group, is not yet solved for the writers, but it can be seen

$$f(\mathfrak{G}) \geqq \dim(\mathfrak{G})/\mathrm{rank}(\mathfrak{G})$$

where rank$(\mathfrak{G})$ is the dimension of a maximal abelian subgroup of $\mathfrak{G}$.

This note will contain the proofs of these two propositions.

For any element x of $\mathfrak{G}$, let $T(x)$ be the abelian closed subgroup of $\mathfrak{G}$ generated by x , and put

(1) $\quad H(x,y,k) = \{ p ; p = \prod_{i=1}^{k} w_i ,$
$\quad w_i \in T(x)$ when i is odd and
$\quad w_i \in T(y)$ when i is even $\}$

(2) $\quad H(x,y) = \bigcup_{k=1}^{\infty} H(x,y,k)$

Then it is clear that

(3) $\quad H(x,y,k) \subseteq \overline{M(x,y,k)}$

If $\mathfrak{G} = \overline{M(x,y)}$ and if $T(x)$ and $T(y)$ are connected, we shall say that x and y constitute a pair of generators of $\mathfrak{G}$. The existence of such x and y is proved in [1].

(1) When $\mathfrak{G}$ is simply connected: Take a pair of generators x, y of $\mathfrak{G}$. Then $H(x,y)$ is an arc-wise connected subgroup of $\mathfrak{G}$ and everywhere dense in $\mathfrak{G}$. It follows from these that $H(x,y) = \mathfrak{G}$ (for the proof see [2]). From