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ON THE BEHAVIOUR OF MOCK θ-FUNCTIONS FOUND IN THE "LOST" NOTE BOOK

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1. Introduction

From Ramanujan's 'lost' note book Andrews and Hickerson [1] stated the seven mock theta functions $\phi(q)$, $\psi(q)$, $\rho(q)$, $\sigma(q)$, $\lambda(q)$, $\mu(q)$ and $\nu(q)$. In their paper Andrews and Hickerson [1] while studying the identities connecting these seven functions did not actually find their asymptotic behaviour in the neighbourhood of a point of a unit circle as $|q| \to 1$. Earlier Watson [6, 7] and Dragonette [3] had discussed in some detail the behaviour of the third and fifth order mock theta functions. The asymptotic behaviour of the seventh order mock theta functions has been discussed by Selberg [5]. The object of this paper is to study in detail the asymptotic behaviour of the seven mock theta functions found in the 'lost' notebook, in their 'bilateral' forms as defined in the next section below.

2. Notation and definitions

If $n \geq 0$, we define

$$(x)_n = (x;q)_n = \prod_{i=0}^{n-1}(1 - q^i x).$$

If $|q| < 1$, we let

$$(x)_\infty = (x;q)_\infty = \lim_{n\to\infty}(x)_n = \prod_{i \geq 0}(1 - q^i x)$$

and more generally

$$\begin{aligned}(x_1, \dots, x_r; q)_\infty &= (x_1)_\infty \cdots (x_r)_\infty \\ &= \prod_{i \geq 0}(1 - q^i x_1) \cdots (1 - q^i x_r).\end{aligned}$$

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For $x \neq 0$, and $|q| < 1$

$$j(x,q) = (x, q/x, q; q)_\infty = \sum_n (-1)^n q^{\binom{n}{2}} x^n.$$

If a and m are integers with $m \geq 1$, then

$$J_{a,m} = j(q^a, q^m)$$

$$\bar{J}_{a,m} = j(-q^a, q^m)$$

and

$$J_m = J_{m,3m} = (q^m; q^m)_\infty.$$

The seven mock theta function found in the 'lost' Note-book Andrews and Hickerson [1, pp. 84, 89]

$$\bar{J}_{0,3}\psi(q) = -\frac{1}{2}J_{1,2}^2 + \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1+q^{3r}}$$

$$\bar{J}_{1,3}\phi(q) = 2\sum_{r=-\infty}^{\infty} \frac{q^{r(3r+1)/2}}{1+q^{3r}}$$

$$qj(-1,q^6)\rho(q) = \frac{J_3^3 j(-q,q^2)}{J_{1,2}j(-1,q^3)} - \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)}}{1+q^{6r}}$$

$$j(-1,q^6)\sigma(q) = \frac{J_6^3 J_{1,6} j(-q,q^2)}{J_{1,2}j(-q,q^6)j(-q^2,q^6)} - \sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)}}{1+q^{6r+2}}$$

$$qj(-1,q^6)\lambda(q) = -\frac{2J_2 J_{2,12} j(q^2,q^6) j(q^4,q^6)}{J_4 j(-1,q^6)} + \frac{qJ_{1,2}\bar{J}_{3,12}}{\bar{J}_{1,4}} j(-1,q^6) + 2\sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)}}{1+q^{6r}}$$

$$2j(z,q^6)\mu(q) = \frac{2J_2 J_{6,12} j(-z,q^6) j(-q^2 z,q^6)}{J_4 j(q^2 z,q^6)} - \frac{J_{1,2}\bar{J}_{1,3}}{\bar{J}_{1,4}} j(z,q^6) - 4\sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{3r(r+1)} z^{r+1}}{(1-q^{6r+2}z)}$$

$$j(-1,q^3)\nu(q) = \frac{J_1^2 j(-1,q^3)}{J_3 j(-1,q)}[wj(-w^2,q) + w^2 j(-w,q)] + 3\sum_{r=-\infty}^{\infty} \frac{q^{3r(r+1)/2}}{1+q^{3r+1}}.$$

3. The behaviour of the mock theta-functions in the neighbourhood of the unit circle

According to Ramanujan the characteristic property of the mock theta-functions is that corresponding to each 'rational point' $q = e^{\pi i(h/k)}$ (h and k integers) of the unit circle $|q| = 1$, there exists a theta function of q whose difference from the mock theta-function is bounded when q approaches this rational point along a radius of the circle.

We prove that the 'complete' mock theta functions of sixth order possess this characteristic property.

The rational numbers h/k, when expressed as fractions in their lowest terms with h positive, can be put into three categories:

(i) h even and k odd
(ii) h and k both odd
(iii) h odd and k even.

The corresponding values of $e^{\pi i(h/k)}$ will be described as points of the first (second or third) categories on the unit circle and if $q = \rho e^{\pi i(h/k)}$, $0 \leq \rho \leq 1$, we shall say that q approaches the circle along a radius when $q \to 1$.

We shall now prove the following theorem.

THEOREM. *For approach to* $|q| = 1$ *along a radius of the first category*

$$\phi_c(q) = O(1), \quad \textit{and} \quad \psi_c(q) = O(1).$$

Proof. Anju Gupta [4, pp. 259–260] has shown that

$$\phi_c(q) = \phi(q) + \frac{2q(1+q)}{1-q} {}_3\phi_1^{(q^2)}\left[\begin{matrix} -q^2 \\ q^3 \end{matrix}, -q^3, q^2; q\right]$$

and

$$\psi_c(q) = \psi(q) + \frac{2q}{1-q} {}_3\phi_1^{(q^2)}\left[\begin{matrix} -q, & -q^2, & q^2 \\ & q^3 & \end{matrix}; q\right].$$

We shall prove the theorem taking the two functions on the right separately. We shall mainly follow the proof given by Dragonette [3, p. 479].

Let $q = \rho e^{i(h/k)}$, $R(\rho) > 0$ and let $\rho \to 1$.

$$\phi(q) = \sum_{m=0}^{\infty} \frac{q^{m^2} \prod_{r=1}^{m}(1 - q^{2r-1})}{\prod_{r=1}^{2m}(1 + q^r)}$$

$$= \sum_{m=0}^{\infty} \frac{\rho^{m^2} e^{\pi i(h/k)m^2} \prod_{r=1}^{m}(1 - \rho^{2r-1} e^{\pi i(h/k)(2r-1)})}{\prod_{r=1}^{2m}(1 + \rho^r e^{\pi i(h/k)r})}$$

Putting $m = \mu k + \nu$, we have

$$\phi(q) = \sum_{\nu=0}^{k-1}\sum_{\mu=0}^{\infty} \frac{\rho^{(\mu k+\nu)^2} e^{\pi i(h/k)(\mu k+\nu)^2} \prod_{r=1}^{\mu k+\nu}(1-\rho^{2r-1}e^{\pi i(h/k)(2r-1)})}{\prod_{r=1}^{2\mu k+2\nu}(1+\rho^r e^{\pi i(h/k)r})}$$

$$= \sum_{\nu=0}^{k-1}\sum_{\mu=0}^{\infty} a_{\nu,\mu} \text{ (say)}.$$

Then we shall show that for $\nu = 0, 1, 2, \ldots, k-1$ and $\rho > 0$, $\sum_\mu a_{\nu,\mu}$ is uniformly convergent. Now

$$\left|\frac{a_{\nu,\mu+1}}{a_{\nu,\mu}}\right| = \frac{\rho^{(\mu k+k+\nu)^2-(\mu k+\nu)^2} \prod_{r=\mu k+\nu+1}^{\mu k+\nu+k} |1-\rho^{2r-1}e^{\pi i(h/k)(2r-1)}|}{\prod_{r=2\mu k+2\nu+1}^{2\mu k+2\nu+2k} |1+\rho^r e^{\pi i(h/k)r}|}.$$

We estimate the denominator using the inequality, Andrews and Hickerson [1, p. 93], for $0 < R' \le R \le 1$ and $|z| = 1$,

$$|1+R^z| \le \sqrt{\frac{R}{R'}}\,|1+R'z|.$$

$$\prod_{r=2\mu k+2\nu+1}^{2\mu k+2\nu+2k} |1+\rho^r e^{\pi i(h/k)r}| = \prod_{r=1}^{2k} |1+\rho^{r+2\mu k+2\nu} e^{\pi i(h/k)(r+2\nu)}|$$

$$\ge \prod_{r=1}^{2k} \rho^{(r+2\mu k-1)/2} |1+\rho^{2\nu+1} e^{\pi i(h/k)(r+2\nu)}|$$

$$= \rho^{(2\mu+1)k^2-(k/2)} \prod_{r=1}^{2k} |1+\rho^{2\nu+1} e^{\pi i(h/k)(r+2\nu)}|$$

$$= \rho^{(2\mu+1)k^2-(k/2)} (1+\rho^{k(2\nu+1)})^2$$

$$\ge \rho^{(2\mu+1)k^2-(k/2)},$$

since $1+\rho^{2\nu+1}e^{\pi i(h/k)(r+2\nu)}$ runs twice through the roots of

$$(x-1)^k - \rho^{k(2\nu+1)}.$$

Now the numerator:

$$\prod_{r=\mu k+\nu+1}^{\mu k+\nu+k} |1-\rho^{2r-1} e^{\pi i(h/k)(2r-1)}|$$

$$= \prod_{r=1}^{k} |1-\rho^{2r+2\mu k+2\nu-1} e^{\pi i(h/k)(2r+2\mu k+2\nu+1)}|$$

$$= \prod_{r=0}^{k-1} |1 - \rho^{2r+2\mu k+2\nu+1} e^{\pi i(h/k)(2r+2\nu+1)}|$$

$$\leq \prod_{r=0}^{k-1} \rho^{r+\mu k-k+1} |1 - \rho^{2\nu+2k-1} e^{\pi i(h/k)(2r+2\nu+1)}|$$

$$= \rho^{k^2(\mu-(1/2))+(k/2)} \prod_{r=0}^{k-1} |(1 - \rho^{2\nu+2k-1} e^{\pi i(h/k)(2r+2\nu+1)}|$$

$$= \rho^{k^2(\mu-(1/2))+(k/2)}(1 - \rho^{k(2\nu+2k-1)}),$$

$$\leq \rho^{k^2(\mu-(1/2))+(k/2)}$$

Since $1 - \rho^{2\nu+2k-1} e^{\pi i(h/k)(2\nu+1+2r)}$ runs through the roots of

$$(x-1)^k + \rho^{k(2\nu+2k-1)}.$$

Hence

$$\left|\frac{a_{\nu,\mu+1}}{a_{\nu,\mu}}\right| \leq \rho^{2\nu k+\mu k+(k/2)} < 1$$

$$\leq \varepsilon \text{ (say) where } 0 < \varepsilon < 1.$$

Hence $\sum_\mu a_{\nu,\mu}$ is uniformly convergent.

Now

$$|\phi(q)| \leq \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{\infty} \varepsilon^\mu |a_{\nu,0}|$$

$$= \frac{1}{1-\varepsilon} \sum_{\nu=0}^{k-1} |a_{\nu,0}|$$

$$= \frac{1}{1-\varepsilon} \sum_{\nu=0}^{k-1} \frac{\rho^{\nu^2} \prod_{r=1}^{\nu} |1 - \rho^{2r-1} e^{\pi i(h/k)(2r-1)}|}{\prod_{r=1}^{2\nu} |1 + \rho^r e^{\pi i(h/k)r}|}$$

$$\leq \frac{1}{1-\varepsilon} \sum_{\nu=0}^{k-1} \frac{1}{|1 + e^{\pi i((k-1)/k)}|^{2\nu}} = O(1)$$

for fixed k as $\rho \to 1$.

To show that $\psi(q)$ is bounded. Let

$$v_n(\rho) = \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}}.$$

Then

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{2n+1}} v_n(\rho)$$

and following the argument given by Andrews and Hickerson [1, p. 96] it can shown that $\psi(q)$ is bounded.

Now the second functions on the right of the definition of $\phi_c(q)$ and $\psi_c(q)$ viz.

$$_3\phi_1^{(q^2)}\left[\begin{matrix} -q^2, -q^3, q^2 \\ q^3 \end{matrix} ; q\right] \quad \text{and} \quad _3\phi_1^{(q^2)}\left[\begin{matrix} -q, -q^2, q^2 \\ q^3 \end{matrix} ; q\right]$$

are bounded functions of q for $|q| < 1$.

Hence $\psi_c(q)$ and $\Phi_c(q)$ are uniformly convergent and bounded when q lies on a radius of the first category.

When q lies on a radius of the second category, it is evident that $-q$ lies on a radius of the first category. Hence by the results proved earlier, for radial approach to points of the second category on the unit circle

$$\psi_c(-q) = O(1) \quad \text{and} \quad \phi_c(-q) = O(1).$$

4. Transformation formulae and asymptotic expansions

We shall now construct the linear transformations of the mock theta functions and follow the method used by Watson [6, pp. 73–76]. Any substitution of the modular group can be resolved into a number of substitutions of the forms

$$\tau' = \tau + 1, \quad \tau' = -\frac{1}{\tau},$$

we shall construct the transformations which express the fourteen functions $\psi(\pm q), \ldots$ in terms of similar functions of q_1 (or powers of q_1), where q and q_1 are connected by the relations

$$q = e^{-\alpha}, \quad \alpha\beta = \pi^2, \quad q_1 = e^{-\beta}.$$

We shall first consider $\psi(q)$. By Cauchy's theorem, we have

$$\bar{J}_{0,3}\psi_c(q) + \frac{1}{2}J_{1,2}^2 = \frac{1}{2\pi i}\left\{\int_{-\infty-\iota c}^{\infty-\iota c} + \int_{\infty+\iota c}^{-\infty+\iota c}\right\} \frac{\pi}{\sin \pi z} \frac{e^{-3\alpha z(z+1)/2}}{1+e^{-3\alpha z}} dz$$

$$= \frac{1}{2\pi i}\left\{\int_{-\infty-\iota c}^{\infty-\iota c} + \int_{\infty+\iota c}^{-\infty+\iota c}\right\} \frac{\pi}{\sin \pi z} \frac{e^{-3\alpha z^2/2}}{2\cosh(3\alpha z/2)} dz$$

where c is a positive integer so small that the zeros of $\sin \pi z$ are the only poles of the integrand between the lines forming the contour. On the higher of these two lines we write

$$\frac{1}{\sin \pi z} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z},$$

so that

$$\frac{1}{2\pi i}\int_{\infty+ic}^{-\infty+ic} \frac{\pi}{\sin \pi z} \frac{e^{-3\alpha z^2/2}}{2\cosh(3\alpha z/2)} dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} 2\pi i \exp\left\{(2n+1)\pi i z - \frac{3\alpha z^2}{2}\right\} \frac{e^{3\alpha z} + e^{-3\alpha z} - 1}{e^{9\alpha z/2} + e^{-3\alpha z/2}} dz$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{-\infty+ic}^{\infty+ic} F_n(z), \quad \text{say.}$$

Now we shall calculate these integrals. The poles of $F_n(z)$ are simple poles at the points

$$z_m = \frac{(2m+1)\pi i}{9\alpha} \quad (m = -\infty, \ldots -1, 0, 1, \ldots +\infty)$$

and the residue at z_m is

$$\frac{4\pi}{9\alpha}(-1)^m \exp\left\{(2n+1)\pi i z_m - \frac{3\alpha z_m^2}{2}\right\}(2\cosh 3\alpha z_m - 1) = \lambda_{n,m}, \quad \text{say.}$$

Now, by Cauchy's theorem

$$\frac{1}{2\pi i}\left\{\int_{-\infty+ic}^{\infty+ic} - P\int_{-\infty+z_n}^{\infty+z_n}\right\} F_n(z)\, dz = \lambda_{n,0} + \lambda_{n,1} + \cdots + \lambda_{n,n-1} + \frac{1}{2}\lambda_{n,n}$$

where P denotes the "principal value" of the integral. On rearranging the repeated series, we have

$$\frac{1}{2}\lambda_{0,0} + \sum_{n=1}^{\infty}\left(\lambda_{n,0} + \lambda_{n,1} + \cdots + \lambda_{n,n-1} + \frac{1}{2}\lambda_{n,n}\right)$$

$$= \sum_{m=0}^{\infty}\left(\frac{1}{2}\lambda_{m,m} + \lambda_{m+1,m} + \lambda_{m+2,m} + \cdots\right)$$

$$= \frac{1}{2}\sum_{m=0}^{\infty}\lambda_{m,m}\frac{1 + e^{2\pi i z_m}}{1 - e^{2\pi i z_m}}$$

$$= \frac{2\pi}{9\alpha}\sum_{m=0}^{\infty}(-1)^m\left(2\cos\frac{2m+1}{3}\pi - 1\right) q_1^{(5/54)(2m+1)^2}\frac{1 + q_1^{(2/9)(2m+1)}}{1 - q_1^{(2/9)(2m+1)}}$$

$$= \frac{2\pi}{3\alpha}\sum_{p=0}^{\infty}(-1)^p q_1^{(5/6)(2p+1)^2}\frac{1 + q_1^{(2/3)(2p+1)}}{1 - q_1^{(2/3)(2p+1)}},$$

where $m = 3p+1$, the terms vanish when $m \neq 3p+1$.

Now

$$P\int_{-\infty+z_n}^{\infty+z_n} F_n(z)\,dz = P\int_{-\infty}^{\infty} F_n(z_n+x)\,dx$$

$$= P\int_{-\infty}^{\infty} 2\pi i \exp\left\{-\frac{5}{54}\frac{(2n+1)^2\pi^2}{\alpha} - \frac{3\alpha x^2}{2} - \frac{2}{3}(2n+1)\pi i x\right\}$$

$$\times \frac{\cosh\{3\alpha x + (2n+1)\pi i/3\} - 1/2}{(-1)^n i \sinh(9\alpha x/2)}\,dx$$

$$= 2\pi i(-1)^n \sin\frac{(2n+1)\pi}{3} q_1^{(5/54)(2n+1)^2}$$

$$\times \int_{-\infty}^{\infty} e^{-(3\alpha x^2/2)-(2/3)(2n+1)\pi i x}\frac{\sinh 3\alpha x}{\sinh(9\alpha x/2)}\,dx.$$

The integral has been simplified by modifying the contour to pass through the stationary point of the function

$$\exp\left\{(2n+1)\pi i z - \frac{3}{2}\alpha z^2\right\}$$

as is done in the "method of steepest descents".

The integral along the lower is evaluated by simply changing the sign of i. Hence we have

$$\bar{J}_{0,3}\psi_0(q) + \frac{1}{2}J_{1,2}^2$$

$$= \frac{4\pi}{3\alpha}\sum_{n=0}^{\infty}(-1)^n q_1^{(5/6)(2n+1)^2}\frac{1+q_1^{(2/3)(2n+1)}}{1-q_1^{(2/3)(2n+1)}}$$

$$+ \sum_{n=0}^{\infty} 4\pi i(-1)^n \frac{\sin(2n+1)\pi}{3} q_1^{(5/54)(2n+1)^2}\int_0^{\infty} e^{-3\alpha x^2/2}$$

$$\times \frac{\sinh 3\alpha x}{\sinh(9\alpha x/2)}\cos\left(\frac{2}{3}(2n+1)\pi x\right)dx$$

which is the transformation formula for $\psi_c(q)$.

We shall now consider the integral on the right hand side of the transformation formula.

Let

$$J(\alpha)=\int_0^\infty e^{-3\alpha x^2/2}\frac{\sinh 3\alpha x}{\sinh(9\alpha x/2)}\cos\left(\frac{2}{3}(2n+1)\pi x\right)dx.$$

$$=\sqrt{\frac{6\beta}{\pi}}\int_0^\infty\int_0^\infty e^{-(3/2)\beta y^2}\cos 3\pi xy\frac{\sinh 3\alpha x}{\sinh(9\alpha x/2)}\cos\left(\frac{2}{3}(2n+1)\pi x\right)dy\,dx$$

$$=\sqrt{\frac{2\beta^3}{\pi^3}}\left[\int_0^\infty e^{-(3/2)\beta y^2}\frac{\cosh\beta((y/3)+(2/27)(2n+1))}{\cosh 3\beta((y/3)+(2/27)(2n+1))}dy\right.$$

$$\left.+\int_0^\infty e^{-(3/2)\beta y^2}\frac{\cosh\beta((y/3)-(2/27)(2n+1))}{\cosh 3\beta((y/3)-(2/27)(2n+1))}dy\right]$$

$$=\sqrt{\frac{2\beta^3}{\pi^3}}[J_1(\beta)+J_2(\beta)]$$

where

$$J_1(\beta)=\int_0^\infty e^{-(3/2)\beta y^2}\frac{\cosh\beta Y_1}{\cosh 3\beta Y_1}dy$$

and

$$J_2(\beta)=\int_0^\infty e^{-(3/2)\beta y^2}\frac{\cosh\beta Y_2}{\cosh 3\beta Y_2}dy$$

where

$$Y_1=\frac{y}{3}+\frac{2}{27}(2n+1)$$

and

$$Y_2=\frac{y}{3}-\frac{2}{27}(2n+1).$$

We obtain the asymptotic expansions for $J(\alpha)$, $J_1(\alpha)$, $J_2(\alpha)$ in ascending powers of α, valid when α is small and $R(\rho)>0$.

$$J(\alpha)=\sqrt{\frac{2\pi}{27\alpha}}\left[1-\left(\frac{5}{8}\alpha+\frac{2}{27}(2n+1)^2\frac{\pi^2}{\alpha}\right)+\cdots\right]$$

$$J_1(\alpha)=\sqrt{\frac{\pi}{6\alpha}}\left[1+\frac{2}{27}\left(\sqrt{\frac{6}{\alpha}}+\frac{2}{27}(2n+1)^2\sqrt{6\alpha}+\frac{4}{3}(2n+1)\right)+\cdots\right]$$

$$J_2(\alpha)=\sqrt{\frac{\pi}{6\alpha}}\left[1+\frac{2}{27}\left(\sqrt{\frac{6}{\alpha}}+\frac{2}{27}(2n+1)^2\sqrt{6\alpha}-\frac{4}{3}(2n+1)\right)+\cdots\right]$$

Similarly we can find the transformation formula for $\phi_c(q)$ viz.

$$\frac{1}{3}J_{1,3}\phi_c(q) = \frac{4\pi}{3\alpha}\sum_{n=0}^{\infty}(-1)^n q_1^{(5/6)(2p+1)^2}\cos\frac{(2p+1)\pi}{3}\frac{1+q_1^{(2/3)(2p+1)}}{1-q_1^{(2/3)(2p+1)}}$$

$$+\sum_{n=0}^{\infty}(-1)^n 4\pi\frac{\sin(2n+1)\pi}{3}q_1^{(5/54)(2n+1)^2}\int_0^{\infty}e^{-(3/2)\alpha x^2+\alpha x}$$

$$\frac{\sinh 3\alpha x}{\sinh(9\alpha x/2)}\cos\left\{\frac{(2n+1)\pi}{9}(6x+1)\right\}dx$$

and the corresponding asymptotic expansions.

Since the other bilateral functions $\rho_c(q)$, $\sigma_c(q)$, $\lambda_c(q)$, $\mu_c(q)$ and $v_c(q)$ are related to $\psi_c(q)$ and $\phi_c(q)$ by means of the relation, Andrews and Hickerson [1, pp. 89–92], Anju Gupta [4, p. 260]

$$\lambda_c(q) = 2\rho_c(q), \quad \mu_c(q) = 2\sigma_c(q)$$

$$2\sigma_c(q) = \frac{\bar{J}_{1,2}\bar{J}_{3,6}}{J_2} - \phi_c(q^2)$$

$$\rho_c(q) = \frac{\bar{J}_{1,2}\bar{J}_{1,6}}{J_2} - q^{-1}\psi(q^2)$$

and

$$\bar{J}_{0,3}[2v_c(q) - 3\phi_c(q)] = \frac{2J_1^2\bar{J}_{0,3}}{J_3\bar{J}_{0,1}}[wj(-w^2,q) + w^2j(-w,q)]$$

it is easy to find their transformation formula and asymptotic behaviour.

5. Lemma

If

$$J(\alpha) = \int_0^{\infty}e^{-(3/2)\alpha x^2}H(\alpha x)\,dx$$

where

$$H(\alpha x) = \frac{\sinh(3\alpha x+\alpha)}{\sinh((9\alpha/2)x+(3\alpha/2))}\cos\left(\frac{2}{3}(2n\pi+1)\pi x+(2n+1)\frac{\pi}{3}\right)$$

then there exists $K > 0$ such that $|\alpha J(\alpha)| < K$ for all $R(\alpha) > 0$ i.e. $\alpha J(\alpha)$ is uniformly bounded in this half-plane.

Proof. Let $z = x+iy(x>0)$, then $H(z) < e^{-(3\alpha x/2)-(\alpha/2)}$ for large x. Let $z = \alpha x$,

$$J(\alpha) = \frac{1}{\alpha}\int_0^{\infty}e^{-(3/2)(1/\alpha)x^2}H(x)\,dx, x \text{ real}.$$

Since $R(\alpha) > 0$, we have $|e^{-(3/2)(1/\alpha)x^2}| < 1$, so

$$|\alpha J(\alpha)| < \int_0^\infty H(x)\,dx < K_1 \int_0^\infty e^{-(3\alpha x/2)-(\alpha/2)}\,dx = K \text{ (say)}$$

which proves the lemma.

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REFERENCES

[1] G. E. ANDREWS AND D. HICKERSON, Ramanujan's "lost" notebook VII: The sixth order mock theta functions, Adv. Math., **89** (1991), 60–105.

[2] G. E. ANDREWS, The Theory of Partitions, Encyclopedia Math. Appl., **2**, Addison-Wesley, Reading, M.A., 1976.

[3] L. A. DRAGONETTE, Some asymptotic formulae for the mock theta series of Ramanujan, Trans. Amer. Math. Soc., **72** (1952), 474–500.

[4] A. GUPTA, On certain Ramanujan's mock theta functions, Proc. Indian Acad. Sci. Math. Sci., **103** (1993), 257–267

[5] A. SELBERG, Über die Mock-Theta funktionen siebenter Ordnung, Arch. Math. Naturvidenskab, **41** (1938), 3–15.

[6] G. N. WATSON, The final problem: An account of the mock theta functions, J. London Math. Soc., **11** (1936), 55–80.

[7] G. N. WATSON, The mock theta functions (2), Proc. London Math. Soc. (2), **42** (1937), 274–304.

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