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MEROMORPHIC FUNCTIONS THAT SHARE TWO SETS

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Abstract

This paper studies the problem of uniqueness of meromorphic functions that share two sets and obtains some unicity theorems which improve some theorems given by H.X. Yi, G.D. Song and N. Li and other authors.

1. Introduction

In this paper, by meromorphic function we always mean a meromorphic function in the complex plane $\boldsymbol{C}$. We adopt the usual notations in the Nevanlinna theory of meromorphic functions as explained in [1]. Let h be a nonconstant meromorphic function and let S be a subset of distinct elements in $\hat{C}$. Define

$$E_h(S)=\bigcup_{a\in S}\{z\mid h(z)-a=0\},$$

where each zero of $h(z)-a=0$ with multiplicity m is repeated m times in $E_h(S)$ (see [2]). The notation $\bar{E}_h(S)$ expresses the set which contains the same points as $E_h(S)$ but without counting multiplicities (see [3]).

Throughout this paper, we assume that f and g are two nonconstant meromorphic functions, S is a subset of distinct elements in $\hat{\boldsymbol{C}}$. If $E_f(S)=E_g(S)$, we say f and g share the set S CM (counting multiplicity). If $\bar{E}_f(S)=\bar{E}_g(S)$, we say f and g share the set S IM (ignoring multiplicity). As a special case, let $S=\{a\}$, where $a\in\hat{\boldsymbol{C}}$. If $E_f(\{a\})=E_g(\{a\})$, we say f and g share the value a CM. If $\bar{E}_f(\{a\})=\bar{E}_g(\{a\})$, we say f and g share the value a IM (see [4]).

In 1976, F. Gross and C.F. Osgood proved the following theorem.

THEOREM A [5]. *Let $S_1=\{-1, 1\}$, $S_2=\{0\}$. If f and g are nonconstant entire functions of finite order such that f and g share the sets S_1 and S_2 CM, then $f\equiv\pm g$ or $f\cdot g\equiv\pm 1$.*

Next, we assume that $S_1=\{1, \omega, \dots, \omega^{n-1}\}$, $S_2=\{\infty\}$, and $S_3=\{0\}$, where $\omega=\cos(2\pi/n)+i\sin(2\pi/n)$, n is a positive integer.

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H.-X. Yi [6, 7], G. Brosch [8], K. Tohge [9], G. Jank and N. Terglane [10] and other authors dealt with the problem of uniqueness of meromorphic functions that share sets S_1, S_2 and S_3 and obtained some results that are improvements of Theorem A. Recently, H.-X. Yi proved the following result.

THEOREM B [3]. *Suppose that f and g share the sets S_j $(j=1, 2, 3)$ IM. If $n \geqq 7$, then*

$$f \equiv tg, \tag{1.1}$$

where $t^n=1$ or

$$f \cdot g \equiv s, \tag{1.2}$$

where 0 and ∞ are lacunary values of f and g, and $s^n=1$.

H.-X. Yi [11, 12], G. D. Song and N. Li [13] and other authors dealt with the problem of uniqueness of meromorphic functions that share sets S_1 and S_2. In 1994, H.-X. Yi proved the following result.

THEOREM C [11]. *Suppose that f and g share the sets S_1 and S_2 CM. If $n \geqq 7$, then f and g satisfy* (1.1) *or* (1.2).

Recently, H.-X. Yi [12] and independently G. D. Song and N. Li [13] proved the following theorem.

THEOREM D. *Suppose that f and g share the sets S_1 and S_2 IM. If $n \geqq 14$, then f and g satisfy* (1.1) *or* (1.2).

In this paper, we prove the following theorems, which are improvements and supplements of the above theorems.

THEOREM 1. *Suppose that f and g share the sets S_1 CM and S_2 IM. If $n \geqq 6$, then f and g satisfy* (1.1) *or* (1.2).

Remark. From Theorem 1 we immediately obtain that the assumption "$n \geqq 7$" in Theorem C can be replaced by "$n \geqq 6$".

THEOREM 2. *Suppose that f and g share the sets S_1 IM and S_2 CM. If $n \geqq 10$, then f and g satisfy* (1.1) *or* (1.2).

THEOREM 3. *Suppose that f and g share the sets S_1 and S_2 IM. If $n \geqq 11$, then f and g satisfy* (1.1) *or* (1.2).

Remark. From Theorem 3 we immediately obtain that the assumption "$n \geqq 14$" in Theorem D can be replaced by "$n \geqq 11$".

2. Some lemmas

Let F and G be two nonconstant meromorphic functions such that F and G share 1 IM. Let z_0 be a 1-point of F of order p, a 1-point of G of order q. We denote by $\overline{N}_L(r, 1/(F-1))$ the counting function of those 1-points of F where $p>q$; by $N_E^{1)}(r, 1/(F-1))$ the counting function of those 1-points of F where $p=q=1$; by $\overline{N}_E^{(2}(r, 1/(F-1))$ the counting function of those 1-points of F where $p=q\geqq 2$, each point in these counting functions is counted only once. In the same way, we can define $\overline{N}_L(r, 1/(G-1))$, $N_E^{1)}(r, 1/(G-1))$ and $\overline{N}_E^{(2}(r, 1/(G-1))$ (see [14]). Particularly, if F and G share 1 CM, then

$$\overline{N}_L\left(r, \frac{1}{F-1}\right)=\overline{N}_L\left(r, \frac{1}{G-1}\right)=0. \tag{2.1}$$

With these notations, it is easy to see that

$$N_E^{1)}\left(r, \frac{1}{F-1}\right)=N_E^{1)}\left(r, \frac{1}{G-1}\right), \tag{2.2}$$

$$\overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right)=\overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) \tag{2.3}$$

and

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F-1}\right)&=N_E^{1)}\left(r, \frac{1}{F-1}\right)+\overline{N}_L\left(r, \frac{1}{F-1}\right)\\&\quad+\overline{N}_L\left(r, \frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right)\\&=\overline{N}\left(r, \frac{1}{G-1}\right).\end{aligned} \tag{2.4}$$

LEMMA 1. *Let*

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right), \tag{2.5}$$

where F and G are two nonconstant meromorphic functions. If F and G share 1 IM and $H\not\equiv 0$, then

$$N_E^{1)}\left(r, \frac{1}{F-1}\right)\leqq N(r, H)+S(r, F)+S(r, G). \tag{2.6}$$

Proof. Suppose that z_0 is a simple 1-point of F and G. Let

$$F(z)=1+a_1(z-z_0)+a_2(z-z_0)^2+O((z-z_0)^3),$$

$$G(z)=1+b_1(z-z_0)+b_2(z-z_0)^2+O((z-z_0)^3),$$

where $a_1\neq 0$ and $b_1\neq 0$. Then an elementary calculation gives that $H(z)=O(z-z_0)$, which proves that z_0 is a zero of H. Thus,

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) \leqq N\left(r, \frac{1}{H}\right) \leqq T(r, H)+O(1). \tag{2.7}$$

From (2.5) we obtain $m(r, H)=S(r, F)+S(r, G)$. Combining this and (2.7), we obtain the conclusion of Lemma 1.

LEMMA 2. *Let F and G be two nonconstant meromorphic functions such that F and G share 1 IM. Then*

$$\begin{aligned} T(r, F) \leqq & \bar{N}\left(r, \frac{1}{F}\right)+\bar{N}(r, F)+\bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+N_E^{1)}\left(r, \frac{1}{F-1}\right) \\ & +\bar{N}_L\left(r, \frac{1}{F-1}\right)-N_0\left(r, \frac{1}{F'}\right)-N_0\left(r, \frac{1}{G'}\right)+S(r, F)+S(r, G), \end{aligned} \tag{2.8}$$

where $N_0(r, 1/F')$ denotes the counting function corresponding to the zeros of F' that are not zeros of F and $F-1$, $N_0(r, 1/G')$ denotes the counting function corresponding to the zeros of G' that are not zeros of G and $G-1$.

Proof. By the second fundamental theorem, we have

$$\begin{aligned} T(r, F)+T(r, G) \leqq & \bar{N}\left(r, \frac{1}{F}\right)+\bar{N}(r, F)+\bar{N}\left(r, \frac{1}{F-1}\right)-N_0\left(r, \frac{1}{F'}\right)+S(r, F) \\ & +\bar{N}\left(r, \frac{1}{G}\right)+\bar{N}(r, G)+\bar{N}\left(r, \frac{1}{G-1}\right)-N_0\left(r, \frac{1}{G'}\right)+S(r, G). \end{aligned} \tag{2.9}$$

Noting that F and G share 1 IM, we get from (2.4)

$$\begin{aligned} \bar{N}\left(r, \frac{1}{F-1}\right)+\bar{N}\left(r, \frac{1}{G-1}\right) = & N_E^{1)}\left(r, \frac{1}{F-1}\right)+\bar{N}_L\left(r, \frac{1}{F-1}\right) \\ & +\bar{N}_L\left(r, \frac{1}{G-1}\right)+\bar{N}_E^{(2}\left(r, \frac{1}{G-1}\right)+\bar{N}\left(r, \frac{1}{G-1}\right) \\ \leqq & N_E^{1)}\left(r, \frac{1}{F-1}\right)+\bar{N}_L\left(r, \frac{1}{F-1}\right)+N\left(r, \frac{1}{G-1}\right) \\ \leqq & N_E^{1)}\left(r, \frac{1}{F-1}\right)+\bar{N}_L\left(r, \frac{1}{F-1}\right)+T(r, G)+O(1). \end{aligned}$$

Combining this and (2.9), we obtain the conclusion of Lemma 2.

LEMMA 3 [15, Lemma 3]. *Suppose that H is given by* (2.5) *and $H \equiv 0$, then*

$$T(r, G)=T(r, F)+O(1).$$

If further suppose that

$$\limsup_{\substack{r \to \infty \\ r \in I}} \frac{\bar{N}(r, 1/F)+\bar{N}(r, F)+\bar{N}(r, 1/G)+\bar{N}(r, G)}{T(r, F)}<1,$$

where I is a set of infinite linear measure of $0<r<\infty$, then $F \equiv G$ or $F \cdot G \equiv 1$.

Remark. Suppose that

$$F=f^n \quad \text{and} \quad G=g^n. \tag{2.10}$$

From Lemma 3 we immediately obtain that if $H\equiv 0$ and $n\geqq 5$, then f and g satisfy (1.1) or (1.2).

LEMMA 4 [3, Lemma 3]. *Let F and G be given by* (2.10). *If F and G share* 1 *IM, then*

$$\bar{N}_L\left(r, \frac{1}{F-1}\right)\leqq \bar{N}\left(r, \frac{1}{f}\right)+\bar{N}(r, f)+S(r, f), \tag{2.11}$$

$$\bar{N}_L\left(r, \frac{1}{G-1}\right)\leqq \bar{N}\left(r, \frac{1}{g}\right)+\bar{N}(r, g)+S(r, g). \tag{2.12}$$

LEMMA 5 [3, Lemma 7]. *Let*

$$V=\left(\frac{F'}{F-1}-\frac{F'}{F}\right)-\left(\frac{G'}{G-1}-\frac{G'}{G}\right), \tag{2.13}$$

where F and G are given by (2.10). *If $V\equiv 0$, and F and G share ∞ IM, then $F\equiv G$.*

LEMMA 6 [3, Lemma 8]. *Let V be given by* (2.13) *and $V\not\equiv 0$. If F and G share ∞ IM, then the poles of F and G are the zeros of V, and*

$$(n-1)\bar{N}(r, f)\leqq N(r, V)+S(r, f)+S(r, g). \tag{2.14}$$

LEMMA 7. *Assume that the conditions of Lemma* 6 *are satisfied.*

(1) *If F and G share* 1 *CM, then*

$$(n-1)\bar{N}(r, f)\leqq \bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{g}\right)+S(r, f)+S(r, g). \tag{2.15}$$

(2) *If F and G share* 1 *IM, then*

$$(n-3)\bar{N}(r, f)\leqq 2\bar{N}\left(r, \frac{1}{f}\right)+2\bar{N}\left(r, \frac{1}{g}\right)+S(r, f)+S(r, g). \tag{2.16}$$

Proof. (1) From (2.13) we have

$$N(r, V)\leqq \bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{g}\right).$$

From this and (2.14) we can obtain (2.15).

(2) From (2.13) we have

$$N(r, V)\leqq \bar{N}\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{g}\right)+\bar{N}_L\left(r, \frac{1}{F-1}\right)+\bar{N}_L\left(r, \frac{1}{G-1}\right). \tag{2.17}$$

By Lemma 4 we can obtain (2.11) and (2.12). Noting F and G share ∞ IM, from (2.11), (2.12), (2.14) and (2.17) we get (2.16).

3. Proofs of Theorem 1, Theorem 2 and Theorem 3

3.1. *Proof of Theorem* 3. Let F and G be given by (2.10). Since f and g share the sets S_1 and S_2 IM, we know from (2.10) that F and G share the values 1 and ∞ IM. By Lemma 2, we can obtain (2.8).

Let H be given by (2.5). If $H\not\equiv 0$, from Lemma 1 we can obtain (2.6). Noting that F and G share the values 1 and ∞ IM, from (2.5) we have

$$N(r, H)\leqq \overline{N}\left(r, \frac{1}{F}\right)+\overline{N}\left(r, \frac{1}{G}\right)+\overline{N}(r, F)+\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right)+N_0\left(r, \frac{1}{F'}\right)+N_0\left(r, \frac{1}{G'}\right). \tag{3.1}$$

Combining (2.6), (2.8) and (3.1), we get

$$T(r, F)\leqq 2\overline{N}\left(r, \frac{1}{F}\right)+2\overline{N}\left(r, \frac{1}{G}\right)+3\overline{N}(r, F)+2\overline{N}_L\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{G-1}\right)+S(r, F)+S(r, G). \tag{3.2}$$

Noting (2.10) and substituting (2.11) and (2.12) into (3.2) we obtain

$$nT(r, f)\leqq 4\overline{N}\left(r, \frac{1}{f}\right)+3\overline{N}\left(r, \frac{1}{g}\right)+6\overline{N}(r, f)+S(r, f)+S(r, g). \tag{3.3}$$

Let V be given by (2.13). Noting $H\not\equiv 0$, by Lemma 5 we have $V\not\equiv 0$. By Lemma 7 we obtain (2.16). Combining (2.16) and (3.3), we get

$$nT(r, f)\leqq \frac{4n}{n-3}T(r, f)+\frac{3n+3}{n-3}T(r, g)+S(r, f)+S(r, g). \tag{3.4}$$

Similarly, we have

$$nT(r, g)\leqq \frac{3n+3}{n-3}T(r, f)+\frac{4n}{n-3}T(r, g)+S(r, f)+S(r, g). \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\frac{n^2-10n-3}{n-3}\{T(r, f)+T(r, g)\}\leqq S(r, f)+S(r, g),$$

which contradicts the assumption $n\geqq 11$. Thus, $H\equiv 0$. Again by Lemma 3, we obtain the conclusion of Theorem 3.

3.2. *Proof of Theorem* 1. Let F and G be given (2.10). Since f and g share the sets S_1 CM and S_2 IM, we know from (2.10) that F and G share the values 1 CM and ∞ IM. Let H be given by (2.5). If $H\not\equiv 0$, proceeding as in the proof of Theorem 3, we can obtain (3.2). Noting (2.1) and (2.10), from (3.2) we get

$$nT(r,\ f)\leqq 2\bar{N}\Big(r,\ \frac{1}{f}\Big)+2\bar{N}\Big(r,\ \frac{1}{g}\Big)+3\bar{N}(r,\ f)+S(r,\ F)+S(r,\ G). \tag{3.6}$$

Let V be given by (2.13). Noting $H\not\equiv 0$, by Lemma 5 we have $V\not\equiv 0$. By Lemma 7 we obtain (2.15). Combining (2.15) and (3.6), we get

$$nT(r,\ f)\leqq\frac{2n+1}{n-1}\{T(r,\ f)+T(r,\ g)\}+S(r,\ f)+S(r,\ g). \tag{3.7}$$

Similarly, we have

$$nT(r,\ g)\leqq\frac{2n+1}{n-1}\{T(r,\ f)+T(r,\ g)\}+S(r,\ f)+S(r,\ g). \tag{3.8}$$

Combining (3.7) and (3.8) we have

$$\frac{n^2-5n-2}{n-1}\{T(r,\ f)+T(r,\ g)\}\leqq S(r,\ f)+S(r,\ g),$$

which contradicts the assumption $n\geqq 6$. Thus, $H\equiv 0$. Again by Lemma 3, we obtain the conclusion of Theorem 1.

3.3. *Proof of Theorem* 2. Let F and G be given (2.10). Since f and g share the sets S_1 IM and S_2 CM, we know from (2.10) that F and G share the values 1 IM and ∞ CM. Let H be given by (2.5). From (2.5) we have

$$\begin{aligned} N(r,\ H)\leqq\ &\bar{N}\Big(r,\ \frac{1}{F}\Big)+\bar{N}\Big(r,\ \frac{1}{G}\Big)+\bar{N}_L\Big(r,\ \frac{1}{F-1}\Big) \\ &+\bar{N}_L\Big(r,\ \frac{1}{G-1}\Big)+N_0\Big(r,\ \frac{1}{F'}\Big)+N_0\Big(r,\ \frac{1}{G'}\Big). \end{aligned} \tag{3.9}$$

Using (3.9) instead of (3.1) and proceeding as in the proof of Theorem 3, we can obtain the conclusion of Theorem 2.

4. Concluding remark

In this section, we shall use w and u to denote the constants $\cos(2\pi/n)+i\sin(2\pi/n)$ and $\cos(2\pi/m)+i\sin(2\pi/m)$ respectively, where n and m are positive integers. Recently, H.-X. Yi [11] proved the following result.

THEOREM E. *Let* $S_1=\{a_1+b_1,\ a_1+b_1w,\ \dots,\ a_1+b_1w^{n-1}\}$, $S_2=\{a_2+b_2,\ a_2+b_2u,\ \dots,\ a_2+b_2u^{m-1}\}$ *and* $S_3=\{\infty\}$, *where* $n\geqq 7$, $m\geqq 7$, a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1b_2\neq 0$ *and* $a_1\neq a_2$. *Suppose that* f *and* g *are nonconstant meromorphic functions. If* f *and* g *share* S_1, S_2 *and* S_3 CM, *then* $f\equiv g$.

Using Theorem 1 and proceeding as in the proof of Theorem E, we can obtain that the assumption "$n \geqq 7$ and $m \geqq 7$" in Theorem E can be replaced by "$n \geqq 6$ and $m \geqq 6$". Using Theorem 3 and proceeding as in the proof of Theorem E, we can obtain the following result.

THEOREM 4. *Let* $S_1 = \{a_1 + b_1,\ a_1 + b_1 w, \dots, a_1 + b_1 w^{n-1}\}$, $S_2 = \{a_2 + b_2,\ a_2 + b_2 u, \dots, a_2 + b_2 u^{m-1}\}$ *and* $S_3 = \{\infty\}$, *where* $n \geqq 11$, $m \geqq 11$, a_1, b_1, a_2 *and* b_2 *are constants such that* $b_1 b_2 \neq 0$ *and* $a_1 \neq a_2$. *Suppose that* f *and* g *are nonconstant meromorphic functions. If* f *and* g *share* S_1, S_2 *and* S_3 IM, *then* $f \equiv g$.

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